Riffle Rank of Even-Order Tensors and Lower Bounds for Arithmetic Formulas

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Abstract

Motivated by the question of VF vs. VBP (arithmetic formulas vs. branching programs), we introduce a complexity measure on even-order tensors $f : [n]^{2k} \to \mathbb{F}$ called *riffle rank*. Denoted by $\mathrm{R}_{\mathrm{riffle}}(f)$, this is the minimum $m \geq 0$ such that f admits a sum-product decomposition of the form

$$f(a_1, b_1, \dots, a_k, b_k) \;=\; \sum_{\ell=1}^{m} B_\ell(b_1, \dots, b_k) \cdot \prod_{i=1}^{k} A_{\ell,i}(b_1, \dots, b_{i-1}, a_i)$$

for some collection of tensors $B_\ell : [n]^k \to \mathbb{F}$ and $A_{\ell,i} : [n]^i \to \mathbb{F}$. Whereas the tensor rank of f may be as large as $\Omega(n^{2k-1}/k)$, riffle rank is bounded by n^k. We conjecture that the upper bound $\mathrm{R}_{\mathrm{riffle}}(f) \leq n^k$ is tight (or nearly tight) with respect to the equality tensor

$$EQ_{n,k}(a_1, b_1, \dots, a_k, b_k) \;=\; \mathbb{1}[\;(a_1, \dots, a_k) = (b_1, \dots, b_k)\;].$$

Motivating this conjecture, we show that an $n^{k-o(\log k)}$ lower bound on $\mathrm{R}_{\mathrm{riffle}}(EQ_{n,k})$ implies an $n^{\Omega(\log k)}$ lower bound on the arithmetic formula size of the *iterated matrix multiplication* polynomial $IMM_{n,k}$ for all $k \leq \frac{\log n}{\log \log n}$. Riffle rank thus provides a new means to potentially separate the power of poly-size arithmetic formulas vs. branching programs.

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1 Introduction

An *order-d tensor* is a function $f : [n_1] \times \cdots \times [n_d] \to \mathbb{F}$ where $\mathbb{F}$ is a field, $n_1, \ldots, n_d$ are positive integers, and $[n_i] = \{1, \ldots, n_i\}$. The *tensor rank* of f, denoted by $\mathrm{R}(f)$, is the minimum $m \geq 0$ such that f admits a sum-product decomposition

$$f(a_1, \ldots, a_d) = \sum_{\ell=1}^{m} \prod_{i=1}^{d} g_{\ell,i}(a_i)$$

for some collection of order-1 tensors $g_{\ell,i} : [n_i] \to \mathbb{F}$. Tensors and tensor rank generalize matrices and matrix rank, which are the case $d = 2$. Whereas matrix rank is well understood and easy to compute, compute tensor rank is NP-complete for all $d \geq 3$ [6]. Elementary dimension arguments show that *almost all* order-d tensors $f : [n]^d \to \mathbb{F}$ have tensor rank $\Omega(n^{d-1}/d)$. Yet, the strongest known lower bound for *explicit* order-d tensors is just $\Omega(n^{\lfloor d/2 \rfloor})$ [1].[1]

An influential theorem of Raz [13] shows that a nearly optimal tensor rank lower bound $\mathrm{R}(f) = n^{d-o(d)}$ for any tensor $f : [n]^d \to \mathbb{F}$ where $\omega(1) \leq d(n) \leq O(\frac{\log n}{\log \log n})$ implies a *super-polynomial* lower bound on the arithmetic formula size of the corresponding set-multilinear polynomial. In particular, showing $\mathrm{R}(f) = n^{d-o(d)}$ for any *explicit* f would yield a major breakthrough: a separation of algebraic complexity classes VF and VP, that is, poly-size arithmetic formulas vs. circuits.[2] Unfortunately, despite the efforts of many researchers over the decade since Raz's theorem, lower bounds for tensor rank have remained stuck at $\Omega(n^{\lfloor d/2 \rfloor})$.

In this paper, we introduce an alternative tensor-rank-like complexity measure on even-order tensors $f : [n]^{2k} \to \mathbb{F}$ called *riffle rank*, which is bounded by $\mathrm{R}_{\text{riffle}}(f) \leq n^k$ for all f. We advance a conjecture that this bound is tight (or nearly tight) with respect to one particularly simple tensor: the equality tensor $EQ_{n,k}$ defined by

$$EQ_{n,k}(a_1, b_1, \ldots, a_k, b_k) := \begin{cases} 1 & \text{if } (a_1, \ldots, a_k) = (b_1, \ldots, b_k), \\ 0 & \text{otherwise.} \end{cases}$$

Our main result (Theorem 4.4) shows that a lower bound $\mathrm{R}_{\text{riffle}}(EQ_{n,k}) = n^{k-o(\log k)}$ implies an $n^{\Omega(\log k)}$ lower bound on the arithmetic formula size of the *iterated matrix multiplication* polynomial $IMM_{n,k}$ for all $k \leq O(\log n / \log \log n)$. This would yield a separation of VF and VBP, that is, poly-size arithmetic formulas vs. branching programs. (A more general version of our main result (Theorem 4.5) — with a statement similar to the aforementioned theorem of Raz — shows that a lower bound $\mathrm{R}_{\text{riffle}}(f) = n^{k-o(\log k)}$ for any order-$2k$ tensor f with $k(n) \leq O(\frac{\log n}{\log \log n})$ implies an $n^{\Omega(\log k)}$ lower bound on the arithmetic formula size of the set-multilinear polynomial corresponding to a certain order-$k+1$ flattening of f.)

[1] A sequence of order-d tensors $f_n : [n]^d \to \mathbb{F}$ is *explicit* if $f_n(a_1, \ldots, a_d)$ can be computed by arithmetic circuits of size at most polynomial in $d \log n$, that is, at most polynomial in the size of the input $(a_1, \ldots, a_d)$.

[2] $\mathsf{VF} \subseteq \mathsf{VBP} \subseteq \mathsf{VP}$ are the classes of n-variable polynomials of $\mathrm{poly}(n)$ degree computable by $\mathrm{poly}(n)$ size arithmetic formulas (resp. branching programs, circuits). (VF is sometimes denoted by $\mathsf{VP_e}$ or VNC^1.) Separating any of these classes (or even VF from the larger class VNP) is a major open problem.

1.1 Related work

The strongest arithmetic formula lower bound for explicit polynomials in VP (or even in VNP) is currently $\Omega(n^2)$ [3]. There are many papers that establish super-polynomial lower bounds for restricted types of arithmetic formulas, including syntactically multilinear formulas [5, 12], non-commutative formulas [8, 11, 16], and low-depth formulas [10].

Some of these results are in fact $n^{\Omega(\log k)}$ lower bounds on the restricted formula size of $IMM_{n,k}$. We remark that $IMM_{n,k}$ has rank $N^{(k-1)/2}$ as an order-k tensor $[N]^k \to \mathbb{F}$, far from the tensor rank $N^{k-o(k)}$ required to produce a formula lower bound via Raz's theorem. Nevertheless, papers [8, 10, 16] (among others) successfully use tensor or matrix rank based arguments — combined with cleverly chosen restrictions and flattenings — to bound the restricted formula size of $IMM_{n,k}$.

The approach to *unrestricted* arithmetic formula lower bounds provided by riffle rank (i.e., the conjecture $\mathrm{R}_{\mathsf{riffle}}(EQ_{n,k}) = n^{k-o(\log k)}$) seems to be distinct from approaches based on tensor or matrix rank. In particular, riffle rank is not a restriction or flattening of tensor rank. Compared to tensor rank $\mathrm{R}(f)$, riffle rank $\mathrm{R}_{\mathsf{riffle}}(f)$ is defined in terms of a more complicated sum-product decomposition (likely much harder to lower bound in general); on the other hand, we only require a lower bound with respect to one specific tensor $EQ_{n,k}$.

1.2 Outline of the paper

§2 reviews the definitions of tensors, tensor rank, and arithmetic formulas.

§3 introduces the notion of *riffle rank* and establishes some basic properties.

§4 presents the main conjecture and main theorem of this paper.

§5 contains the proof of the main theorem.

§6 discusses some generalizations and special cases of the main conjecture.

2 Preliminaries

Throughout this paper, we fix an arbitrary field $\mathbb{F}$. We write $\mathbb{F}[\mathcal{X}]$ for the set of polynomials over $\mathbb{F}$ with variables from a set $\mathcal{X}$.

We regard n as a growing parameter. Objects that depend on n (tensors, formulas, and parameters like $d = d(n)$ and $k = k(n)$) should be understood as sequences of objects, one for each $n \in \{1, 2, 3, \dots\}$.

A *partition* of $[d]$ $(= \{1, \dots, d\})$, denoted $\mathcal{P} \vdash [d]$, is a set $\mathcal{P} = \{S_1, \dots, S_c\}$ of nonempty mutually disjoint sets $S_1, \dots, S_c$ such that $S_1 \cup \dots \cup S_c = [d]$. Without loss of generality, we index sets $S_1, \dots, S_c$ such that $1 \le \max(S_1) < \dots < \max(S_c) = d$.

Let $\langle \cdot, \cdot \rangle$ be a standard bijection from $[n]^2$ to $[n^2]$, such as $\langle a_1, a_2 \rangle = a_1 + (a_2 - 1)n$. More generally, for a tuple $(a_1, \dots, a_d) \in [n]^d$, we write $\langle a_1, \dots, a_d \rangle$ for the corresponding integer $\sum_{i=1}^d (a_i - 1)n^{i-1} + 1 \in [n^d]$.

For a d-tuple $(a_1, \dots, a_d) \in [n]^d$ and a set $S = \{i_1 < \dots < i_{|S|}\} \subseteq [d]$, let a_S denote the s-tuple $(a_{i_1}, \dots, a_{i_s}) \in [n]^{|S|}$ and let $\langle a_S \rangle$ denote the corresponding element of $[n^{|S|}]$.

▶ **Definition 2.1** (Tensors and tensor rank). *An* order-d tensor *is a function* $f : [n_1] \times \dots \times [n_d] \to \mathbb{F}$.

We say that f is basic *if there exist order-*1 *tensors $g_i : [n_i] \to \mathbb{F}$ (i=1,...,d) such that $f(a_1, \ldots, a_d) = g_1(a_1) \cdots g_d(a_d)$ for all $a \in [n_1] \times \cdots \times [n_d]$.*

The tensor rank *of f, denoted $\mathrm{R}(f)$, is the minimum number $m \geq 0$ of basic tensors $f_1, \ldots, f_m$ such that $f = \sum_{\ell=1}^{m} f_\ell$.*

The tensor product *of $f : [n_1] \times \cdots \times [n_d] \to \mathbb{F}$ and $g : [o_1] \times \cdots \times [o_e] \to \mathbb{F}$ is the order-$d+e$ tensor defined by $(f \otimes g)(x_1, \ldots, x_d, y_1, \ldots, y_e) = f(x_1, \ldots, x_d) \cdot g(y_1, \ldots, y_e)$.*

Note that tensor rank is sub-additive (i.e., $\mathrm{R}(f_1 + f_2) \leq \mathrm{R}(f_1) + \mathrm{R}(f_2)$ for tensors f_1, f_2 of the same format) and sub-multiplicative under $\otimes$ (i.e., $\mathrm{R}(f \otimes g) \leq \mathrm{R}(f) \cdot \mathrm{R}(g)$).

▶ **Definition 2.2** (The set-multilinear polynomial corresponding to a tensor). *We identify a tensor $f : [n_1] \times \cdots \times [n_d] \to \mathbb{F}$ with the corresponding set-multilinear polynomial in $\mathbb{F}[\mathcal{X}]$, where $\mathcal{X}$ is the set of variables $\{X_{i,a_i} : i \in [d],\, a_i \in [n_i]\}$. That is, we identify f with the polynomial*

$$f(\mathcal{X}) = \sum_{(a_1,\ldots,a_d) \in [n_1] \times \cdots \times [n_d]} f(a_1, \ldots, a_d) \cdot \prod_{i=1}^{d} X_{i,a_i}.$$

▶ **Definition 2.3** (Arithmetic formulas). *An* arithmetic formula *over a set of variables $\mathcal{X}$ is a finite rooted tree whose leaves are labeled by elements of $\mathcal{X} \cup \mathbb{F}$ and whose non-leaves are labeled by $+$ or $\times$. The* formula size *of a polynomial $P \in \mathbb{F}[\mathcal{X}]$ is the minimum number of leaves in an arithmetic formula that computes P.*

▶ **Definition 2.4** (Flattening). *For an order-d tensor $f : [n]^d \to \mathbb{F}$ and a partition $\mathcal{P} = (S_1, \ldots, S_c) \vdash [d]$, the $\mathcal{P}$-*flattening *of f is the order-c tensor $f^{\mathcal{P}} : [n^{|S_1|}] \times \cdots \times [n^{|S_c|}] \to \mathbb{F}$ defined by the equation*

$$f(a_1, \ldots, a_d) = f^{\mathcal{P}}(\langle a_{S_1} \rangle, \ldots, \langle a_{S_c} \rangle).$$

*We say that f is $\mathcal{P}$-*basic *if $f^{\mathcal{P}}$ is basic. Equivalently, f is $\mathcal{P}$-*basic *if there exist tensors $g_j : [n]^{|S_c|} \to \mathbb{F}$ ($j = 1, \ldots, c$) such that $f(a_1, \ldots, a_d) = g_1(a_{S_1}) \cdots g_c(a_{S_c})$ for all $(a_1, \ldots, a_d) \in [n]^d$.*

Note that $f = f^{\{\{1\},\ldots,\{d\}\}}$ and $\mathrm{R}(f^{\mathcal{Q}}) \leq \mathrm{R}(f^{\mathcal{P}})$ whenever $\mathcal{P}, \mathcal{Q} \vdash [d]$ and $\mathcal{P}$ refines $\mathcal{Q}$.

The following lemma (essentially Lemma 10 of [16]) gives a "log-product decomposition" of a tensor, induced by any arithmetic formula that computes the associated polynomial. There are many variants of this basic lemma in the literature (see [5, 7, 14]).

▶ **Lemma 2.5.** *Suppose that $f : [n]^d \to \mathbb{F}$ is computed by an arithmetic formula of size s. There exist a number $m = \mathrm{poly}(s) \cdot O(\log s)^d$, tensors $f_1, \ldots, f_m : [n]^d \to \mathbb{F}$ and partitions $\mathcal{P}_1, \ldots, \mathcal{P}_m \vdash [d]$ such that*

- *$f = \sum_{\ell=1}^{m} f_\ell$,*
- *each f_ℓ is $\mathcal{P}_\ell$-basic,*
- *each $\mathcal{P}_\ell$ has size $\lceil \log_2(d) \rceil$.*

Proof. Let Φ be an arithmetic formula of size s that computes the polynomial associated with f. We first convert Φ to an equivalent formula Φ_1 of depth $\Delta_1 = O(\log s)$ and size $s_1 = s^{O(1)}$ by the "Formula Balancing Theorem" of Brent [2]. We next convert Φ_1 to an equivalent syntactically multilinear formula Φ_2 of size $s_2 = O((\Delta_1 + 2)^d \cdot s_1)$ by the "Multilinearization

Theorem" of Raz [13] (Theorem 3). Finally, we obtain the desired decomposition $f = \sum_{\ell=1}^{m} f_\ell$ into $\mathcal{P}_\ell$-basic tensors f_ℓ where $|\mathcal{P}_\ell| = \lceil \log_2(d) \rceil$ and $m = s_2 = s^{O(1)} \cdot O(\log s)^d$ via the "Log-Product Lemma" of Limaye, Srinivasan and Tavenas [16] (Lemma 10). Applying these same steps in the non-commutative setting results in partitions $\mathcal{P}_\ell \vdash [n+1]$ where all sets are intervals. ◀

3 Riffle rank

We now define *riffle rank* and examine its relationship to tensor rank.

▶ **Definition 3.1** (Riffle rank). *Let $f : [n]^{2k} \to \mathbb{F}$ be an even-order tensor.*

We say that f is riffle-basic *if there exist tensors $A_i : [n]^i \to \mathbb{F}$ $(i = 1, \dots, k)$ and $B : [n]^k \to \mathbb{F}$ such that*

$$f(a_1, b_1, \dots, a_k, b_k) = B(b_1, \dots, b_k) \cdot \prod_{i=1}^{k} A_i(b_1, \dots, b_{i-1}, a_i).$$

The riffle rank *of f, denoted $\mathrm{R}_{\mathrm{riffle}}(f)$, is the minimum number $m \geq 0$ of riffle-basic tensors $f_1, \dots, f_m$ such that $f = \sum_{\ell=1}^{m} f_\ell$.*

(Although we don't consider it in this paper, the definition of riffle rank extends to general even-order tensors $f : [n_1] \times [n_2] \times \cdots \times [n_{2k-1}] \times [n_{2k}] \to \mathbb{F}$.)

Note that riffle rank is not a flattened tensor rank, due to the multiple occurrences of coordinates $b_1, \dots, b_{k-1}$ in different terms $B, A_1, \dots, A_k$ of the product $B \cdot \prod_{i=1}^{k} A_i$ in the definition of riffle-basic tensors.

The next lemma records a simple observation that riffle rank is bounded by n^k.

▶ **Lemma 3.2.** *For all $f : [n]^{2k} \to \mathbb{F}$, we have $\mathrm{R}_{\mathrm{riffle}}(f) \leq n^k$.*

Proof. This is shown by the following sum-product decomposition of f:

$$f(a_1, b_1, \dots, a_k, b_k) = \sum_{\alpha \in [n]^k} \overbrace{f(\alpha_1, b_1, \dots, \alpha_k, b_k)}^{B_\alpha(b_1,\dots,b_k)} \cdot \prod_{i=1}^{k} \overbrace{\mathbb{1}[a_i = \alpha_i]}^{A_{\alpha,i}(b_1,\dots,b_{i-1},a_i)}.$$ ◀

3.1 Relationship between riffle rank and tensor rank

It is easy to see that $\mathrm{R}_{\mathrm{riffle}}(f) \leq \mathrm{R}(f)$ (i.e., riffle rank is at most tensor rank), since every basic tensor $(a_1, b_1, \dots, a_k, b_k) \mapsto \prod_{i=1}^{k} A_i(a_i) \cdot B_i(b_i)$ is clearly also riffle-basic. We slightly strengthen this observation by noticing that $\mathrm{R}_{\mathrm{riffle}}(f)$ is in fact bounded by the tensor rank of a certain order-$k+1$ flattening of f.

▶ **Definition 3.3** (Flattening $f^\flat$ of f). *For an order-$2k$ tensor $f : [n]^{2k} \to \mathbb{F}$, let $f^\flat : [n] \times [n^2]^{k-1} \times [n] \to \mathbb{F}$ denote the order-$k+1$ flattening of f defined by the equation*

$$f(a_1, b_1, a_2, b_2, a_3, \dots, b_{k-1}, a_k, b_k) = f^\flat(a_1, \langle b_1, a_2 \rangle, \langle b_2, a_3 \rangle, \dots, \langle b_{k-1}, a_k \rangle, b_k).$$

(Recall that $\langle \cdot, \cdot \rangle$ is a bijection from $[n]^2$ to $[n^2]$.) That is, in the notation of Def. 2.4, $f^\flat$ is the flattening of f with respect to the partition

$$\flat = \{\{1\}, \{2,3\}, \{4,5\}, \dots, \{2k-2, 2k-1\}, \{k\}\} \vdash [2k].$$

The next lemma records a few (in)equalities relating riffle rank and tensor rank.

▶ **Lemma 3.4** (Relating riffle rank and tensor rank)**.**

1. *For all matrices* $f : [n]^2 \to \mathbb{F}$ *(i.e., the case* $k = 1$*),*

$$\mathrm{R}_{\text{riffle}}(f) \;=\; \mathrm{R}(f^\flat) \;=\; \mathrm{R}(f) \;=\; \text{the matrix rank of } f.$$

2. *For all tensors* $f : [n]^{2k} \to \mathbb{F}$*,*

$$\mathrm{R}_{\text{riffle}}(f) \;\leq\; \mathrm{R}(f^\flat) \;\leq\; \mathrm{R}(f) \;=\; O(n^{2k-1}).$$

3. *For almost all tensors* $f : [n]^{2k} \to \mathbb{F}$*,*

$$\mathrm{R}(f^\flat) \;=\; \Omega(n^{2k-2}/k), \qquad \mathrm{R}(f) \;=\; \Omega(n^{2k-1}/k).$$

Proof. (1) and (2) are immediate from definitions. (3) is a standard fact which is established by counting/dimension arguments (see [1, 9, 15]). ◀

For $k \geq 2$, notice the gap between the upper bound $\mathrm{R}_{\text{riffle}}(f) = O(n^{2k-1})$ for all f (Lemma 3.2) and lower bound $\mathrm{R}(f^\flat) = \Omega(n^{2k-2}/k)$ for almost all f (Lemma 3.4(3)).

4 Conjecture and Main Theorem

We shall now focus on one specific order-$2k$ tensor.

▶ **Definition 4.1.** *The* equality tensor $EQ_{n,k} : [n]^{2k} \to \mathbb{F}$ *is defined by*

$$EQ_{n,k}(a_1, b_1, \ldots, a_k, b_k) := \begin{cases} 1 & \text{if } (a_1, \ldots, a_k) = (b_1, \ldots, b_k), \\ 0 & \text{otherwise.} \end{cases}$$

Another way to view $EQ_{n,k}$: it is the k-fold tensor product $I_n^{\otimes k} = I_n \otimes \cdots \otimes I_n$ (k times) of the identity matrix $I_n : [n]^2 \to \mathbb{F}$. By further flattening $EQ^\flat_{n,k}$ to a matrix $[n^k] \times [n^k] \to \mathbb{F}$, we see that

$$\mathrm{R}(EQ^\flat_{n,k}) = \mathrm{R}(EQ_{n,k}) = n^k.$$

This raises the question: what about the riffle rank of $EQ_{n,k}$? We conjecture that $\mathrm{R}_{\text{riffle}}(EQ_{n,k})$ is nearly (if not exactly) n^k.

▶ **Conjecture 4.2** (Strong version)**.** $\mathrm{R}_{\text{riffle}}(EQ_{n,k}) = n^k$*, or at least* $\mathrm{R}_{\text{riffle}}(EQ_{n,k}) = \Omega(n^k)$*.*

Conjecture 4.2 is true for $k = 1$ (where $\mathrm{R}_{\text{riffle}}$ coincides with matrix rank by Lemma 3.4(1)). We examine the first open case $k = 2$ in Section 6.1.

The following weaker version of Conjecture 4.2 suffices for our application to arithmetic formula lower bounds.

▶ **Conjecture 4.3** (Weaker version)**.** $\mathrm{R}_{\text{riffle}}(EQ_{n,k}) = n^{k-o(\log k)}$*, if not for all* n *and* k*, then at least for some growing* $k(n) = \omega(1)$*.*

As motivation for studying the riffle rank of $EQ_{n,k}$, we show that the weaker Conjecture 4.3 implies nearly tight lower bounds on the formula size of the *iterated matrix multiplication* polynomial $IMM_{n,k}$, and hence a separation of classes $\mathsf{VF} \neq \mathsf{VBP}$.

Before stating our main result, let us first formally define $IMM_{n,k}$. This is usually regarded as a set-multilinear polynomial in variables $\mathcal{X} = \{X_{i,\langle s,t\rangle} : i \in [k],\ s,t \in [n]\}$ defined by

$$IMM_{n,k}(\mathcal{X}) = \sum_{(p_1,\dots,p_{k-1})\in[n]^{k-1}} X_{1,\langle 1,p_1\rangle} X_{2,\langle p_1,p_2\rangle} \cdots X_{k-1,\langle p_{k-1},p_{k-1}\rangle} X_{k,\langle p_{k-1},1\rangle}.$$

If we view $X_{i,\langle s,t\rangle}$ as representing the (s,t)-entry in an $n \times n$ matrix X_i, then $IMM_{n,k}(\mathcal{X})$ gives the $(1,1)$-entry of the iterated matrix product $X_1 \cdots X_d$. As a set-multilinear polynomial in $\mathbb{F}[\mathcal{X}]$, this corresponds to an order-k tensor of format $[n^2]^k \to \mathbb{F}$.

However, since variables $X_{1,\langle s,t\rangle}$ for $s \neq 1$ and $X_{k,\langle s,t\rangle}$ for $t \neq 1$ do not occur in any monomial of $IMM_{n,k}(\mathcal{X})$, we prefer to regard $IMM_{n,k}$ as a set-multilinear polynomial over variables

$$\mathcal{X}' = \{X_{1,\langle 1,t\rangle} : t \in [n]\} \cup \{X_{i,\langle s,t\rangle} : 2 \leq i \leq k-1,\ s,t \in [n]\} \cup \{X_{k,\langle s,1\rangle} : s \in [n]\}.$$

This polynomial instead corresponds to an order-k tensor of format $[n] \times [n^2]^{k-1} \times [n] \to \mathbb{F}$, which is how we shall regard $IMM_{n,k}$ as a tensor.

Note that $IMM_{n,k}$ is precisely the flattened tensor $EQ^\flat_{n,k-1}$. That is, for all $(a_1, b_1, \dots, a_{k-1}, b_{k-1}) \in [n]^{2(k-1)}$, we have

$$IMM_{n,k}(a_1, \langle b_1, a_2\rangle, \dots, \langle b_{k-2}, a_{k-1}\rangle, b_{k-1}) = EQ_{n,k-1}(a_1, b_1, \dots, a_{k-1}, b_{k-1}).$$

Based on this relationship, our main theorem lower bounds the arithmetic formula size of $IMM_{n,k}(\mathcal{X}')$ in terms of the riffle rank of tensors $EQ_{n,k}$.

▶ **Theorem 4.4** (Main Theorem). *If $\mathrm{R}_{\mathrm{riffle}}(EQ_{n,k}) = n^{k-o(\log k)}$, then $IMM_{n,k}$ has arithmetic formula size $n^{\Omega(\log k)}$ for all $k \leq \frac{\log n}{\log\log n}$, which as a consequence implies $\mathsf{VF} \neq \mathsf{VBP}$.*

Theorem 4.4 in fact follows from the $f = EQ_{n,k}$ case of the following more general theorem.

▶ **Theorem 4.5.** *For any even-order tensor $f : [n]^{2k} \to \mathbb{F}$, if the flattened tensor $f^\flat : [n] \times [n^2]^{k-1} \times [n]$ is computable by an arithmetic formula of size s, then*

$$\mathrm{R}_{\mathrm{riffle}}(f) \leq n^{k-O(\log k)} \cdot s^{O(1)} \cdot O(\log s)^{k+1}.$$

5 Proof of the Main Theorem

5.1 Riffle rank with respect to a partition $\mathcal{P} \vdash [k+1]$

In order to prove Theorem 4.4, we consider a generalization of riffle rank $\mathrm{R}^{\mathcal{P}}_{\mathrm{riffle}}(f)$ with respect to a partition $\mathcal{P} \vdash [k+1]$.

▶ **Definition 5.1.** *For a nonempty set $\emptyset \neq S \subseteq [k+1]$, we define sets $S^\uparrow \subseteq [k]$ and $S^\downarrow \subseteq S^\Downarrow \subseteq [k]$ by*

$$\begin{aligned} S^\uparrow &:= \{i \in [k] : i \in S\} &&= S \cap [k], \\ S^\downarrow &:= \{i \in [k] : i+1 \in S\} &&= (S-1) \cap [k], \\ S^\Downarrow &:= \{i \in [k] : i+1 \leq \max(S)\} &&= [\max(S)-1]. \end{aligned}$$

For example, for an interval $[s,t] = \{s, s+1, \dots, t\}$ where $1 \le s \le t \le k+1$, we have

$$[s,t]^{\uparrow} = [s, \max(t,k)], \qquad [s,t]^{\downarrow} = [\min(1, s-1), t-1], \qquad [s,t]^{\Downarrow} = [1, t-1].$$

▶ **Definition 5.2.** *Let $f : [n]^{2k} \to \mathbb{F}$ be an even-order tensor, and let $\mathcal{P}$ be a partition of $[k+1]$. We say that f is:*

- $\mathcal{P}$-flat-basic *if there exist tensors $g_S : [n]^{|S^{\downarrow}|} \times [n]^{|S^{\uparrow}|} \to \mathbb{F}$ $(S \in \mathcal{P})$ such that for all $a, b \in [n]^k$,*

$$f(a_1, b_1, \dots, a_k, b_k) = \prod_{S \in \mathcal{P}} g_S(b_{S^{\downarrow}}, a_{S^{\uparrow}}),$$

- $\mathcal{P}$-riffle-basic *if there exist tensors $g_S : [n]^{|S^{\Downarrow}|} \times [n]^{|S^{\uparrow}|} \to \mathbb{F}$ $(S \in \mathcal{P})$ such that for all $a, b \in [n]^k$,*

$$f(a_1, b_1, \dots, a_k, b_k) = \prod_{S \in \mathcal{P}} g_S(b_{S^{\Downarrow}}, a_{S^{\uparrow}}).$$

(The only difference between these two definitions is the switch from $S^{\downarrow}$ to $S^{\Downarrow}$.)

$\mathcal{P}$-flat rank *(resp.* $\mathcal{P}$-riffle rank*), denoted $\mathrm{R}^{\mathcal{P}}_{\text{flat}}(f)$ (resp. $\mathrm{R}^{\mathcal{P}}_{\text{riffle}}(f)$), is the minimum number of $\mathcal{P}$-flat-basic tensors (resp. $\mathcal{P}$-riffle-basic tensors) that sum to f.*

▶ **Lemma 5.3.** *The following statements hold for every even-order tensor $f : [n]^{2k} \to \mathbb{F}$ and partition $\mathcal{P} \vdash [k+1]$.*

1. $\mathrm{R}^{\mathcal{P}}_{\text{riffle}}(f) \le \mathrm{R}^{\mathcal{P}}_{\text{flat}}(f)$.
2. *With respect to the singleton partition $\{\{1\}, \dots, \{k+1\}\}$, we have $\mathrm{R}^{\{\{1\},\dots,\{k+1\}\}}_{\text{riffle}}(f) = \mathrm{R}_{\text{riffle}}(f)$ and $\mathrm{R}^{\{\{1\},\dots,\{k+1\}\}}_{\text{flat}}(f) = \mathrm{R}(f^{\flat})$.*
3. *$\mathrm{R}^{\mathcal{P}}_{\text{flat}}$ is a flattened tensor rank, namely $\mathrm{R}^{\mathcal{P}}_{\text{flat}}(f) = \mathrm{R}(f^{\mathcal{P}^*})$ for the partition $\mathcal{P}^* \vdash [2k]$ defined by*

$$\mathcal{P}^* := \{\{2i-1 : i \in S^{\downarrow}\} \cup \{2i : i \in S^{\uparrow}\} : S \in \mathcal{P}\}.$$

4. *$f^{\mathcal{P}^*} = (f^{\flat})^{\mathcal{P}}$, that is, the $\mathcal{P}^*$-flattening of f coincides with the $\mathcal{P}$-flattening of the $\flat$-flattening of f.*
5. *f is $\mathcal{P}$-flat-basic if, and only if, $f^{\flat}$ is $\mathcal{P}$-basic (in the sense of Def. 2.4).*

Proof. (1) follows from the implication *$\mathcal{P}$-flat-basic* $\Rightarrow$ *$\mathcal{P}$-riffle-basic*, which is a consequence of the fact that $S^{\downarrow} \subseteq S^{\Downarrow}$ for all $\emptyset \ne S \subseteq [k+1]$.

(2) follows from the coincidence of definitions *$\flat$-basic* $\Leftrightarrow$ *$\{\{1\}, \dots, \{k+1\}\}$-flat-basic* and *riffle-basic* $\Leftrightarrow$ *$\{\{1\}, \dots, \{k+1\}\}$-riffle-basic*.

(3) are (4) are straightforward from definitions. In particular, note that $\mathcal{P}^*$ is a partition of $[2k]$, since both $\{S^{\downarrow} : S \in \mathcal{P}\} \setminus \{\emptyset\}$ and $\{S^{\uparrow} : S \in \mathcal{P}\} \setminus \{\emptyset\}$ are partitions of $[k]$.

(5) now follows from the chain of equivalences: f is $\mathcal{P}$-flat-basic $\Leftrightarrow$ $f^{\mathcal{P}^*}$ is basic $\Leftrightarrow$ $(f^{\flat})^{\mathcal{P}}$ is basic $\Leftrightarrow$ $f^{\flat}$ is $\mathcal{P}$-basic. ◀

The final lemma needed for our proof of Theorem 4.4 gives an upper bound on riffle rank in terms of $\mathcal{P}$-riffle rank for any partition $\mathcal{P}$.

▶ **Lemma 5.4.** *For every even-order tensor $f : [n]^{2k} \to \mathbb{F}$ and partition $\mathcal{P} = \{S_1, \dots, S_c\} \vdash [k+1]$, we have $\mathrm{R}_{\text{riffle}}(f) \le n^{k-c+1} \cdot \mathrm{R}^{\mathcal{P}}_{\text{riffle}}(f)$.*

Proof. Without loss of generality, assume that the sets $S_1, \dots, S_c$ in $\mathcal{P}$ are ordered such that $1 \le t_1 < t_2 < \dots < t_c = k+1$ where $t_j := \max(S_j)$. In particular, note that $S_c^{\Downarrow} = [k]$.

Let $m := \mathrm{R}^{\mathcal{P}}_{\mathrm{riffle}}(f)$ as witnessed by a sum-product decomposition

$$f(a_1, b_1, \dots, a_k, b_k) = \sum_{\ell \in [m]} \prod_{j \in [c]} g_{\ell,j}(b_{S_j^{\Downarrow}}, a_{S_j^{\uparrow}}).$$

For each $\ell \in [m]$ and $\alpha \in [n]^{[k] \setminus \{t_1, \dots, t_{c-1}\}}$, we define tensors

$$B_{\ell,\alpha} : [n]^k \to \mathbb{F} \quad \text{and} \quad A_{\ell,\alpha,i} : [n]^i \to \mathbb{F} \ (i = 1, \dots, k)$$

as follows:

$$B_{\ell,\alpha}(b_1, \dots, b_k) := g_{\ell,c}(b_{S_c^{\Downarrow}}, \alpha_{S_c^{\uparrow}}) \quad (= g_{\ell,c}(b_1, \dots, b_k, \alpha_{S_c^{\uparrow}}),$$

$$A_{\ell,\alpha,i}(b_1, \dots, b_{i-1}, a_i) := \begin{cases} \mathbb{1}[\, a_i = \alpha_i \,] & \text{if } i \in [k] \setminus \{t_1, \dots, t_{c-1}\}, \\ g_{\ell,j}(b_{S_j^{\Downarrow}}, \alpha_{S_j^{\uparrow} \setminus \{t_j\}}, a_{t_j}) & \text{if } i = t_j \text{ for some (unique) } j \in [c-1]. \end{cases}$$

Notice that

$$\begin{aligned} &B_{\ell,\alpha}(b_1, \dots, b_k) \cdot \prod_{i \in [k]} A_{\ell,\alpha,i}(b_1, \dots, b_{i-1}, a_i) \\ &\quad = g_{\ell,c}(b_{S_c^{\Downarrow}}, \alpha_{S_c^{\uparrow}}) \cdot \prod_{i \in [k] \setminus \{t_1, \dots, t_{c-1}\}} \mathbb{1}[\, a_i = \alpha_i \,] \cdot \prod_{j \in [c-1]} g_{\ell,j}(b_{S_j^{\Downarrow}}, \alpha_{S_j^{\uparrow} \setminus \{t_j\}}, a_{t_j}) \\ &\quad = \begin{cases} g_{\ell,c}(b_{S_c^{\Downarrow}}, \alpha_{S_c^{\uparrow}}) \cdot \displaystyle\prod_{j \in [c-1]} g_{\ell,j}(b_{S_j^{\Downarrow}}, \alpha_{S_j^{\uparrow} \setminus \{t_j\}}, a_{t_j}) & \text{if } \alpha = a_{[k] \setminus \{t_1, \dots, t_{c-1}\}}, \\ 0 & \text{otherwise,} \end{cases} \\ &\quad = \begin{cases} \displaystyle\prod_{j \in [c]} g_{\ell,j}(b_{S_j^{\Downarrow}}, a_{S_j^{\uparrow}}) & \text{if } \alpha = a_{[k] \setminus \{t_1, \dots, t_{c-1}\}}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We now obtain an upper bound on $\mathrm{R}_{\mathrm{riffle}}(f)$ via the sum-product decomposition:

$$\begin{aligned} &\sum_{(\ell,\alpha) \in [m] \times [n]^{[k] \setminus \{t_1, \dots, t_{c-1}\}}} B_{\ell,\alpha}(b_1, \dots, b_k) \cdot \prod_{i \in [k]} A_{\ell,\alpha,i}(b_1, \dots, b_{i-1}, a_i) \\ &\qquad = \sum_{(\ell,\alpha) \in [m] \times [n]^{[k] \setminus \{t_1, \dots, t_{c-1}\}}} \mathbb{1}[\, \alpha = a_{[k] \setminus \{t_1, \dots, t_{c-1}\}} \,] \cdot \prod_{j \in [c]} g_{\ell,j}(b_{S_j^{\Downarrow}}, a_{S_j^{\uparrow}}) \\ &\qquad = \sum_{\ell \in [m]} \prod_{j \in [c]} g_{\ell,j}(b_{S_j^{\Downarrow}}, a_{S_j^{\uparrow}}) \\ &\qquad = f(a_1, b_1, \dots, a_k, b_k). \end{aligned}$$

This shows that $\mathrm{R}_{\mathrm{riffle}}(f) \le \#([m] \times [n]^{[k] \setminus \{t_1, \dots, t_{c-1}\}}) = m \cdot n^{k-c+1} = \mathrm{R}^{\mathcal{P}}_{\mathrm{riffle}}(f) \cdot n^{k-c+1}$, as required. ◀

▶ Remark 5.5. The bound $\mathrm{R}(f^\flat) \le n^{2(k-c+1)} \cdot \mathrm{R}^{\mathcal{P}}_{\mathrm{flat}}(f)$ can be shown by an argument similar to the proof of Lemma 5.4. However, the additional factor 2 in the exponent of n prevents lower bounds on $\mathrm{R}(EQ^\flat_{n,k})$ (in particular, the tight bound $\mathrm{R}(EQ^\flat_{n,k}) = n^k$) from implying nontrivial arithmetic formulas lower bound in the manner of Theorem 4.4.

5.2 Proof of Theorems 4.4 and 4.5

We are now ready to prove our main theorem relating the riffle rank of $EQ_{n,k}$ to lower bounds on the formula size of $IMM_{n,k}$.

Proof of Theorem 4.4. Suppose that $IMM_{n,k+1} : [n] \times [n^2]^{k-1} \times [n] \to \mathbb{F}$ is computed by an arithmetic formula of size s. By Lemma 2.5, there exist a number $m = \text{poly}(s) \cdot O(\log s)^{k+1}$, tensors $h_1, \ldots, h_m : [n] \times [n^2]^{k-1} \times [n] \to \mathbb{F}$ and partitions $\mathcal{P}_1, \ldots, \mathcal{P}_m \vdash [k+1]$ such that

- $IMM_{n,k+1} = \sum_{\ell=1}^{m} h_\ell$,
- each h_ℓ is $\mathcal{P}_\ell$-basic,
- each $\mathcal{P}_\ell$ has size $\lceil \log_2(k+1) \rceil$.

Recall that the flattening $f \mapsto f^\flat$ is a bijection from tensors of format $[n]^{2k}$ to tensors of format $[n] \times [n^2]^{k-1} \times [n]$. We shall denote the inverse bijection by $h \mapsto h^\sharp$. In particular, for each $\ell \in [m]$, let $h_\ell^\sharp : [n]^{2k} \to \mathbb{F}$ be the unique order-$2k$ tensor such that $(h_\ell^\sharp)^\flat = h_\ell$. Note that $h_\ell^\sharp$ is $\mathcal{P}_\ell$-flat-basic by Lemma 5.3(5).

Since $IMM_{n,k+1} = EQ_{n,k}^\flat$, it follows that

$$EQ_{n,k} = IMM_{n,k+1}^\sharp = \sum_{\ell \in [m]} h_\ell^\sharp.$$

We now have

$$\begin{aligned}
\mathrm{R}_{\text{riffle}}(EQ_{n,k}) &\le \sum_{\ell \in [m]} \mathrm{R}_{\text{riffle}}(h_\ell^\sharp) && \text{(by sub-additivity of } \mathrm{R}_{\text{riffle}}) \\
&\le \sum_{\ell \in [m]} n^{k-|\mathcal{P}_\ell|+1} \cdot \mathrm{R}_{\text{riffle}}^{\mathcal{P}_\ell}(h_\ell^\sharp) && \text{(by Lemma 5.4)} \\
&\le n^{k-\lceil \log_2(k+1) \rceil+1} \sum_{\ell \in [m]} \mathrm{R}_{\text{flat}}^{\mathcal{P}_\ell}(h_\ell^\sharp) && \text{(since } \mathrm{R}_{\text{riffle}}^{\mathcal{P}_\ell} \le \mathrm{R}_{\text{flat}}^{\mathcal{P}_\ell} \text{ by Lemma 5.3(1))} \\
&\le n^{k-\lceil \log_2(k+1) \rceil+1} \cdot m && \text{(since each } h_\ell^\sharp \text{ is } \mathcal{P}_\ell\text{-flat-basic)} \\
&= n^{k-\lceil \log_2(k+1) \rceil+1} \cdot s^{O(1)} \cdot O(\log s)^{k+1}.
\end{aligned}$$

We conclude that the desired implication

$$\mathrm{R}_{\text{riffle}}(EQ_{n,k}) = n^{k-o(\log k)} \implies s = n^{\Omega(\log k)}$$

holds for all $k \le \frac{\log n}{\log \log n}$, which finishes the proof. ◀

For any even-order tensor $f : [n]^{2k} \to \mathbb{F}$, the same argument shows that $\mathrm{R}_{\text{riffle}}(f) \le n^{k-O(\log k)} \cdot s^{O(1)} \cdot O(\log s)^{k+1}$ where s is the arithmetic formula size of $f^\flat : [n] \times [n^2]^{k-1} \times [n]$. That is, the proof of Theorem 4.4 directly generalizes to Theorem 4.5.

6 Generalizations and special cases

We conclude this paper by discussing some generalizations and special cases of Conjectures 4.2 and 4.3.

6.1 The first open case $k = 2$ of Conjecture 4.3

The following is a restatement of Conjecture 4.2 in the first open case $k = 2$:

▶ **Conjecture 6.1.** *Suppose we have a family of tensors* $f_\ell : [n] \to \mathbb{F}$ *and* $g_\ell, h_\ell : [n]^2 \to \mathbb{F}$ $(\ell = 1, \dots, m)$ *such that, for all* $a_1, a_2, b_1, b_2 \in [n]$,

$$\sum_{\ell=1}^{m} f_\ell(a_1) \cdot g_\ell(b_1, a_2) \cdot h_\ell(b_1, b_2) = \mathbb{1}[\,(a_1, a_2) = (b_1, b_2)\,].$$

Then $m \geq n^2$, *or at least* $m = \Omega(n^2)$.

A conceivable way to prove Conjecture 6.1 is by transforming an arbitrary sequence of tensors $\{(f_\ell, g_\ell, h_\ell)\}_{\ell\in[m]}$ into some "normal form" (where the desired lower bound $m \geq n^2$ should be easy to prove) via operations $\{(f_\ell, g_\ell, h_\ell)\}_{\ell\in[m]} \mapsto \{(f'_\ell, g'_\ell, h'_\ell)\}_{\ell\in[m]}$ that act locally by modifying triples (f_ℓ, g_ℓ, h_ℓ) at only a few indices $\ell \in [m]$, while preserving both the size m and the tensor $\sum_{\ell=1}^{m} f_\ell(a_1) \cdot g_\ell(b_1, a_2) \cdot h_\ell(b_1, b_2)$.

As a challenge to this strategy, we describe a few distinct constructions where $\sum_{\ell=1}^{m} f_\ell(a_1)\cdot g_\ell(b_1, a_2) \cdot h_\ell(b_1, b_2) = \mathbb{1}[\,(a_1, a_2) = (b_1, b_2)\,]$, all with $m = n^2$. These examples might be a helpful starting point in thinking about the extremal constructions which characterize the equality $\mathrm{R}_{\text{riffle}}(EQ_{n,2}) = n^2$, if this strongest version of Conjecture 6.1 turns out to be correct.

▶ **Example 6.2.** Three obvious ways of realizing the upper bound $\mathrm{R}_{\text{riffle}}(EQ_{n,2}) \leq n^2$ are:

$$\begin{aligned}
\mathbb{1}[\,(a_1, a_2) = (b_1, b_2)\,] &= \sum_{(i,j)\in[n]^2} \overbrace{\mathbb{1}[\,a_1 = i\,]}^{f_{i,j}(a_1)} \cdot \overbrace{\mathbb{1}[\,(b_1, a_2) = (i, j)\,]}^{g_{i,j}(b_1,a_2)} \cdot \overbrace{\mathbb{1}[\,(b_1, b_2) = (i, j)\,]}^{h_{i,j}(b_1,b_2)} \\
&= \sum_{(i,j)\in[n]^2} \overbrace{\mathbb{1}[\,a_1 = i\,]}^{f_{i,j}(a_1)} \cdot \overbrace{\mathbb{1}[\,a_2 = j\,]}^{g_{i,j}(b_1,a_2)} \cdot \overbrace{\mathbb{1}[\,(b_1, b_2) = (i, j)\,]}^{h_{i,j}(b_1,b_2)} \\
&= \sum_{(i,j)\in[n]^2} \overbrace{\mathbb{1}[\,a_1 = i\,]}^{f_{i,j}(a_1)} \cdot \overbrace{\mathbb{1}[\,(b_1, a_2) = (i, j)\,]}^{g_{i,j}(b_1,a_2)} \cdot \overbrace{\mathbb{1}[\,b_2 = j\,]}^{h_{i,j}(b_1,b_2)}.
\end{aligned}$$

▶ **Example 6.3.** Let $L_1, \dots, L_n, M_1, \dots, M_n, N_1, \dots, N_n$ be any $n \times n$ matrices such that $L_i \bullet M_i$ and N_i are invertible for all $i \in [n]$, where $\bullet$ denotes the Hadamard (entry-wise) product and $O[i, j]$ denotes (i, j)-entry of a matrix O. Then

$$\begin{aligned}
&\sum_{(i,j)\in[n]^2} \overbrace{(L_j \bullet M_j)^{-1}[a_1, i]}^{f_{i,j}(a_1)} \cdot \overbrace{L_j[i, b_1] \cdot N_{b_1}^{-1}[a_2, j]}^{g_{i,j}(b_1,a_2)} \cdot \overbrace{M_j[i, b_1] \cdot N_{b_1}[j, b_2]}^{h_{i,j}(b_1,b_2)} \\
&\quad= \sum_{j\in[n]} N_{b_1}^{-1}[a_2, j] \cdot N_{b_1}[j, b_2] \cdot \sum_{i\in[n]} (L_j \bullet M_j)^{-1}[a_1, i] \cdot (L_j \bullet M_j)[i, b_1] \\
&\quad= \sum_{j\in[n]} N_{b_1}^{-1}[a_2, j] \cdot N_{b_1}[j, b_2] \cdot \mathbb{1}[\,a_1 = b_1\,] \\
&\quad= \mathbb{1}[\,(a_1, a_2) = (b_1, b_2)\,].
\end{aligned}$$

▶ **Example 6.4.** Let L and M and $N_{i,j}$ $(i,j \in [n])$ be any $n \times n$ matrices such that $L \bullet M$ and all $N_{i,j}$ are invertible. Then

$$\sum_{(i,j)\in[n]^2} \overbrace{(L \bullet M)^{-1}[a_1,i]}^{f_{i,j}(a_1)} \cdot \overbrace{L[i,b_1] \cdot N_{i,b_1}^{-1}[a_2,j]}^{g_{i,j}(b_1,a_2)} \cdot \overbrace{M[i,b_1] \cdot N_{i,b_1}[j,b_2]}^{h_{i,j}(b_1,b_2)}$$

$$= \sum_{i\in[n]} (L \bullet M)^{-1}[a_1,i] \cdot (L \bullet M)[i,b_1] \cdot \sum_{j\in[n]} N_{i,b_1}^{-1}[a_2,j] \cdot N_{i,b_1}[j,b_2]$$

$$= \sum_{i\in[n]} (L \bullet M)^{-1}[a_1,i] \cdot (L \bullet M)[i,b_1] \cdot \mathbb{1}[\, a_2 = b_2 \,]$$

$$= \mathbb{1}[\,(a_1,a_2) = (b_1,b_2)\,].$$

6.2 Is riffle rank multiplicative under tensor products of matrices?

As noted earlier, $EQ_{n,k}$ is a tensor product of identity matrices $EQ_{n,k} = I_n \otimes \cdots \otimes I_n$ (k times). This raises a few questions in connection to Conjecture 4.2.

▶ Question 1. Is riffle rank multiplicative under tensor products of matrices? That is, does $\mathrm{R}_{\text{riffle}}(M_1 \otimes \cdots \otimes M_k) = \prod_{i=1}^{k} \mathrm{R}(M_i)$ hold for all matrices $M_1,\dots,M_k : [n]^2 \to \mathbb{F}$? (A positive answer clearly implies $\mathrm{R}_{\text{riffle}}(EQ_{n,k}) = n^k$.)

Note that tensor rank is multiplicative under tensor products of matrices: we have $\mathrm{R}(M_1 \otimes \cdots \otimes M_k) = \prod_{i=1}^{k} \mathrm{R}(M_i)$ for all matrices $M_1,\dots,M_k : [n]^2 \to \mathbb{F}$. However, tensor rank is not multiplicative under tensor products of higher-order tensors [4].

▶ Question 2. Is riffle rank multiplicative under the tensor products of the form $f \otimes I_n$ (or $I_n \otimes f$)? That is, does $\mathrm{R}_{\text{riffle}}(f \otimes I_n) = \mathrm{R}_{\text{riffle}}(f) \cdot n$ (or $\mathrm{R}_{\text{riffle}}(I_n \otimes f) = \mathrm{R}_{\text{riffle}}(f) \cdot n$) hold for all even-order tensors $f : [n]^{2k} \to \mathbb{F}$? (Here again a positive answer implies $\mathrm{R}_{\text{riffle}}(EQ_{n,k}) = n^k$.)

6.3 Multi-party communication complexity

There is a natural analogue of riffle rank in the setting of $k+1$-party communication complexity. Consider a problem where cooperating players $\mathrm{P}_1,\dots,\mathrm{P}_{k+1}$ have the goal of computing a function $f(a_1,b_1,\dots,a_k,b_k)$ of inputs $a_1,b_1,\dots,a_k,b_k$, each visible to a different subset of players. Imagine that $\mathrm{P}_1,\dots,\mathrm{P}_{k+1}$ are arranged in a line, with P_i facing in the direction of $\mathrm{P}_1,\dots,\mathrm{P}_{i-1}$. For $i = 1,\dots,k$, player P_i holds input a_i *in hand* (visible only to himself) and has input b_i *on back of his head* (visible to players $\mathrm{P}_{i+1},\dots,\mathrm{P}_{k+1}$). Player P_{k+1} holds no input, but sees $b_1,\dots,b_k$.

For tensors $f : [n]^{2k} \to \mathbb{F}$, lower bounds on the communication complexity of this $k+1$-party communication problem imply lower bounds on $\mathrm{R}_{\text{riffle}}(f)$. For instance, if $\mathbb{F}$ is the 2-element field and players $\mathrm{P}_1,\dots,\mathrm{P}_{k+1}$ each simultaneously broadcast a single bit in each round, then the number of rounds required for all players to learn the value of $f(a_1,b_1,\dots,a_k,b_k)$ clearly gives a lower bound on riffle rank.

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