§4 The Integers and the Real Numbers

Up to now we have been discussing what might be called the *logical foundations* for our study of topology—the elementary concepts of set theory. Now we turn to what we might call the *mathematical foundations* for our study—the integers and the real number system. We have already used them in an informal way in the examples and exercises of the preceding sections. Now we wish to deal with them more formally.

One way of establishing these foundations is to *construct* the real number system, using only the axioms of set theory—to build them with one’s bare hands, so to speak. This way of approaching the subject takes a good deal of time and effort and is of greater logical than mathematical interest.

A second way is simply to assume a set of axioms for the real numbers and work from these axioms. In the present section, we shall sketch this approach to the real numbers. Specifically, we shall give a set of axioms for the real numbers and shall indicate how the familiar properties of real numbers and the integers are derived from them. But we shall leave most of the proofs to the exercises. If you have seen all this before, our description should refresh your memory. If not, you may want to work through the exercises in detail in order to make sure of your knowledge of the mathematical foundations.

First we need a definition from set theory.

**Definition.** A *binary operation* on a set $A$ is a function $f$ mapping $A 	imes A$ into $A$.

When dealing with a binary operation $f$ on a set $A$, we usually use a notation different from the standard functional notation introduced in §2. Instead of denoting the value of the function $f$ at the point $(a, a')$ by $f(a, a')$, we usually write the symbol for the function *between* the two coordinates of the point in question, writing the value of the function at $(a, a')$ as $af a'$. Furthermore (just as was the case with relations), it is more common to use some symbol other than a letter to denote an operation. Symbols often used are the plus symbol $+$, the multiplication symbols $\cdot$ and $\circ$, and the asterisk $*$; however, there are many others.

**Assumption**

We assume there exists a set $\mathbb{R}$, called the set of *real numbers*, two binary operations $+$ and $\cdot$ on $\mathbb{R}$, called the addition and multiplication operations, respectively, and an order relation $<$ on $\mathbb{R}$, such that the following properties hold:

**Algebraic Properties**

1. $(x + y) + z = x + (y + z),$
   
   $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z$ in $\mathbb{R}$.
2. $x + y = y + x,$
   
   $x \cdot y = y \cdot x$ for all $x, y$ in $\mathbb{R}$. 
(3) There exists a unique element of \( \mathbb{R} \) called **zero**, denoted by 0, such that \( x + 0 = x \) for all \( x \in \mathbb{R} \).

There exists a unique element of \( \mathbb{R} \) called **one**, different from 0 and denoted by 1, such that \( x \cdot 1 = x \) for all \( x \in \mathbb{R} \).

(4) For each \( x \) in \( \mathbb{R} \), there exists a unique \( y \) in \( \mathbb{R} \) such that \( x + y = 0 \).

For each \( x \) in \( \mathbb{R} \) different from 0, there exists a unique \( y \) in \( \mathbb{R} \) such that \( x \cdot y = 1 \).

(5) \( x \cdot (y + z) = (x \cdot y) + (x \cdot z) \) for all \( x, y, z \in \mathbb{R} \).

**A Mixed Algebraic and Order Property**

(6) If \( x > y \), then \( x + z > y + z \).

If \( x > y \) and \( z > 0 \), then \( x \cdot z > y \cdot z \).

**Order Properties**

(7) The order relation \(<\) has the least upper bound property.

(8) If \( x < y \), there exists an element \( z \) such that \( x < z \) and \( z < y \).

From properties (1)–(5) follow the familiar “laws of algebra.” Given \( x \), one denotes by \(-x\) that number \( y \) such that \( x + y = 0 \); it is called the **negative** of \( x \). One defines the **subtraction operation** by the formula \( z - x = z + (-x) \). Similarly, given \( x \neq 0 \), one denotes by \( 1/x \) that number \( y \) such that \( x \cdot y = 1 \); it is called the **reciprocal** of \( x \). One defines the **quotient** \( z/x \) by the formula \( z/x = z \cdot (1/x) \). The usual laws of signs, and the rules for adding and multiplying fractions, follow as theorems. These laws of algebra are listed in Exercise 1 at the end of the section. We often denote \( x \cdot y \) simply by \( xy \).

When one adjoins property (6) to properties (1)–(5), one can prove the usual “laws of inequalities,” such as the following:

If \( x > y \) and \( z < 0 \), then \( x \cdot z < y \cdot z \).

\( -1 < 0 \) and \( 0 < 1 \).

The laws of inequalities are listed in Exercise 2.

We define a number \( x \) to be **positive** if \( x > 0 \), and to be **negative** if \( x < 0 \). We denote the positive reals by \( \mathbb{R}_+ \) and the nonnegative reals (for reasons to be explained later) by \( \mathbb{R}_+ \). Properties (1)–(6) are familiar properties in modern algebra. Any set with two binary operations satisfying (1)–(5) is called by algebraists a **field**; if the field has an order relation satisfying (6), it is called an **ordered field**.

Properties (7) and (8), on the other hand, are familiar properties in topology. They involve only the order relation; any set with an order relation satisfying (7) and (8) is called by topologists a **linear continuum**.

Now it happens that when one adjoins to the axioms for an ordered field [properties (1)–(6)] the axioms for a linear continuum [properties (7) and (8)], the resulting list contains some redundancies. Property (8), in particular, can be proved as a consequence of the others; given \( x < y \) one can show that \( z = (x + y)/(1 + 1) \) satisfies the requirements of (8). Therefore, in the standard treatment of the real numbers, properties (1)–(7) are taken as axioms, and property (8) becomes a theorem. We have
included (8) in our list merely to emphasize the fact that it and the least upper bound property are the two crucial properties of the order relation for \( \mathbb{R} \). From these two properties many of the topological properties of \( \mathbb{R} \) may be derived, as we shall see in Chapter 3.

Now there is nothing in this list as it stands to tell us what an integer is. We now define the integers, using only properties (1)–(6).

**Definition.** A subset \( A \) of the real numbers is said to be **inductive** if it contains the number 1, and if for every \( x \) in \( A \), the number \( x + 1 \) is also in \( A \). Let \( \mathcal{A} \) be the collection of all inductive subsets of \( \mathbb{R} \). Then the set \( \mathbb{Z}_+ \) of **positive integers** is defined by the equation

\[
\mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A.
\]

Note that the set \( \mathbb{R}_+ \) of positive real numbers is inductive, for it contains 1 and the statement \( x > 0 \) implies the statement \( x + 1 > 0 \). Therefore, \( \mathbb{Z}_+ \subseteq \mathbb{R}_+ \), so the elements of \( \mathbb{Z}_+ \) are indeed positive, as the choice of terminology suggests. Indeed, one sees readily that 1 is the smallest element of \( \mathbb{Z}_+ \), because the set of all real numbers \( x \) for which \( x \geq 1 \) is inductive.

The basic properties of \( \mathbb{Z}_+ \), which follow readily from the definition, are the following:

1. \( \mathbb{Z}_+ \) is inductive.
2. (Principle of induction). If \( A \) is an inductive set of positive integers, then \( A = \mathbb{Z}_+ \).

We define the set \( \mathbb{Z} \) of **integers** to be the set consisting of the positive integers \( \mathbb{Z}_+ \), the number 0, and the negatives of the elements of \( \mathbb{Z}_+ \). One proves that the sum, difference, and product of two integers are integers, but the quotient is not necessarily an integer. The set \( \mathbb{Q} \) of quotients of integers is called the set of **rational numbers**.

One proves also that, given the integer \( n \), there is no integer \( a \) such that \( n < a < n + 1 \).

If \( n \) is a positive integer, we use the symbol \( S_n \) to denote the set of all positive integers less than \( n \); we call it a **section** of the positive integers. The set \( S_1 \) is empty, and \( S_{n+1} \) denotes the set of positive integers between 1 and \( n \), inclusive. We also use the notation

\[
\{1, \ldots, n\} = S_{n+1}
\]

for the latter set.

Now we prove two properties of the positive integers that may not be quite so familiar, but are quite useful. They may be thought of as alternative versions of the induction principle.

**Theorem 4.1 (Well-ordering property).** Every nonempty subset of \( \mathbb{Z}_+ \) has a smallest element.
Proof. We first prove that, for each \( n \in \mathbb{Z}_+ \), the following statement holds: Every nonempty subset of \( \{1, \ldots, n\} \) has a smallest element.

Let \( A \) be the set of all positive integers \( n \) for which this statement holds. Then \( A \) contains 1, since if \( n = 1 \), the only nonempty subset of \( \{1, \ldots, n\} \) is the set \( \{1\} \) itself. Then, supposing \( A \) contains \( n \), we show that it contains \( n+1 \). So let \( C \) be a nonempty subset of the set \( \{1, \ldots, n+1\} \). If \( C \) consists of the single element \( n+1 \), then that element is the smallest element of \( C \). Otherwise, consider the set \( C \cap \{1, \ldots, n\} \), which is nonempty. Because \( n \in A \), this set has a smallest element, which will automatically be the smallest element of \( C \) also. Thus \( A \) is inductive, so we conclude that \( A = \mathbb{Z}_+ \); hence the statement is true for all \( n \in \mathbb{Z}_+ \).

Now we prove the theorem. Suppose that \( D \) is a nonempty subset of \( \mathbb{Z}_+ \). Choose an element \( n \) of \( D \). Then the set \( A = D \cap \{1, \ldots, n\} \) is nonempty, so that \( A \) has a smallest element \( k \). The element \( k \) is automatically the smallest element of \( D \) as well.

\[\textbf{Theorem 4.2 (Strong induction principle).} \] Let \( A \) be a set of positive integers. Suppose that for each positive integer \( n \), the statement \( S_n \subset A \) implies the statement \( n \in A \). Then \( A = \mathbb{Z}_+ \).

\[\text{Proof.} \quad \text{If } A \text{ does not equal all of } \mathbb{Z}_+, \text{ let } n \text{ be the smallest positive integer that is not in } A. \text{ Then every positive integer less than } n \text{ is in } A, \text{ so that } S_n \subset A. \text{ Our hypothesis implies that } n \in A, \text{ contrary to assumption.} \]

Everything we have done up to now has used only the axioms for an ordered field, properties (1)–(6) of the real numbers. At what point do you need (7), the least upper bound axiom?

For one thing, you need the least upper bound axiom to prove that the set \( \mathbb{Z}_+ \) of positive integers has no upper bound in \( \mathbb{R} \). This is the Archimedean ordering property of the real line. To prove it, we assume that \( \mathbb{Z}_+ \) has an upper bound and derive a contradiction. If \( \mathbb{Z}_+ \) has an upper bound, it has a least upper bound \( b \). There exists \( n \in \mathbb{Z}_+ \) such that \( n > b - 1 \); for otherwise, \( b - 1 \) would be an upper bound for \( \mathbb{Z}_+ \) smaller than \( b \). Then \( n + 1 > b \), contrary to the fact that \( b \) is an upper bound for \( \mathbb{Z}_+ \).

The least upper bound axiom is also used to prove a number of other things about \( \mathbb{R} \). It is used for instance to show that \( \mathbb{R} \) has the greatest lower bound property. It is also used to prove the existence of a unique positive square root \( \sqrt{x} \) for every positive real number. This fact, in turn, can be used to demonstrate the existence of real numbers that are not rational numbers; the number \( \sqrt{2} \) is an easy example.

We use the symbol \( 2 \) to denote \( 1 + 1 \), the symbol \( 3 \) to denote \( 2 + 1 \), and so on through the standard symbols for the positive integers. It is a fact that this procedure assigns to each positive integer a unique symbol, but we never need this fact and shall not prove it.

Proofs of these properties of the integers and real numbers, along with a few other properties we shall need later, are outlined in the exercises that follow.
Exercises

1. Prove the following “laws of algebra” for $\mathbb{R}$, using only axioms (1)–(5):
   (a) If $x + y = x$, then $y = 0$.
   (b) $0 \cdot x = 0$. [Hint: Compute $(x + 0) \cdot x$.]
   (c) $-0 = 0$.
   (d) $-(-x) = x$.
   (e) $x(-y) = -(xy) = (-x)y$.
   (f) $(-1)x = -x$.
   (g) $x(y - z) = xy - xz$.
   (h) $-(x + y) = -x - y$; $-(x - y) = -x + y$.
   (i) If $x \neq 0$ and $x \cdot y = x$, then $y = 1$.
   (j) $x/x = 1$ if $x \neq 0$.
   (k) $x/1 = x$.
   (l) $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$.
   (m) $(1/y)(1/z) = 1/(yz)$ if $y, z \neq 0$.
   (n) $(x/y)(w/z) = (xw)/(yz)$ if $y, z \neq 0$.
   (o) $(x/y) + (w/z) = (xz + wy)/(yz)$ if $y, z \neq 0$.
   (p) $x \neq 0 \Rightarrow 1/x \neq 0$.
   (q) $1/(w/z) = z/w$ if $w, z \neq 0$.
   (r) $(x/y)/(w/z) = (xz)/(yw)$ if $y, w, z \neq 0$.
   (s) $(ax)/y = a(x/y)$ if $y \neq 0$.
   (t) $(-x)/y = x/(-y) = -(x/y)$ if $y \neq 0$.

2. Prove the following “laws of inequalities” for $\mathbb{R}$, using axioms (1)–(6) along with the results of Exercise 1:
   (a) $x > y$ and $w > z \Rightarrow x + w > y + z$.
   (b) $x > 0$ and $y > 0 \Rightarrow x + y > 0$ and $x \cdot y > 0$.
   (c) $x > 0 \Leftrightarrow -x < 0$.
   (d) $x > y \Leftrightarrow -x < -y$.
   (e) $x > y$ and $z < 0 \Rightarrow xz < yz$.
   (f) $x \neq 0 \Rightarrow x^2 > 0$, where $x^2 = x \cdot x$.
   (g) $-1 < 0 < 1$.
   (h) $xy > 0 \Leftrightarrow x$ and $y$ are both positive or both negative.
   (i) $x > 0 \Rightarrow 1/x > 0$.
   (j) $x > y > 0 \Rightarrow 1/x < 1/y$.
   (k) $x < y \Rightarrow x < (x + y)/2 < y$.

3. (a) Show that if $A$ is a collection of inductive sets, then the intersection of the elements of $A$ is an inductive set.
   (b) Prove the basic properties (1) and (2) of $\mathbb{Z}$.

4. (a) Prove by induction that given $n \in \mathbb{Z}$, every nonempty subset of $\{1, \ldots, n\}$ has a largest element.
   (b) Explain why you cannot conclude from (a) that every nonempty subset of $\mathbb{Z}$ has a largest element.
5. Prove the following properties of $\mathbb{Z}$ and $\mathbb{Z}_+$:
   (a) $a, b \in \mathbb{Z}_+ \Rightarrow a + b \in \mathbb{Z}_+$. [Hint: Show that given $a \in \mathbb{Z}_+$, the set $X = \{x \mid x \in \mathbb{R} \text{ and } a + x \in \mathbb{Z}_+\}$ is inductive.]
   (b) $a, b \in \mathbb{Z}_+ \Rightarrow a \cdot b \in \mathbb{Z}_+$. 
   (c) Show that $a \in \mathbb{Z}_+ \Rightarrow a - 1 \in \mathbb{Z}_+ \cup \{0\}$. [Hint: Let $X = \{x \mid x \in \mathbb{R} \text{ and } x - 1 \in \mathbb{Z}_+ \cup \{0\}\}$; show that $X$ is inductive.]
   (d) $c, d \in \mathbb{Z} \Rightarrow c + d \in \mathbb{Z}$ and $c - d \in \mathbb{Z}$. [Hint: Prove it first for $d = 1$.]
   (e) $c, d \in \mathbb{Z} \Rightarrow c \cdot d \in \mathbb{Z}$.

6. Let $a \in \mathbb{R}$. Define inductively
   
   \[
   a^1 = a, \\
   a^{n+1} = a^n \cdot a
   \]

   for $n \in \mathbb{Z}_+$. (See §7 for a discussion of the process of inductive definition.)

   Show that for $n, m \in \mathbb{Z}_+$ and $a, b \in \mathbb{R}$,
   
   \[
   a^n a^m = a^{n+m}, \\
   (a^n)^m = a^{nm}, \\
   a^n b^m = (ab)^m.
   \]

   These are called the laws of exponents. [Hint: For fixed $n$, prove the formulas by induction on $m$.]

7. Let $a \in \mathbb{R}$ and $a \neq 0$. Define $a^0 = 1$, and for $n \in \mathbb{Z}_+$, $a^{-n} = 1/a^n$. Show that the laws of exponents hold for $a, b \neq 0$ and $n, m \in \mathbb{Z}$.

8. (a) Show that $\mathbb{R}$ has the greatest lower bound property.
   (b) Show that $\inf\{1/n \mid n \in \mathbb{Z}_+\} = 0$.
   (c) Show that given $a$ with $0 < a < 1$, $\inf\{a^n \mid n \in \mathbb{Z}_+\} = 0$. [Hint: Let $h = (1-a)/a$, and show that $(1+h)^n \geq 1+nh$.]

9. (a) Show that every nonempty subset of $\mathbb{Z}$ that is bounded above has a largest element.
   (b) If $x \notin \mathbb{Z}$, show there is exactly one $n \in \mathbb{Z}$ such that $n < x < n + 1$.
   (c) If $x - y > 1$, show there is at least one $n \in \mathbb{Z}$ such that $y < n < x$.
   (d) If $y < x$, show there is a rational number $z$ such that $y < z < x$.

10. Show that every positive number $a$ has exactly one positive square root, as follows:
    (a) Show that if $x > 0$ and $0 \leq h < 1$, then
        \[
        (x + h)^2 \leq x^2 + h(2x + 1), \\
        (x - h)^2 \geq x^2 - h(2x).
        \]
    (b) Let $x > 0$. Show that if $x^2 < a$, then $(x + h)^2 < a$ for some $h > 0$; and if $x^2 > a$, then $(x - h)^2 > a$ for some $h > 0$. 

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(c) Given \( a > 0 \), let \( B \) be the set of all real numbers \( x \) such that \( x^2 < a \). Show that \( B \) is bounded above and contains at least one positive number. Let \( b = \sup B \); show that \( b^2 = a \).

(d) Show that if \( b \) and \( c \) are positive and \( b^2 = c^2 \), then \( b = c \).

11. Given \( m \in \mathbb{Z} \), we say that \( m \) is even if \( m/2 \in \mathbb{Z} \), and \( m \) is odd otherwise.

(a) Show that if \( m \) is odd, \( m = 2n + 1 \) for some \( n \in \mathbb{Z} \). [Hint: Choose \( n \) so that \( n < m/2 < n + 1 \).]

(b) Show that if \( p \) and \( q \) are odd, so are \( p \cdot q \) and \( p^n \), for any \( n \in \mathbb{Z}_+ \).

(c) Show that if \( a > 0 \) is rational, then \( a = m/n \) for some \( m, n \in \mathbb{Z}_+ \) where not both \( m \) and \( n \) are even. [Hint: Let \( n \) be the smallest element of the set \( \{x \mid x \in \mathbb{Z}_+ \text{ and } x \cdot a \in \mathbb{Z}_+ \} \}].

(d) Theorem. \( \sqrt{2} \) is irrational.

§5  Cartesian Products

We have already defined what we mean by the cartesian product \( A \times B \) of two sets. Now we introduce more general cartesian products.

Definition. Let \( \mathcal{A} \) be a nonempty collection of sets. An indexing function for \( \mathcal{A} \) is a surjective function \( f \) from some set \( J \), called the index set, to \( \mathcal{A} \). The collection \( \mathcal{A} \), together with the indexing function \( f \), is called an indexed family of sets. Given \( \alpha \in J \), we shall denote the set \( f(\alpha) \) by the symbol \( A_\alpha \). And we shall denote the indexed family itself by the symbol

\[
\{A_\alpha\}_{\alpha \in J},
\]

which is read "the family of all \( A_\alpha \), as \( \alpha \) ranges over \( J \)." Sometimes we write merely \( \{A_\alpha\} \), if it is clear what the index set is.

Note that although an indexing function is required to be surjective, it is not required to be injective. It is entirely possible for \( A_\alpha \) and \( A_\beta \) to be the same set of \( \mathcal{A} \), even though \( \alpha \neq \beta \).

One way in which indexing functions are used is to give a new notation for arbitrary unions and intersections of sets. Suppose that \( f : J \to \mathcal{A} \) is an indexing function for \( \mathcal{A} \); let \( A_\alpha \) denote \( f(\alpha) \). Then we define

\[
\bigcup_{\alpha \in J} A_\alpha = \{x \mid \text{for at least one } \alpha \in J, x \in A_\alpha \},
\]

and

\[
\bigcap_{\alpha \in J} A_\alpha = \{x \mid \text{for every } \alpha \in J, x \in A_\alpha \}.
\]
These are simply new notations for previously defined concepts; one sees at once (using the surjectivity of the index function) that the first equals the union of all the elements of $\mathcal{A}$ and the second equals the intersection of all the elements of $\mathcal{A}$.

Two especially useful index sets are the set $\{1, \ldots, n\}$ of positive integers from 1 to $n$, and the set $\mathbb{Z}_+$ of all positive integers. For these index sets, we introduce some special notation. If a collection of sets is indexed by the set $\{1, \ldots, n\}$, we denote the indexed family by the symbol $\{A_1, \ldots, A_n\}$, and we denote the union and intersection, respectively, of the members of this family by the symbols

$$A_1 \cup \cdots \cup A_n \quad \text{and} \quad A_1 \cap \cdots \cap A_n.$$

In the case where the index set is the set $\mathbb{Z}_+$, we denote the indexed family by the symbol $\{A_1, A_2, \ldots\}$, and the union and intersection by the respective symbols

$$A_1 \cup A_2 \cup \cdots \quad \text{and} \quad A_1 \cap A_2 \cap \cdots.$$

**Definition.** Let $m$ be a positive integer. Given a set $X$, we define an $m$-tuple of elements of $X$ to be a function

$$\mathbf{x} : \{1, \ldots, m\} \to X.$$

If $\mathbf{x}$ is an $m$-tuple, we often denote the value of $\mathbf{x}$ at $i$ by the symbol $x_i$ rather than $x(i)$ and call it the $i$th coordinate of $\mathbf{x}$. And we often denote the function $\mathbf{x}$ itself by the symbol

$$(x_1, \ldots, x_m).$$

Now let $\{A_1, \ldots, A_m\}$ be a family of sets indexed with the set $\{1, \ldots, m\}$. Let $X = A_1 \cup \cdots \cup A_m$. We define the cartesian product of this indexed family, denoted by

$$\prod_{i=1}^{m} A_i \quad \text{or} \quad A_1 \times \cdots \times A_m,$$

to be the set of all $m$-tuples $(x_1, \ldots, x_m)$ of elements of $X$ such that $x_i \in A_i$ for each $i$.

**Example 1.** We now have two definitions for the symbol $A \times B$. One definition is, of course, the one given earlier, under which $A \times B$ denotes the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$. The second definition, just given, defines $A \times B$ as the set of all functions $\mathbf{x} : \{1, 2\} \to A \cup B$ such that $\mathbf{x}(1) \in A$ and $\mathbf{x}(2) \in B$. There is an obvious bijective correspondence between these two sets, under which the ordered pair $(a, b)$ corresponds to the function $\mathbf{x}$ defined by $\mathbf{x}(1) = a$ and $\mathbf{x}(2) = b$. Since we commonly denote this function $\mathbf{x}$ in "tuple notation" by the symbol $(a, b)$, the notation itself suggests the correspondence. Thus for the cartesian product of two sets, the general definition of cartesian product reduces essentially to the earlier one.

**Example 2.** How does the cartesian product $A \times B \times C$ differ from the cartesian products $A \times (B \times C)$ and $(A \times B) \times C$? Very little. There are obvious bijective correspondences between these sets, indicated as follows:

$$(a, b, c) \longleftrightarrow (a, (b, c)) \longleftrightarrow ((a, b), c).$$
Definition. Given a set $X$, we define an $\omega$-tuple of elements of $X$ to be a function

$$x : \mathbb{Z}_+ \longrightarrow X;$$

we also call such a function a sequence, or an infinite sequence, of elements of $X$. If $x$ is an $\omega$-tuple, we often denote the value of $x$ at $i$ by $x_i$ rather than $x(i)$, and call it the $i$th coordinate of $x$. We denote $x$ itself by the symbol

$$(x_1, x_2, \ldots) \quad \text{or} \quad (x_n)_{n \in \mathbb{Z}_+}.$$ 

Now let $\{A_1, A_2, \ldots\}$ be a family of sets, indexed with the positive integers; let $X$ be the union of the sets in this family. The cartesian product of this indexed family of sets, denoted by

$$\prod_{i \in \mathbb{Z}_+} A_i \quad \text{or} \quad A_1 \times A_2 \times \cdots,$$

is defined to be the set of all $\omega$-tuples $(x_1, x_2, \ldots)$ of elements of $X$ such that $x_i \in A_i$ for each $i$.

Nothing in these definitions requires the sets $A_i$ to be different from one another. Indeed, they may all equal the same set $X$. In that case, the cartesian product $A_1 \times \cdots \times A_m$ is just the set of all $m$-tuples of elements of $X$, which we denote by $X^m$. Similarly, the product $A_1 \times A_2 \times \cdots$ is just the set of all $\omega$-tuples of elements of $X$, which we denote by $X^\omega$.

Later we will define the cartesian product of an arbitrary indexed family of sets.

Example 3. If $\mathbb{R}$ is the set of real numbers, then $\mathbb{R}^m$ denotes the set of all $m$-tuples of real numbers; it is often called euclidean $m$-space (although Euclid would never recognize it). Analogously, $\mathbb{R}^\omega$ is sometimes called "infinite-dimensional euclidean space"; it is the set of all $\omega$-tuples $(x_1, x_2, \ldots)$ of real numbers, that is, the set of all functions $x : \mathbb{Z}_+ \rightarrow \mathbb{R}$.

Exercises

1. Show there is a bijective correspondence of $A \times B$ with $B \times A$.

2. (a) Show that if $n > 1$ there is bijective correspondence of

$$A_1 \times \cdots \times A_n \quad \text{with} \quad (A_1 \times \cdots \times A_{n-1}) \times A_n.$$

(b) Given the indexed family $\{A_1, A_2, \ldots\}$, let $B_i = A_{2i-1} \times A_{2i}$ for each positive integer $i$. Show there is bijective correspondence of $A_1 \times A_2 \times \cdots$ with $B_1 \times B_2 \times \cdots$.

3. Let $A = A_1 \times A_2 \times \cdots$ and $B = B_1 \times B_2 \times \cdots$.

(a) Show that if $B_i \subset A_i$ for all $i$, then $B \subset A$. (Strictly speaking, if we are given a function mapping the index set $\mathbb{Z}_+$ into the union of the sets $B_i$, we must change its range before it can be considered as a function mapping $\mathbb{Z}_+$ into the union of the sets $A_i$. We shall ignore this technicality when dealing with cartesian products).
(b) Show the converse of (a) holds if \( B \) is nonempty.
(c) Show that if \( A \) is nonempty, each \( A_i \) is nonempty. Does the converse hold? (We will return to this question in the exercises of §19.)
(d) What is the relation between the set \( A \cup B \) and the cartesian product of the sets \( A_i \cup B_i \)? What is the relation between the set \( A \cap B \) and the cartesian product of the sets \( A_i \cap B_i \)?

4. Let \( m, n \in \mathbb{Z}_+ \). Let \( X \neq \emptyset \).
(a) If \( m \leq n \), find an injective map \( f : X^m \to X^n \).
(b) Find a bijective map \( g : X^m \times X^n \to X^{m+n} \).
(c) Find an injective map \( h : X^n \to X^\omega \).
(d) Find a bijective map \( k : X^n \times X^{\omega} \to X^{\omega} \).
(e) Find a bijective map \( l : X^{\omega} \times X^{\omega} \to X^{\omega} \).
(f) If \( A \subset B \), find an injective map \( m : (A^{\omega})^n \to B^{\omega} \).

5. Which of the following subsets of \( \mathbb{R}^\omega \) can be expressed as the cartesian product of subsets of \( \mathbb{R}^? \)?
(a) \( \{ x \mid x_i \text{ is an integer for all } i \} \).
(b) \( \{ x \mid x_i \geq i \text{ for all } i \} \).
(c) \( \{ x \mid x_i \text{ is an integer for all } i \geq 100 \} \).
(d) \( \{ x \mid x_2 = x_3 \} \).

§6 Finite Sets

Finite sets and infinite sets, countable sets and uncountable sets, these are types of sets that you may have encountered before. Nevertheless, we shall discuss them in this section and the next, not only to make sure you understand them thoroughly, but also to elucidate some particular points of logic that will arise later on. First we consider finite sets.

Recall that if \( n \) is a positive integer, we use \( S_n \) to denote the set of positive integers less than \( n \); it is called a section of the positive integers. The sets \( S_n \) are the prototypes for what we call the finite sets.

**Definition.** A set is said to be **finite** if there is a bijective correspondence of \( A \) with some section of the positive integers. That is, \( A \) is finite if it is empty or if there is a bijection

\[
f : A \longrightarrow \{1, \ldots, n\}
\]

for some positive integer \( n \). In the former case, we say that \( A \) has **cardinality 0**; in the latter case, we say that \( A \) has **cardinality** \( n \).

For instance, the set \( \{1, \ldots, n\} \) itself has cardinality \( n \), for it is in bijective correspondence with itself under the identity function.