Three proofs today:

1. Reforming the post office (by induction)

2. It’s irrational! (by induction and contradiction)

3. A horse of a different color

These may be the first non-trivial proofs you have ever seen!

First, a reminder of the *induction recipe*.

**Step 1.** What variable?

**Step 2.** What property?

**Step 3.** Base case

**Step 4.** Inductive step

### 1 Postage Stamp Problem

The post office is trying to save money by printing fewer categories of stamps. They wonder if it would be sufficient for them to print only 3¢ and 5¢ stamps.

**Claim.** *Using 3¢ and 5¢ stamps suffices to make any postage greater than 7¢.*

**Proof.** **Step 1.** Induction on \( n \), the amount of postage.

**Step 2.** We show \( P(n) : n = 3a + 5b \) for some \( a, b \in \mathbb{N} \), for all \( n > 7 \).

**Step 3.** Base case.

\[
\begin{align*}
8 &= 3 + 5 \\
9 &= 3 + 3 + 3 \\
10 &= 5 + 5
\end{align*}
\]

Why do we need three base cases?

**Step 4.** Inductive step.

Inductive Hypothesis: Assume for all \( k \), \( 7 < k < n \), there exists \( a, b \in \mathbb{N} \) such
that $k = 3a + 5b$ (note that $a$ and $b$ depend on $k$).

We are using strong induction (why?)

Need only show that $n = 3\alpha + 5\beta$ for some $\alpha, \beta \in \mathbb{N}$. Well... by the inductive hypothesis we know that

$$n - 3 = 3a + 5b$$

so

$$n = (n - 3) + 3$$
$$= 3a + 5b + 3$$
$$= 3(a + 1) + 5b$$
$$= 3\alpha + 5\beta$$

where $\alpha = a + 1$ and $\beta = b$. 

$\square$
2 \( \sqrt{2} \) is irrational

Consider the following relationship (don’t worry, you don’t have to have seen all these yet).

\[
\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}
\]

where

\[
\mathbb{N} = \{0, 1, 2, 3, \ldots\}\\
\mathbb{Z} = \{\pm k \mid k \in \mathbb{N}\}\\
\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}
\]

It is easy to see that \(-1\) is in \(\mathbb{Z}\) but not in \(\mathbb{N}\). Formally, \(-1 \in \mathbb{Z} - \mathbb{N} = \{k \mid k \in \mathbb{Z}, k \notin \mathbb{N}\} \neq \emptyset\).

It’s also easy to see that \(\frac{1}{2} \in \mathbb{Q} - \mathbb{Z} \neq \emptyset\).

But how do we know that \(\mathbb{R} - \mathbb{Q} \neq \emptyset\)? It’s not obvious if there is a real number which is not rational. We will show that \(\sqrt{2}\) is one such number. You may have been told this before, but now we will prove it!

NB This is the first part of our study of the deep structure of sets and numbers. Later in the course, we will also be interested in other fascinating questions about these sets. For example, what is the size of each of them?

**Theorem 1.** \(\sqrt{2}\) is irrational.

**Proof.** By induction. We show \((\frac{p}{q})^2 \neq 2\) for all \(p, q \in \mathbb{N}\), by induction on \(q\).

**Step 1.** Induction on \(q \in \mathbb{N}\).

**Step 2.** \(P(q) : (\frac{p}{q})^2 \neq 2\) \(\forall p \in \mathbb{N}\).

**Step 3.** Base Case: \(q = 1\). We see that \((1/q)^2 = 1 \neq 2\), \((2/q)^2 = 4 > 2\), and for all \(p > 2\), \((p/q)^2 > 4\). This proves the base case since we have accounted for all \(p \in \mathbb{N}\).

**Step 4.** Inductive Step:

Our inductive hypothesis: Assume that (and note that we are using strong induction)

\[
\left(\frac{p}{q}\right)^2 \neq 2 \quad \forall p, q \in \mathbb{N}, \quad q < n
\]

We will show that \((\frac{p}{n})^2 \neq 2\) for all \(p \in \mathbb{N}\). We will do this by contradiction!

**Proof Plan**

- Assume the inductive hypothesis

- Now assume that \((\frac{p}{n})^2 = 2\) for some \(p \in \mathbb{N}\) we are assuming the opposite of what we want to prove).

- NOS that \(\exists p', q' \in \mathbb{N}\) with \(q' < n\) such that \((\frac{p'}{q'})^2 = 2\). This contradicts the inductive hypothesis (which we are allowed to assume, since we have shown the base case is true). Therefore, the assumption that \((\frac{p}{n})^2 = 2\) must be false!

How do we do the “NOS” step? We will use two claims.
Claim 1 : If \( (\frac{p}{n})^2 = 2 \) then \( 0 < p - n < n \).

Claim 2 : If \( (\frac{p}{n})^2 = 2 \) then \( (\frac{2n-p}{p-n})^2 = 2 \).

Claim 1 implies that the denominator in claim 2 is less than \( n \). So we can take

\[
p' = 2n - p \quad q' = p - n
\]

and the proof is done! (Look back at the proof plan now and check that you understand this)

NB. The inductive hypothesis is not needed to prove either claim 1 or claim 2. They are both “if ... then ...” statements that are proven as written. But, then they are used with the IH to prove the contradiction.

You should try and prove the claims yourself before moving on.
2.1 Proof of Claim 1

Claim 1. If \((\frac{p}{n})^2 = 2\) then \(0 < p - n < n\).

Proof. There are two parts. We first prove that if \((\frac{p}{n})^2 = 2\) then \(0 < p - n\). The proof is by contradiction. Suppose that \(0 \geq p - n\). Then

\[
\begin{align*}
0 & \geq p - n \\
n & \geq p \\
1 & \geq \frac{p}{n} \\
1^2 = 1 & \geq \left(\frac{p}{n}\right)^2 = 2
\end{align*}
\]

Which is a contradiction, as desired.

We next show that if \((\frac{p}{n})^2 = 2\) then \(p - n < 2\), again by contradiction. Suppose that \(p - n \geq n\). Then

\[
\begin{align*}
p - n & \geq n \\
p & \geq 2n \\
\frac{p}{n} & \geq 2 \\
2 = \left(\frac{p}{n}\right)^2 & \geq 4
\end{align*}
\]

Which is a contradiction, as desired. This completes the proof. \(\square\)
2.2 Proof of Claim 2

Claim 2. If \((\frac{p}{n})^2 = 2\) then \((\frac{2n-p}{p-n})^2 = 2\).

Proof. We know that \((\frac{p}{n})^2 = 2\). So we can do the following:

\[
\left(\frac{2n-p}{p-n}\right)^2 = \frac{\frac{1}{n}(2n-p)^2}{\frac{1}{n}(p-n)^2}
\]

\[
= \frac{(\frac{1}{n}(2n-p))^2}{(\frac{1}{n}(p-n))^2}
\]

\[
= \frac{(2 - \frac{p}{n})^2}{(\frac{p}{n} - 1)^2}
\]

\[
= \left(\frac{2 - \frac{p}{n}}{\frac{p}{n} - 1}\right)^2
\]

\[
= \left(\frac{2 - \sqrt{2}}{\sqrt{2} - 1}\right)^2 \quad \text{since} \quad \left(\frac{p}{n}\right)^2 = 2
\]

\[
= \frac{4 - 4\sqrt{2} + 2}{2 - 2\sqrt{2} + 1}
\]

\[
= \frac{2(2 - 2\sqrt{2} + 1)}{2 - 2\sqrt{2} + 1}
\]

\[= 2 \quad \square\]

For simplicity, we treat all rational numbers as if they are in lowest terms (i.e., \((\text{gcd} \, p \, q) = 1\)). But this is not necessary for the proof. Think about why. \(\square\)
3 All horses are the same color

Theorem 2. All horses are the same color.

Corollary 3. All students in this class will get the same grade.

Corollary 4. If one student fails then all students will fail.

Proof. Step 1. By induction on \( n \), the number of horses in a set.

Step 2. \( P(n) \) : in a set of horses of size \( n \), all horses are the same color.

Step 3. Base case. \( n = 1 \). In a set of size 1 all horses are the same color (there’s only one horse so of course this is true).

Step 4. Inductive step. Inductive hypothesis: Assume that any set of \( n - 1 \) horses are the same color. Need only show that a set of \( n \) horses are all the same color. So take a set of \( n \) horses. There are three special horses we will name: Bob, Ted and Mitch.

Suppose we take Bob out of the group. Then there are now \( n - 1 \) horses, which, by the IH, are all the same color. In particular, Ted and Mitch are still in the group and so Ted and Mitch have the same color.

Now put Bob back in, and take out Mitch. So we have a different group of \( n - 1 \) horses (which are all the same color, by the IH). In particular, Bob and Ted are part of this new group, and so they both have the same color.

So Bob has the same color as Ted who has the same color as Mitch. By transitivity, Bob has the same color as Mitch. But there is nothing special about Bob and Mitch; we could have chosen any two horses! So all the horses have the same color. \( \square \)
Obviously, this is not a true theorem! What’s wrong? The base case is fine, but the induction step requires the existence of AT LEAST 3 horses (Bob, Ted and Mitch). So we can’t use the inductive step to go from $n = 1$ to $n = 2$.

NB If we could somehow show that any two horses have the same color, then the inductive step would work just fine to take us from $n = 2$ to any $n$ we wanted. But (of course), this is not the case!