ON THE MODULAR EQUATION OF ORDER 11

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Extended Abstract

1. INTRODUCTION

In this paper we give the explicit form of the modular equation of order 11: $\Phi_{11}(x,y) = 0$, computed using the computer algebra system MACSYMA [10].

The modular equation $\Phi_n(x,y) = 0$ $(n \geq 2)$ was introduced by Kronecker and used by Kronecker and Weber in the theory of complex multiplication to prove the (algebraic) integrality of the "class invariants". The equation $\Phi_n(x,y) = 0$ defines a (singular) affine curve over **Z**. We hope that our result will be of some use for the study of its geometrical as well as arithmetical properties (e.g. irreducibility, singularities and desingularization).

In 1878, Smith [11] computed Φ_3 (see also Fricke [3, II.4]). Φ_5 was first computed by Berwick [1] in 1916. In 1974, Herrmann [5] determined Φ_7 explicitly. Yui [13] described an algorithm which we used in [7] to compute Φ_5 and Φ_7, being unaware of previous work. The equation we next aimed to determine was Φ_{11}. However, our algorithm when applied to Φ_{11} became inefficient, and in fact, we ran out of storage after 7 hours of VAX-780 CPU-time. Herrmann, using a slightly different algorithm, stated that his program would consume unjustifiable much of computing time to produce Φ_{11}. In spite of this pessimistic forecast, owing to a

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very lucky, so far unnoticed, mathematical property of the coefficients of the modular equation (see section 3) we were able to modify our algorithm in such a way that it requires much less space. The renewed attack, running in the background of UNIX on a VAX-780, finally produced Φ_{11}. Because of several system failures, which, though partial information was retained, destroyed our time keeping records, we cannot tell how much CPU-time was consumed. However, we are not too far off to say that the time was 20 ± 5 hours.

We present a hard copy of $\Phi_{11}(x, y)$ in the appendix. We factored out primes ≤ 1000 in the coefficients, but the remaining factors are still of substantial size (e.g. 60 digits). Readers who are interested in using Φ_{11} can obtain either a FORTRAN-style source file or a MACSYMA save module from the authors.

Since the coefficients of Φ_{11} are rather large, we felt that it was paramount to provide an independent test to check its correctness. In section 4, we describe such a test, based on a theorem of Kronecker (the Kronecker relation) (cf. Weber [12]) and our previous work [7] on the determination of class equations. This test verified our computation. We recommend that readers who are interested in using our result apply this test to avoid typographical or transmission errors when defining the polynomial.

2. MATHEMATICAL PREREQUISITES

We first introduce the elliptic modular j-invariant. For each complex number z with non-negative imaginary part, let $q = e^{2\pi i z}$ and let

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad \sigma_3(n) = \sum_{\substack{t \mid n \\ t>0}} t^3.$$

Furthermore, let

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) = q^{\frac{1}{24}} \left[1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{\frac{n(3n-1)}{2}} + q^{\frac{n(3n+1)}{2}} \right) \right].$$

The elliptic modular j-invariant $j(z)$ is defined as

$$j(z) = \left(\frac{E_4(z)}{\eta(z)^8} \right)^3.$$

We see that $j(z)$ has the q-expansion with integer coefficients

$$j(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots .$$

Now let $GL_2^+(\mathbf{Z})$ denote the set of 2×2 matrices with entries in $\mathbf{Z}$ and positive determinant. If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{Z})$, we say that α is primitive if $\mathrm{GCD}(a, b, c, d) = 1$. For a prime p, let Δ_p^* denote the subset of $GL_2^+(\mathbf{Z})$ consisting of primitive matrices with determinant p. Then $SL_2(\mathbf{Z})$ acts on Δ_p^* (indeed, the multiplication on the left or right by elements of $SL_2(\mathbf{Z})$ maps Δ_p^* into itself). The left coset representatives of Δ_p^* modulo $SL_2(\mathbf{Z})$ are given by the set A of the $p+1$ matrices:

$$A = \{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix} \text{ with } 0 \le i < p \}.$$

For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A$ and for $z = x + \sqrt{-1}\,y$ with $y > 0$, we write $j \circ \alpha$ for

$$(j \circ \alpha)(z) = j(\alpha(z)) = j\left(\frac{az+b}{cz+d}\right),$$

and form the polynomial

$$\Phi_p(x) = \prod_{\alpha \in A} (x - j \circ \alpha) = (x - j(pz)) \prod_{i=0}^{p-1} \left(x - j\left(\frac{z+i}{p}\right)\right) = x^{p+1} + \sum_{i=0}^{p} S_i(x)$$

with an indeterminate x, where $S_i(x)$ are the elementary symmetric functions in the $j \circ \alpha$. Then the coefficients of $\Phi_p(x)$ are in $\mathbf{Z}[j]$, i.e., polynomials in $j(z)$ with integral coefficients. Thus, we may view $\Phi_p(x)$ as a polynomial in two variables x and j, and we write it as

$$\Phi_p(x) = \Phi_p(x, j) \in \mathbf{Z}[x, j].$$

We call this the modular polynomial of order p. The equation $\Phi_p(x, j) = 0$ is called the modular equation of order p.

The properties of $\Phi_p(x, j)$, which are relevant in our discussion, are collected in the following theorem.

Theorem (see, e.g. Weber [12, §69], Fricke [3, II.4] and Lang [9]).

(a) *$\Phi_p(x, j)$ is symmetric with respect to x and j, i.e., $\Phi_p(x, j) = \Phi_p(j, x)$.*

(b) $\Phi_p(x, x) \in \mathbb{Z}[x]$ *and the leading term is* $-x^{2p}$.

(c) $\Phi_p(x, j)$ *satisfies the congruence* $\Phi_p(x, j) \equiv (x^p - j)(x - j^p) \pmod p$.

By virtue of the properties of $\Phi_p(x, j)$ stated in the theorem above, we can write

$$\Phi_p(x, j) = (x^p - j)(x - j^p) - \sum_{m,n=0}^{p} c_{m,n} x^m j^n$$

where $c_{m,n}$ are integers such that

$$\begin{aligned} c_{m,n} &= c_{n,m} \\ c_{m,n} &\equiv 0 \pmod p \end{aligned} \quad \text{for all } m, n = 0, 1, \ldots, p$$

and $c_{p,p} = 0$. Putting all the above together we get the following result.

Theorem (Yui [13]). *Let* $x = j(pz)$. *Then*

$$0 = \Phi_p(x, j) = (x^p - j)(x - j^p) - p \sum_{m=1}^{p} \sum_{n=0}^{m-1} d_{m,n}(x^m j^n + x^n j^m) - p \sum_{m=0}^{p-1} d_{m,m} x^m j^m$$

where $d_{m,n}$ *and* $d_{m,m}$ *are integers.*

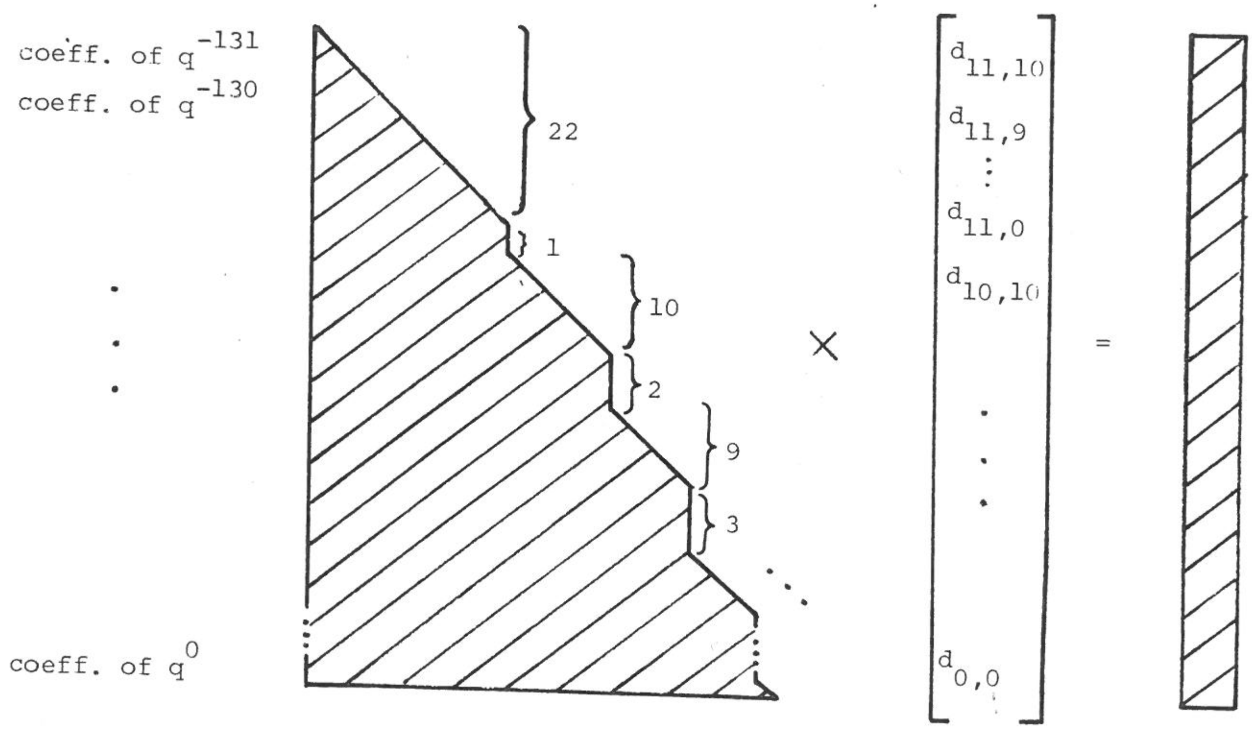
3. THE ALGORITHM

The above theorem is the basis for our algorithm. In order to determine $d_{m,n}$ and $d_{m,m}$, we substitute for j and x their q-expansions $j(q)$ and $j(q^p)$, and then equate the coefficients of the power of q in

$$(j(q^p)^p - j(q))(j(q^p) - j(q)^p) =$$
$$p \sum_{m=1}^{p} \sum_{n=0}^{m-1} d_{m,n}(j(q^p)^m j(q)^n + j(q^p)^n j(q)^m) + p \sum_{m=0}^{p-1} d_{m,m} j(q^p)^m j(q)^m.$$

Note that in this expression the term of lowest order is q^{-p^2-p}, and that $d_{0,0}$ occurs in the term of order zero. Therefore, one needs the q-expansion of j to order $p^2 + p - 1$. This leaves us with a linear system of $(p^2 + 3p)/2$ variables and $p^2 + p$ equations (the lowest term coefficient is 0). Smith [11], Berwick [1], Herrmann [5] and Yui [13] all suggest setting up this linear system and solving it for $d_{m,n}$ and $d_{m,m}$.

For $p = 11$ we get an expansion with 132 terms in 77 variables whose integer coefficients are typically 80 digits long. This expression is much too large, even before one attempts to solve the resulting system. However, on inspecting this system for Φ_5 and Φ_7, one quickly discovers the following computationally important fact: The resulting linear system is subdiagonal from lowest to highest coefficient, i.e. has the shape.



Though it might appear that this observation is important for the linear system solver, we make use of it long before that step. The idea is to set up the system for, say, the first 11 unknowns, $d_{11,10}, \ldots, d_{11,0}$. To do this we only need the q-expansion of j to order 10. After having found the correct values, we repeat this procedure for the next 11 unknowns, $d_{10,10}, \ldots, d_{10,0}$, now already using the values for the known coefficients. The q-expansion of j is needed to order 21, but the number of unknowns does not grow. In fact, one could introduce one variable at a time, instead of 11 new unknowns, thus reducing the storage requirement approximately 77-fold. Actually, we broke up the system into only two parts, since our available computing resources are abundant. Our observation also resolves an old question, namely, whether the linear system obtained from the q-expansion sufficiently determines the unknowns. It could have been that the system (of even infinitely many equations) was underdetermined, but this is not the case.

4. THE VERIFICATION

We use another property of the modular polynomial, known as the Kronecker relation.

Theorem (Kronecker, see e.g. Weber [12, 115]). *We have*

$$\Phi_p(x,x) = -\prod_D H_D(x)^{r'(D)}$$

where the quantities in the right-hand side are defined as follows. The product ranges over all $D \in \mathbf{Z}$, $D < 0$, such that $y^2 - Dx^2 = 4p$ has a solution $(x,y) \in \mathbf{Z} \times \mathbf{Z}$ with $x > 0$. Denoting by $r(D)$ the number of such solutions, the multiplicity $r'(D)$ is equal to $r(D)$ if $D < -4$, $r(D)/2$ if $D = -4$ and $r(D)/3$ if $D = -6$. $H_D(x)$ denotes the class equation for the imaginary quadratic order of discriminant D; it is an integral polynomial of degree h_D (the class number of order).

In case $-D$ is a prime (so necessarily $\equiv 3 \bmod 4$, since it must be a discriminant), we can determine the class equation $H_D(x)$ using the algorithm developed in Kaltofen and Yui [7]. For composite D the theory is more complicated and readers are referred to our full paper [8] for the explanation (see, also Weber [12] and Lang [9, §10]).

For $p = 11$, we list the discriminants D, class numbers h_D, and the corresponding class equations $H_D(x)$ with their multiplicities $r'(D)$ in the table below.

Our results satisfies the relation of Kronecker:

$$-\Phi_{11}(x,x) = H_{-7}(x)^2\, H_{-8}(x)^2\, H_{-11}(x)\, H_{-19}(x)^2\, H_{-28}(x)^2$$
$$\times H_{-35}(x)^2\, H_{-40}(x)^2\, H_{-43}(x)^2\, H_{-44}(x).$$

This verifies that the equation $\Phi_{11}(x,y) = 0$ is indeed correct.

D	$y^2 - Dx^2 = 44$	h_D	z	$H_D(x) = \prod(x - j(z))$	$r'(D)$
-7	$y = \pm 4,\ x = 2$	1	$\frac{1+\sqrt{-7}}{2}$	$x + 3^3 5^3$	2
-8	$y = \pm 6,\ x = 1$	1	$3+\sqrt{-2}$	$x - 2^6 5^3$	2
-11	$y = 0,\ x = 2$	1	$\frac{1+\sqrt{-11}}{2}$	$x + 2^{15}$	1
-19	$y = \pm 5,\ x = 1$	1	$\frac{1+\sqrt{-19}}{2}$	$x + 2^{15} 3^3$	2
-28	$y = \pm 4,\ x = 1$	1	$2+\sqrt{-7}$	$x - 3^3 5^3 17^3$	2
-35	$y = \pm 3,\ x = 1$	2	$\left\{\begin{array}{l}\frac{1+\sqrt{-35}}{6}\\ \frac{3+\sqrt{-35}}{2}\end{array}\right.$	$x^2 + 2^{19} 3^2 5^2 x$ $- 2^{30} 5^3$	2
-40	$y = \pm 2,\ x = 1$	2	$\left\{\begin{array}{l}1+\sqrt{-40}\\ \frac{\sqrt{-10}}{2}\end{array}\right.$	$x^2 - 2^7 3^2 5^2 13 \cdot 379\, x$ $+ 2^{12} 3^6 5^3 29^3$	2
-43	$y = \pm 1,\ x = 1$	1	$\frac{1+\sqrt{-43}}{2}$	$x + 2^{18} 3^3 5^3$	2
-44	$y = 0,\ x = 1$	3	$\left\{\begin{array}{l}\sqrt{-11}\\ \frac{1+\sqrt{-11}}{4}\\ \frac{3+\sqrt{-11}}{4}\end{array}\right.$	$x^3 - 2^4 1709 \cdot 41057\, x^2$ $+ 2^8 3 \cdot 11^4 24049\, x$ $- 2^{12} 11^3 17^3 29^3$	1

5. ADDITIONAL PROPERTIES

It is known that $\Phi_p(x,y)$ is absolutely irreducible. This fact can be easily proved for $\Phi_{11}(x,y)$ with the help of a criterion developed by Kaltofen [6] stating that if $\Phi_{11}(x,y)$ is irreducible over $\mathbf{Q}$ and $\Phi_{11}(x,r)$ has a linear factor for some $r \in \mathbf{Q}$, then $\Phi_{11}(x,y)$ is absolutely irreducible. Choosing $r = j((1+\sqrt{-11})/2) = -2^{15}$ we get a linear factor $H_{-11}(x) = x + 2^{15}$ dividing $\Phi_{11}(x,-2^{15})$. The irreducibility of $\Phi_{11}(x,y)$ over $\mathbf{Q}$ may be verified directly on MACSYMA.

David Masser communicated to us that Paula Cohen [2] had recently established the following bound for the absolutely largest coefficient of Φ_n, $\|\Phi_n\|$:

$$\log \|\Phi_n\| = 6\psi(n)(\log n - 2\kappa(n) + O(1)) \quad \text{as } n \to \infty$$

where

$$\psi(n) = n \prod_{\substack{p \mid n \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right), \quad \kappa(n) = \sum_{\substack{p \mid n \\ p \text{ prime}}} \frac{\log p}{p}.$$

Her estimate (ignoring $O(1)$ term) leads to $\log\|\Phi_{11}\| = 141.25$, whereas the true $\log\|\Phi_{11}\| = 289.09$. The difference by a factor of 2 can, perhaps, be explained by the fact that our n is rather small.

6. CONCLUSION

The modular equation $\Phi_{11}(x,y) = 0$ represents the (modular) algebraic correspondence

$$\{(j(z), j(\alpha(z))) \mid \alpha \in A,\ z = x + \sqrt{-1}\,y \text{ with } y > 0\} \subset \mathbf{P}^1 \times \mathbf{P}^1$$

and it defines an affine curve over $\mathbf{Z}$. After desingularization, this yields a (modular) elliptic curve with conductor 11. (For $p < 11$, $\Phi_p(x,y) = 0$, after desingularization, gives rise to a rational curve). It, therefore, seemed important to us to compute this equation explicitly to be used in future investigations.

Finally, we remark that the methods recently developed by Gross and Zagier [4] for computing values of class equations also seem to yield a very efficient algorithm for determining the explicit form of Φ_p with $p \leq 13$.

Acknowledgements

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APPENDIX

$\Phi_{11}(x, y) = 0 =$

$x^{12} + y^{12} + 2^3 \cdot 3 \cdot 11 \cdot 31 (y^{10}x^{11} + y^{11}x^{10})$

$- 71411 \cdot 2^2 \cdot 3^2 \cdot 11 (y^9x^{11} + y^{11}x^9)$

$+ 152519383 \cdot 2^5 \cdot 11 (y^8x^{11} + y^{11}x^8)$

$- 185027238353 \cdot 2 \cdot 3 \cdot 5 \cdot 11 (y^7x^{11} + y^{11}x^7)$

$+ 2443204381063 \cdot 2^4 \cdot 3^2 \cdot 11^2 (y^6x^{11} + y^{11}x^6)$

$- 803967223898807 \cdot 2^3 \cdot 11^2 \cdot 23 (y^5x^{11} + y^{11}x^5)$

$+ 24009920521667 \cdot 2^6 \cdot 3 \cdot 5 \cdot 11^2 \cdot 23 \cdot 67 (y^4x^{11} + y^{11}x^4)$

$- 2491107819565 6531 \cdot 3^2 \cdot 5 \cdot 11^2 \cdot 47 \cdot 83 (y^3x^{11} + y^{11}x^3)$

$+ 1302864869715323531 \cdot 2^3 \cdot 5^2 \cdot 11^2 \cdot 863 (y^2x^{11} + y^{11}x^2)$

$- 28353616561976008 34891 \cdot 2^2 \cdot 3 \cdot 7 \cdot 11^2 \cdot 13 (yx^{11} + y^{11}x)$

$+ 204842039071 \cdot 2^{15} \cdot 3^4 \cdot 5^5 \cdot 11 \cdot 29 \cdot 547 (x^{11} + y^{11}) - y^{11} \cdot x^{11}$

$+ 304071601918951 \cdot 2^6 \cdot 3^2 \cdot 7 \cdot 11^3 \cdot 59 \cdot 313 (y^9x^{10} + y^{10}x^9)$

$+ 2136328579151531252537261237 \cdot 2^4 \cdot 3 \cdot 11^2 (y^8x^{10} + y^{10}x^8)$

$+ 1390024623964499806523710733 \cdot 2^4 \cdot 5 \cdot 7^2 \cdot 11^3 \cdot 89 (y^7x^{10} + y^{10}x^7)$

$+ 216211652871283314750652744726722052 09 \cdot 3 \cdot 11^2 (y^6x^{10} + y^{10}x^6)$

$+ 1198618662080385562294052403766384436 3 \cdot 2^4 \cdot 3 \cdot 5 \cdot 11^2 \cdot 83 (y^5x^{10} + y^{10}x^5)$

$+ 213507160242946938854998919923028533 3001$

$\cdot 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11^2 \cdot 163 (y^4x^{10} + y^{10}x^4)$

$+ 4009436914258508906988957285878140697$

$\cdot 2^3 \cdot 3^2 \cdot 5^2 \cdot 11^2 \cdot 13 \cdot 41 \cdot 97 \cdot 313 (y^3x^{10} + y^{10}x^3)$

$+ 126413487710766963183099189805278133505334 69967$

$\cdot 2 \cdot 3 \cdot 11^2 \cdot 173 (y^2x^{10} + y^{10}x^2)$

$+ 5706913341490117715230609475565 3851$

$\cdot 2^{16} \cdot 3^4 \cdot 5^5 \cdot 7 \cdot 11^2 \cdot 53 \cdot 787 (yx^{10} + y^{10}x)$

$+ 1770507108874086630732300610321 9 \cdot 2^{32} \cdot 3^7 \cdot 5^8 \cdot 11 \cdot 41 (x^{10} + y^{10})$

$$+2310043787617 \cdot 2 \cdot 3 \cdot 7 \cdot 11^2 \cdot 137 y^{10} x^{10}$$

$$+958986152668874595646678295950027977749 \cdot 2^3 \cdot 3^2 \cdot 11^2 \cdot 29 (y^8 x^9 + y^9 x^8)$$

$$-227023852347378294634000352833934025481847$$

$$\cdot 2^2 \cdot 3 \cdot 5^3 \cdot 11^2 \cdot 17 \cdot 73 (y^7 x^9 + y^9 x^7)$$

$$+398218210423415599112603061821999718129105297253$$

$$\cdot 2^4 \cdot 5 \cdot 11^2 \cdot 37 \cdot 103 (y^6 x^9 + y^9 x^6)$$

$$-15636348956916775374459962506623439044625336536043561$$

$$\cdot 2^2 \cdot 3^2 \cdot 5 \cdot 11^2 \cdot 23 \cdot 127 (y^5 x^9 + y^9 x^5)$$

$$+5084592561048113954497357458608235518570914753005936792557$$

$$\cdot 2^5 \cdot 3^2 \cdot 5^3 \cdot 11^2 (y^4 x^9 + y^9 x^4)$$

$$-2631641938847849826248466004202500462392871407080228827723$$

$$\cdot 2^2 \cdot 5 \cdot 7 \cdot 11^2 \cdot 73 \cdot 97 \cdot 631 (y^3 x^9 + y^9 x^3)$$

$$+35281844588726974505190069979409367904482134933419992191$$

$$\cdot 2^{16} \cdot 3^4 \cdot 5^5 \cdot 11^2 \cdot 37 \cdot 307 (y^2 x^9 + y^9 x^2)$$

$$-99829907842493508262141389376076076063117836229817429$$

$$\cdot 2^{31} \cdot 3^7 \cdot 5^7 \cdot 7 \cdot 11^2 \cdot 47 (y x^9 + y^9 x)$$

$$+6201360168079554794154776324781254624005839317983$$

$$\cdot 2^{47} \cdot 3^9 \cdot 5^{10} \cdot 11 \cdot 523 (x^9 + y^9)$$

$$-5549102003290133646182846491 \cdot 11^2 \cdot 23 \cdot 107 \cdot 347 y^9 x^9$$

$$+6538603459601786748399998328460836913035658866376243$$

$$\cdot 2^5 \cdot 3^2 \cdot 5 \cdot 11^2 \cdot 13 \cdot 43 (y^7 x^8 + y^8 x^7)$$

$$+39575823334648243045699771757262514374336453051410244101837$$

$$\cdot 2^3 \cdot 3^3 \cdot 5 \cdot 11^2 \cdot 191 (y^6 x^8 + y^8 x^6)$$

$$+55184946694943711741085559572229904964746360798010979607934039691$$

$$\cdot 2^5 \cdot 3 \cdot 5^2 \cdot 11^2 \cdot 13 (y^5 x^8 + y^8 x^5)$$

$$+47667893763427400590733682246640762520305038257533404702442273428197$$

$$\cdot 3^2 \cdot 5 \cdot 11^2 \cdot 137 \cdot 239 (y^4 x^8 + y^8 x^4)$$

$$+ 797486113325666051207144427556019607967175266756951523267265691$$

$$\cdot 2^{15} \cdot 3^{5} \cdot 5^{6} \cdot 11^{2} \cdot 37 \cdot 179\,(y^{3}x^{8} + y^{8}x^{3})$$

$$+ 8009960374040182967007753370486902998209765597116211 1387839$$

$$\cdot 2^{30} \cdot 3^{9} \cdot 5^{8} \cdot 11^{2} \cdot 19 \cdot 113\,(y^{2}x^{8} + y^{8}x^{2})$$

$$+ 6977584036201576781364737231326406838149469394527541 44989$$

$$\cdot 2^{45} \cdot 3^{11} \cdot 5^{10} \cdot 11^{2} \cdot 13\,(yx^{8} + y^{8}x)$$

$$+ 683730432108521215394229342308931081392608349144 41$$

$$\cdot 2^{61} \cdot 3^{14} \cdot 5^{12} \cdot 11 \cdot 661 \cdot (x^{8} + y^{8})$$

$$+ 551175148962314491470689831868719645976312445171 \cdot 2 \cdot 3 \cdot 11^{2} \cdot 73\,y^{8}x^{8}$$

$$+ 196241595866137309132558183605234495713289835477894050448122714531 11$$

$$\cdot 2^{4} \cdot 3^{2} \cdot 5^{2} \cdot 11^{2} \cdot 29\,(y^{6}x^{7} + y^{7}x^{6})$$

$$- 3790303489503132780553935283224487717585072428050639027598798831125635 28117$$

$$\cdot 2^{3} \cdot 3^{2} \cdot 11^{2} \cdot 23\,(y^{5}x^{7} + y^{7}x^{5})$$

$$+ 160824577406730860273789387165024001737732576198609272168579968043 6681$$

$$\cdot 2^{16} \cdot 3^{5} \cdot 5^{5} \cdot 11^{2} \cdot 307\,(y^{4}x^{7} + y^{7}x^{4})$$

$$- 159957843527415451830589180941436548552071371986278493322441015 207$$

$$\cdot 2^{31} \cdot 3^{8} \cdot 5^{7} \cdot 11^{2} \cdot 17 \cdot 61\,(y^{3}x^{7} + y^{7}x^{3})$$

$$+ 184961896721807024750026898291235482859370554860027721990488 99$$

$$\cdot 2^{48} \cdot 3^{11} \cdot 5^{10} \cdot 11^{2} \cdot 41\,(y^{2}x^{7} + y^{7}x^{2})$$

$$- 323205032897532515202275404436991311479324105085890077 97$$

$$\cdot 2^{62} \cdot 3^{14} \cdot 5^{12} \cdot 11^{2} \cdot 31 \cdot 37\,(yx^{7} + y^{7}x)$$

$$+ 687009021714920181070182211269378797568514277601 491$$

$$\cdot 2^{76} \cdot 3^{17} \cdot 5^{17} \cdot 11^{2}\,(x^{7} + y^{7})$$

$$- 196770037447127085470395892412591349684010213581554712179017 357$$

$$\cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11^{2} \cdot 13\,y^{7}x^{7}$$

$$+ 523227457741748896478712662293468229488602403462904134065506855983 0069$$

$$\cdot 2^{19} \cdot 3^{5} \cdot 5^{5} \cdot 7 \cdot 11^{2} \cdot 127\,(y^{5}x^{6} + y^{6}x^{5})$$

$+\,2639597822321660760623986551115137186103254693853880722408285800749$

$\cdot\,2^{30}\cdot 3^{8}\cdot 5^{7}\cdot 11^{2}\cdot 41\cdot 269\,(y^4x^6+y^6x^4)$

$-\,3390006450591977441574987173885881895826454398227872777126649$

$\cdot\,2^{45}\cdot 3^{10}\cdot 5^{10}\cdot 11^{2}\cdot 13\cdot 191\cdot 349\,(y^3x^6+y^6x^3)$

$+\,4645016400370491862707802951436041667403655592108532123045 31$

$\cdot\,2^{60}\cdot 3^{14}\cdot 5^{12}\cdot 11^{2}\cdot 13\cdot 31\,(y^2x^6+y^6x^2)$

$-\,2116697753055142065178040088792429290699740691877530773 01$

$\cdot\,2^{75}\cdot 3^{17}\cdot 5^{17}\cdot 11^{2}\,(yx^6+y^6x)$

$+\,270909647855313899315632002810352263119290522273 03$

$\cdot\,2^{92}\cdot 3^{19}\cdot 5^{20}\cdot 11^{2}\cdot 53\,(x^6+y^6)$

$+\,17929822479611682569015747211559561628347489460983284512397278954317612 1251$

$\cdot\,2^{2}\cdot 3\cdot 7\cdot 11^{2}\cdot 641\,y^6x^6$

$-\,604196099441808431074554236446836961540340911056070363759033226213 3$

$\cdot\,2^{47}\cdot 3^{11}\cdot 5^{10}\cdot 11^{2}\,(y^4x^5+y^5x^4)$

$-\,50989444320695027911825310821174628867090275457599551374195427 1$

$\cdot\,2^{64}\cdot 3^{14}\cdot 5^{12}\cdot 11^{2}\,(y^3x^5+y^5x^3)$

$+\,288273875757574108718257118868547016275500534534111371$

$\cdot\,2^{76}\cdot 3^{18}\cdot 5^{19}\cdot 11^{2}\cdot 499\,(y^2x^5+y^5x^2)$

$-\,554253659534181630845812048633091751608705133761316 1$

$\cdot\,2^{91}\cdot 3^{20}\cdot 5^{19}\cdot 11^{2}\cdot 71\,(yx^5+y^5x)$

$-\,1653476895503145332636396574661852948285989619$

$\cdot\,2^{107}\cdot 3^{23}\cdot 5^{22}\cdot 7\cdot 11^{2}\cdot 61\,(x^5+y^5)$

$-\,19226241612254832195313713477276757020637669730798645838780745261595 3$

$\cdot\,2^{33}\cdot 3^{9}\cdot 5^{7}\cdot 7^{2}\cdot 11^{2}\,y^5x^5$

$-\,231269172271974353664230209620036871044329015363345898573 1$

$\cdot\,2^{75}\cdot 3^{17}\cdot 5^{17}\cdot 7^{2}\cdot 11^{2}\,(y^3x^4+y^4x^3)$

$+\,568778932684149150734806516764852153707646931172588 11$

$$\cdot 2^{92} \cdot 3^{20} \cdot 5^{19} \cdot 11^{2} \cdot 167\,(y^{2}x^{4} + y^{4}x^{2})$$

$$+ 5415225397677834475422807358887936494076700872475 9$$

$$\cdot 2^{105} \cdot 3^{23} \cdot 5^{22} \cdot 11^{2}\,(yx^{4} + y^{4}x)$$

$$+ 179394759835202390842768047676772279232606213 7$$

$$\cdot 2^{120} \cdot 3^{26} \cdot 5^{24} \cdot 11^{2}\,(x^{4} + y^{4})$$

$$+ 7440188171658385376358337002121255117746294641223361 39999$$

$$\cdot 2^{61} \cdot 3^{14} \cdot 5^{12} \cdot 11^{2} \cdot 13 \cdot 71^{2} \cdot 947\,y^{4}x^{4}$$

$$+ 4985689196260039104574994880749571567063178837 79$$

$$\cdot 2^{105} \cdot 3^{23} \cdot 5^{22} \cdot 11^{2} \cdot 31 \cdot 61\,(y^{2}x^{3} + y^{3}x^{2})$$

$$+ 4584255170679832459479690138586689073359163$$

$$\cdot 2^{122} \cdot 3^{26} \cdot 5^{25} \cdot 11^{2} \cdot 13 \cdot 17\,(yx^{3} + y^{3}x)$$

$$- 1098837621190731896352705522344284221 7 \cdot 2^{135} \cdot 3^{27} \cdot 5^{29} \cdot 11^{3} \cdot 373\,(x^{3} + y^{3})$$

$$- 1756916645734597206593739283186846037202205214735 3$$

$$\cdot 2^{92} \cdot 3^{19} \cdot 5^{19} \cdot 11^{2} \cdot 17 \cdot 263 \cdot 887\,y^{3}x^{3}$$

$$- 3718315996872737698045165105650113507860 3$$

$$\cdot 2^{135} \cdot 3^{28} \cdot 5^{30} \cdot 11^{2}\,(yx^{2} + y^{2}x)$$

$$+ 1646536955955348221662739 \cdot 2^{153} \cdot 3^{31} \cdot 5^{33} \cdot 7 \cdot 11^{3} \cdot 17^{3} \cdot 29^{3}\,(x^{2} + y^{2})$$

$$- 2613350213961239479483298763842596729317481 3$$

$$\cdot 2^{121} \cdot 3^{27} \cdot 5^{24} \cdot 11^{2} \cdot 79\,y^{2}x^{2}$$

$$- 162899624593 \cdot 2^{171} \cdot 3^{34} \cdot 5^{34} \cdot 11^{3} \cdot 17^{6} \cdot 29^{6} \cdot 41\,(x + y)$$

$$+ 26094174253158533018911091 \cdot 2^{153} \cdot 3^{31} \cdot 5^{31} \cdot 17^{3} \cdot 29^{3} \cdot 139 \cdot 487\,yx$$

$$+ 2^{189} \cdot 3^{36} \cdot 5^{36} \cdot 11^{3} \cdot 17^{9} \cdot 29^{9}$$