Growth of functions: asymptotic notation

To characterize the time cost of algorithms, we focus on functions that map input size to (typically, worst-case) running time. (Similarly for space costs.)

We are interested in precise notation for characterizing running-time differences that are likely to be significant across different platforms and different implementations of the algorithms.

This naturally leads to an interest in the “asymptotic growth” of functions. We focus on how the function behaves as its input grows large.

Asymptotic notation is a standard means for describing families of functions that share similar asymptotic behavior.

Asymptotic notation allows us to ignore small input sizes, constant factors, lower-order terms in polynomials, and so forth.
Big O notation (asymptotic upper bound)

**Definition**  \( f(n) \in O(g(n)) \) if there are positive \( c, n_0 \) s.t.

\[
|f(n)| \leq c|g(n)| \quad \text{for all } n \geq n_0 .
\]

We assume that these functions map reals to reals. They may be partial functions, but, as we’ll explain shortly, we will assume that they are “asymptotically-defined on \( \mathbb{N} \)”. 

It is common to write \( f(n) = O(g(n)) \) instead of \( f(n) \in O(g(n)) \). 

Beware: this is not the same as writing \( O(g(n)) = f(n) \). 

What about the awkwardness of writing \( f(n) \) instead of \( f \), and \( g(n) \) instead of \( g \)? (That is, writing an expression for the value of function \( f \) at \( n \), when we wish to name the function \( f \) itself.) 

It corresponds to the standard (and convenient) practice of writing, say, \( O(n^2) \) to stand for the set of functions \( f : \mathbb{R} \to \mathbb{R} \) s.t. for some positive \( c, n_0 \), 

\[
|f(n)| \leq c|n^2| \quad \text{for all } n \geq n_0 .
\]

Is \( g(n) = O(g(n)) \)?
Asymptotically defined on \( \mathbb{N} \)

We say we’re interested in functions from \( \mathbb{R} \) to \( \mathbb{R} \), but . . .

. . . what about a function like \( \log_2 n \)?

Problem: not defined for \( n \leq 0 \).

An approach that couldn’t talk about log-space and log-time would not be very useful.

And what about a function like \( n! \)?  (Commonly defined with domain \( \mathbb{N} \).)

It is convenient to consider any (partial) function from \( \mathbb{R} \) to \( \mathbb{R} \) that is “asymptotically defined on \( \mathbb{N} \)” — that is, defined for all natural numbers \( n \geq n_0 \), for some constant \( n_0 \).

So let’s assume that in asymptotic notation we consider (partial) functions from \( \mathbb{R} \) to \( \mathbb{R} \) that are asymptotically defined on \( \mathbb{N} \).
Ignoring constants

Asymptotic notation is supposed to allow us to ignore constants. Let’s look at how this works... Is \( n + k = O(n) \)?

\[
f(n) \in O(g(n)) \text{ if there are positive } c, n_0 \text{ s.t. } |f(n)| \leq c|g(n)| \text{ for all } n \geq n_0.
\]

If \( k = 0 \), the result is trivial. (Take \( c = 1, n_0 = 1 \).)

If \( k \) is positive:
We can take \( c = 2 \) and \( n_0 = k \). Notice that
\[
|n + k| \leq 2|n| \text{ for all } n \geq k.
\]

What if \( k \) is negative?
Then we can take \( c = 1 \) and \( n_0 = -k \), and notice that
\[
|n + k| \leq |n| \text{ for all } n \geq -k.
\]

So, for all \( k \in \mathbb{R} \),
\[
n + k = O(n).
\]
If \( f(n) = O(g(n)) \), is \( kf(n) = O(g(n)) \)?

\( f(n) \in O(g(n)) \) if there are positive \( c, n_0 \) s.t.

\[
|f(n)| \leq c|g(n)| \quad \text{for all } n \geq n_0.
\]

If \( k = 0 \), the result is immediate, so assume otherwise.

Assume \( f(n) = O(g(n)) \).

So there are positive constants \( c', n'_0 \) s.t.

\[
|f(n)| \leq c'|g(n)| \quad \text{for all } n \geq n'_0.
\]

Take \( c = c'|k| \) and \( n_0 = n'_0 \).

Observe that, for all \( n \geq n_0 \),

\[
|kf(n)| = |k| \cdot |f(n)| \\
\leq |k| \cdot c'|g(n)| \\
= c|g(n)|
\]

So

\[
kf(n) = O(g(n))
\]

whenever \( f(n) = O(g(n)) \).
If \( f(n) = O(g(n)) \), is \( f(n) + k = O(g(n)) \)?

\( f(n) \in O(g(n)) \) if there are positive \( c, n_0 \) s.t.

\[
|f(n)| \leq c|g(n)| \quad \text{for all } n \geq n_0.
\]

Assume that \( f(n) = O(g(n)) \).

We need positive \( c \) and \( n_0 \) s.t.

\[
|f(n) + k| \leq c|g(n)| \quad \text{for all } n \geq n_0.
\]

In general, such \( c \) and \( n_0 \) may fail to exist. But here’s a simple condition that will guarantee their existence.

Assume that \( |g(n)| \) has an asymptotic nonzero lower bound. That is, for some positive \( b \) and \( m_0 \),

\[
|g(n)| \geq b \quad \text{for all } n \geq m_0.
\]

Since \( f(n) = O(g(n)) \), there are \( c', n'_0 \) s.t.

\[
f(n) \leq c'g(n) \quad \text{for all } n \geq n'_0.
\]

It suffices to take \( c = \frac{|k|}{b} + c' \) and \( n_0 = m_0 + n'_0 \). Let’s check that...
Given: For some positive $b$ and $m_0$,
\[ |g(n)| \geq b \quad \text{for all } n \geq m_0 , \]
and, for some positive $c'$, $n'_0$,
\[ |f(n)| \leq c'|g(n)| \quad \text{for all } n \geq n'_0 . \]

NTS: There are positive $c$ and $n_0$ s.t.
\[ |f(n) + k| \leq c|g(n)| \quad \text{for all } n \geq n_0 . \]

Take $c = |k/b| + c'$ and $n_0 = m_0 + n'_0$.

Notice that, for all $n \geq n_0$,
\[
|f(n) + k| \leq |f(n)| + |k| \\
\leq c'|g(n)| + |k| \\
\leq c'|g(n)| + |k| \cdot \frac{|g(n)|}{b} \\
= c'|g(n)| + |k/b| \cdot |g(n)| \\
= (c' + |k/b|) \cdot |g(n)| \\
= c \cdot |g(n)|
\]

So, if $|g(n)|$ has a nonzero asymptotic lower bound, then
\[ f(n) + k = O(g(n)) \]
whenever $f(n) = O(g(n))$. 

Polynomials and big O

Our results thus far can explain, for instance, the well-known fact that
\[ an^k = O(n^k) \]

and, similarly, that
\[ n^k + b = O(n^k) \quad (k \geq 0) \]

But what about drawing the conclusion that
\[ an^k + b = O(n^k) \quad (k \geq 0) \]

It looks like this should be true, but we’d need another argument.

There is a much more general fact about polynomials and Big O that you should know:

Any polynomial expression whose highest order term is
\[ an^k \]

belongs to
\[ O(n^k) \].

Let’s prove it...
Polynomials of degree at most $k$ belong to $O(n^k)$

Consider an arbitrary expression of the form

$$a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n + a_0 n^0$$

with $k \in \mathcal{N}$ and $a_k, \ldots, a_0 \in \mathcal{R}$. (Notice that any of $a_k, \ldots, a_0$ may equal zero. If all do, the result is trivial, so we'll assume that at least one does not.)

For convenience, we'll write this expression in the form

$$\sum_{i=0}^{k} a_i n^i.$$

Notice that, for all $n \geq 1$,

$$\left| \sum_{i=0}^{k} a_i n^i \right| \leq \sum_{i=0}^{k} |a_i n^i| = \sum_{i=0}^{k} |a_i| n^i \leq \left( \sum_{i=0}^{k} |a_i| \right) \cdot n^k = \left( \sum_{i=0}^{k} |a_i| \right) \cdot |n^k|$$

which shows that

$$\sum_{i=0}^{k} a_i n^i = O(n^k).$$

(Take $n_0 = 1$ and $c = \sum_{i=0}^{k} |a_i|$.)
If \( f(n) = O(g(n)) \) and \( g(n) = O(h(n)) \), then \( f(n) = O(h(n)) \)

Assume \( f(n) = O(g(n)) \) and \( g(n) = O(h(n)) \).

It follows that there are positive \( c', n'_0, c'', n''_0 \) s.t.

\[
|f(n)| \leq c' |g(n)| \quad \text{for all } n \geq n'_0
\]

and

\[
|g(n)| \leq c'' |h(n)| \quad \text{for all } n \geq n''_0.
\]

Take \( c = c' c'' \) and \( n_0 = n'_0 + n''_0 \). Notice that

\[
|f(n)| \leq c' |g(n)| \leq c' c'' |h(n)| = c |h(n)| \quad \text{for all } n \geq n_0.
\]
If \( f(n) = O(h(n)) \) and \( g(n) = O(h(n)) \), then \( f(n) + g(n) = O(h(n)) \)

Assume \( f(n) = O(h(n)) \) and \( g(n) = O(h(n)) \).

It follows that there are positive \( c', n'_0, c'', n''_0 \) s.t.

\[
|f(n)| \leq c'|h(n)| \quad \text{for all } n \geq n'_0
\]

and

\[
|g(n)| \leq c''|h(n)| \quad \text{for all } n \geq n''_0.
\]

Take \( c = c' + c'' \) and \( n_0 = n'_0 + n''_0 \). Then

\[
|f(n) + g(n)| \leq |f(n)| + |g(n)| \leq c|h(n)| \quad \text{for all } n \geq n_0.
\]

That is, \( f(n) + g(n) = O(h(n)) \).

It follows that if \( f_i(n) = O(g(n)) \) for all \( i \in \mathcal{N} \), then

\[
\sum_{i=0}^{k} f_i(n) = O(g(n))
\]

for any \( k \in \mathcal{N} \).
If $f(n) + g(n) = O(h(n))$ and $f(n)$ and $g(n)$ are asymptotically nonnegative, then $f(n) = O(h(n))$

Assume that $f(n) + g(n) = O(h(n))$ and that both $f(n)$ and $g(n)$ are asymptotically nonnegative.

It follows that there are positive $c'$, $n'_0$, $m_0$, $m'_0$ s.t.

$$|f(n) + g(n)| \leq c'|h(n)| \quad \text{for all } n \geq n'_0$$

and

$$0 \leq f(n) \quad \text{for all } n \geq m_0$$

and

$$0 \leq g(n) \quad \text{for all } n \geq m'_0$$

Take $c = c'$ and $n_0 = n'_0 + m_0 + m'_0$. Then

$$|f(n)| \leq c|h(n)| \quad \text{for all } n \geq n_0.$$
If \( f(n) \cdot g(n) = O(h(n)) \) and \( g(n) \) has a positive asymptotic lower bound, then \( f(n) = O(h(n)) \).

Assume that \( f(n) \cdot g(n) = O(h(n)) \) and that \( g(n) \) has a positive asymptotic lower bound. So there are positive \( c', n'_0, b, m_0 \) s.t.

\[
|f(n) \cdot g(n)| \leq c'|h(n)| \quad \text{for all} \quad n \geq n'_0
\]

and

\[
b \leq g(n) \quad \text{for all} \quad n \geq m_0.
\]

Take \( c = c'/b \) and \( n_0 = n'_0 + m_0 \). Notice that

\[
b|f(n)| = |bf(n)| \leq |f(n) \cdot g(n)| \leq c'|h(n)| \quad \text{for all} \quad n \geq n_0
\]

Therefore,

\[
|f(n)| \leq \frac{c'}{b} |h(n)| = c|h(n)| \quad \text{for all} \quad n \geq n_0
\]

Is it true that if \( f(n) \cdot g(n) = O(h(n)) \), then at least one of \( f(n), g(n) \) belongs to \( O(h(n)) \)? Suppose

\[
f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases} \quad g(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \quad h(n) = 0
\]
If \( f(n) = O(g(n)) \) and \( f'(n) = O(g'(n)) \), then \( f(n) \cdot f'(n) = O(g(n) \cdot g'(n)) \).

Assume \( f(n) = O(g(n)) \) and \( f'(n) = O(g'(n)) \).

So there are positive \( c', n'_0, c'', n''_0 \) s.t.
\[
|f(n)| \leq c'|g(n)| \quad \text{for all } n \geq n'_0
\]
and
\[
|f'(n)| \leq c''|g'(n)| \quad \text{for all } n \geq n''_0.
\]

Take \( c = c' c'' \) and \( n_0 = n'_0 + n''_0 \). Then, for all \( n \geq n_0 \),
\[
|f(n) \cdot f'(n)| = |f(n)| \cdot |f'(n)| \\
\leq c'|g(n)| \cdot c''|g'(n)| \\
= (c' \cdot c'') \cdot |g(n)| \cdot |g'(n)| \\
= c(|g(n)| \cdot |g'(n)|) \\
= c|g(n) \cdot g'(n)|
\]

It follows that if \( f_i(n) = O(g_i(n)) \) for all \( i \in \mathcal{N} \), then
\[
\prod_{i=0}^{k} f_i(n) = O\left(\prod_{i=0}^{k} g_i(n)\right)
\]
for any \( k \in \mathcal{N} \).
\( n^j \neq O(n^k) \) if \( j > k \)

Assume \( j > k \). Suppose there are positive \( c, n_0 \) s.t.

\[
|n^j| \leq c|n^k| \quad \text{for all } n \geq n_0.
\]

Then

\[
n^{j-k} \leq c \quad \text{for all } n \geq n_0.
\]

But this is not possible, since \( j - k \) is positive. Contradiction. We conclude that the supposition was false. Hence \( n^j \neq O(n^k) \) when \( j > k \).
Asymptotic lower bound, asymptotically tight bound

**Definition** (asymptotic lower bound)

\( f(n) \in \Omega(g(n)) \) if there are positive \( c, n_0 \) s.t.

\[
c|g(n)| \leq |f(n)| \quad \text{for all } n \geq n_0.
\]

**Claim:** \( f(n) = \Omega(g(n)) \) iff \( g(n) = O(f(n)) \). Easy to show. (Try it.)

**Definition** (asymptotically tight bound)

\( f(n) \in \Theta(g(n)) \) if there are positive \( c, d, n_0 \) s.t.

\[
c |g(n)| \leq |f(n)| \leq d |g(n)| \quad \text{for all } n \geq n_0.
\]

**Theorem**  \( f(n) = \Theta(g(n)) \) iff \( f(n) = \Omega(g(n)) \) and \( f(n) = O(g(n)) \).

\( f(n) \overset{?}{=} \Omega(f(n)) \)

\( f(n) \overset{?}{=} \Theta(f(n)) \)

*You might want to consider whether other properties of big \( O \) hold for \( \Omega, \Theta \).*
Other asymptotic notations

**Definition** (asymptotically strict upper bound)  \(f(n) \in o(g(n))\) if for every positive \(c\) there is a positive \(n_0\) s.t.

\[|f(n)| < c|g(n)| \quad \text{for all } n \geq n_0.\]

Notice: If \(f(n) = o(g(n))\) then \(f(n) = O(g(n))\).

**Claim:** If \(f(n) = o(g(n))\) then \(f(n) \neq \Omega(g(n))\).

**Definition** (asymptotically strict lower bound)  \(f(n) \in \omega(g(n))\) if for every positive \(c\) there is a positive \(n_0\) s.t.

\[c|g(n)| < |f(n)| \quad \text{for all } n \geq n_0.\]

Notice: If \(f(n) = \omega(g(n))\) then \(f(n) = \Omega(g(n))\).

**Claim:** If \(f(n) = \omega(g(n))\) then \(f(n) \neq O(g(n))\).

**Claim:** \(f(n) = o(g(n))\) iff \(g(n) = \omega(f(n))\).
A few more laws

If \( f(n) = \Theta(g(n)) \) and \( g(n) = \begin{cases} 
\mathcal{O}(h(n)) \\
\Theta(h(n)) \\
\Omega(h(n)) \\
\omega(h(n)) 
\end{cases} \), then \( f(n) = \begin{cases} 
\mathcal{O}(h(n)) \\
\Theta(h(n)) \\
\Omega(h(n)) \\
\omega(h(n)) 
\end{cases} \).

If \( f(n) = \begin{cases} 
\mathcal{o}(g(n)) \\
\omega(g(n)) 
\end{cases} \) and \( g(n) = \begin{cases} 
\mathcal{O}(h(n)) \\
\Omega(h(n)) 
\end{cases} \), then \( f(n) = \begin{cases} 
\mathcal{o}(h(n)) \\
\omega(h(n)) 
\end{cases} \).

If \( f(n) = \begin{cases} 
\mathcal{O}(g(n)) \\
\Omega(g(n)) 
\end{cases} \) and \( g(n) = \begin{cases} 
\mathcal{o}(h(n)) \\
\omega(h(n)) 
\end{cases} \), then \( f(n) = \begin{cases} 
\mathcal{o}(h(n)) \\
\omega(h(n)) 
\end{cases} \).
Some facts about hierarchy of asymptotic classes

\( n^n \) beats factorial:
\[
n! = o(n^n)
\]

Factorial beats exponential:
\[
k^n = o(n!)
\]

Exponential beats polynomial:
\[
n^k = o(b^n) \quad \text{if } b > 1
\]

Higher-degree polynomial beats lower-degree polynomial:
\[
n^j = o(n^k) \quad \text{if } 0 \leq j < k
\]

Polynomial beats (poly)logarithmic:
\[
\log_b^a n = o(n^k) \quad \text{if } b > 1 \text{ and } k > 0
\]

Note: \( \log_b^a n \) is an abbreviation for \( (\log_b n)^a \).
The base of a (poly)logarithm doesn’t matter:

\[ \log_a^k n = \Theta(\log_b^k n) \quad \text{if } a, b > 1 \]

The base of an exponential matters:

\[ a^n = o(b^n) \quad \text{if } 1 \leq a < b \]

Adding a constant to the exponent (of an exponential) doesn’t matter:

\[ a^n = \Theta(a^{n+k}) \]

A constant factor in the exponent matters:

\[ a^n = o(a^{bn}) \quad \text{if } a, b > 1 \]
A hierarchy of asymptotically strict bounds:

- $2^{2^n}$
- $n^n$
- $n!$
- $3^n$
- $n \cdot 2^n$
- $2^n$
- $n^3$
- $n^2 \log n$
- $n^2$
- $n \log n$
- $n$
- $\sqrt{n}$
- $\log^3 n$
- $\log^2 n$
- $\log n$
- $\sqrt{\log n}$
- $\log \log n$
- $\log \log \log n$
- $1$
- $1/n$
- $0$