1 Bijective/Injective Functions

Let $A$ be a set, and let $X \subseteq A$.
We say $r : A \to X$ is a retraction if $r(x) = x$ for all $x \in X$.
We say $i : X \to A$ is an inclusion if $i(x) = x$ for all $x \in X$.

Prove that all retractions are surjective and the inclusion is injective. Do this two ways:

1. Directly, using the definitions of surjectivity and injectivity.
2. Using the fact that if $f \circ g$ is bijective, then $f$ is surjective and $g$ is injective (from SA-13).
2 Streams

Define the stream integers with naturals and stream-map. Assume naturals is the stream (1, 2, 3, 4, ...).
3 Streams (Cont.)

Write a function that takes in a stream of numbers $a = (a_1, a_2, \ldots)$ and returns the unique stream $b = (b_1, b_2, \ldots)$ such that

\[
\begin{align*}
    a_1 + b_1 &= a_2 \\
    a_2 + b_2 &= a_3 \\
    & \vdots
\end{align*}
\]
4 Countability

Is the set of irrational numbers countable? (or: is the set difference of an uncountable set with a countable set countable?)
5 Structural Induction

Use structural induction to prove reverse-tree produces a mirror-image copy of the given tree.

(define reverse-tree
  (lambda (t) ;(t <obj>)
    (if (leaf? t)
        t
        (make <btree>
            :left (reverse-tree (right t))
            :right (reverse-tree (left t))))))
Show that for any function $f : A \to B$ and a set $S \subseteq B$, show that:

$$f(f^{-1}(S)) \subseteq S$$

Give an example of a function $f$ and a set $S$ for which equality doesn’t hold. (Bonus: give an equivalent condition to when equality holds for all sets $S$.)
Sometimes in math we will take the arbitrary union of a collection of sets. These don’t have to be finite (\( \mathbb{N} \)), countable (\( \mathbb{R} \)), or even well-ordered (\( \mathbb{C} \)). The goal with this problem is to be able to prove the following claims for any choice of index \( i \).

Claim:

\[
A - \left( \bigcup_i B_i \right) = \bigcap_i (A - B_i)
\]

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8 Enrichment; Harder than Exams

Define a *Andy Space* $A$ on a set $X$ as follows:

$T$ is a collection (set) of subsets of $X$. It’s a set of sets, isn’t that cool?

1. Both $\emptyset$ and $X$ are in $A$.
2. Arbitrary Union: If each $U_i$ is in $A$ then $\bigcup_i U_i$ is in $A$.
3. Finite Intersection: If each $U_i$ is in $A$ then $\bigcap_{i=0}^n U_i$ is in $A$.

What is the smallest *Andy Space* you can make on any set $X$? The largest?

Consider the set $\mathbb{R}$. Let us define $T$ to be the collection of subsets of $\mathbb{R}$ such that $\mathbb{R} - U$ is countable $\iff U \in T$. Let us also say that $\emptyset \in T$. Prove that $T$ is a *Andy Space* (DeMorgan’s laws will be your best friend).