For this entire short assignment, let \( f : A \rightarrow B \) and \( g : B \rightarrow C \)

1. Let \( C_0 \subset C \). Consider \((g \circ f)^{-1}(C_0)\):

\[
(g \circ f)^{-1}(C_0) = \{ a : (g \circ f)(a) \in C_0 \} = \{ a : g(f(a)) \in C_0 \} = \{ a : f(a) \in g^{-1}(C_0) \} = f^{-1}(g^{-1}(C_0))
\]

So, \((g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))\). ■

2. Let \( f, g \) be injective functions. Then, \( f(a) = f(a') \Rightarrow a = a' \) and \( g(b) = g(b') \Rightarrow b = b' \). Now, consider \( g \circ f \). Suppose BWOC \( \exists a, a' \) such that \((g \circ f)(a) = (g \circ f)(a') \) but \( a \neq a' \). Then, we have:

\[
\begin{align*}
(g \circ f)(a) &= g(f(a)) \\
(g \circ f)(a') &= g(f(a'))
\end{align*}
\]

By the contradictory assumption, we know \( g(f(a)) = g(f(a')) \). Since \( g \) is injective, this then implies that \( f(a) = f(a') \). However, \( a \neq a' \) by assumption, which contradicts the fact that \( f \) is an injective function. Thus, if \( f \) and \( g \) are injective, \( g \circ f \) must also be injective. ■

3. Now, suppose \( g \circ f \) is injective. Then, I assert that \( f \) is injective. To see this, suppose BWOC that \( f \) is not injective. Then, \( \exists a, a' \subset A \) s.t. \( a \neq a' \) but \( f(a) = f(a') \). But then \( (g \circ f)(a) = g(f(a)) = g(f(a')) = (g \circ f)(a') \). But \( a \neq a' \), which violates the assumption that \( g \circ f \) is injective.

However, it is not necessarily the case that \( g \) must be injective. To see this, suppose \( A = \{1, 2\} \) and \( B = C = \mathbb{R} \). Set \( f(a) = a \). Set \( g(a) = a^2 \). Clearly, \( g \) is not injective, as \( 2 \neq -2 \) but \( g(2) = g(-2) \). However, \( g(f(a)) \) takes on values 4 if \( a = 2 \) and 1 if \( a = 1 \).

Thus, if \( g \circ f \) is injective, \( f \) must be injective but \( g \) need not be. ■

4. Suppose \( f, g \) are surjective functions. Then, \( b \in B \Rightarrow b = f(a) \) for at least one \( a \in A \) and \( c \in C \Rightarrow c = g(b) \) for at least one \( b \in B \). Now, consider \( g \circ f \). Suppose BWOC \( \exists c \in C \) s.t. \( \nexists a \in A \) s.t. \((g \circ f)(a) = c\). Then, \( f^{-1}(g^{-1}(c)) = \emptyset \). However, because \( g \) is surjective, \( \exists b \in B \) s.t. \( g(b) = c \), so \( g^{-1}(c) \neq \emptyset \). Thus, if \( f^{-1}(g^{-1}(c)) = \emptyset \), then \( \forall a \in A \) s.t. \( f(a) \in g^{-1}(c) \neq \emptyset \). But, because \( f \) is surjective, such an \( a \in A \) always exists. Thus, we have a contradiction. So, if \( f, g \) are surjective functions, so is \( g \circ f \). ■

5. Suppose that \( g \circ f \) is surjective. For some \( c \in C \), since \( g \circ f \) is surjective, \( \exists a \in A \) s.t. \((g \circ f)(a) = g(f(a)) = c \). Defining \( b = f(a) \), it then follows that \( g(b) = c \). Since this holds \( \forall c \in C \), we have shown that there always exists \( b \in B \) s.t. \( g(b) = c \), which is the precise definition of a surjective function.

However, it is not necessarily the case that \( f \) must be surjective. To see this, let \( A = \{2\}, B = \{1, 2\}, \) and \( C = \{4\} \). Define \( f(a) = a \), \( g(b) = b^2 \). Then, \((g \circ f)\) only takes on the value 4, so is surjective (because for any \( c \in C \) (there is only one), \( \exists b \in B \) \((b = 2) \), s.t. \( g(b) = c \)). However, \( f \) is not surjective, as there does not exist an \( a \in A \) such that \( f(a) = 1 \in B \).

Thus, if \( g \circ f \) is surjective, \( g \) must be surjective but \( f \) need not be. ■

6. **Theorem:** Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \). Then,

1. If \( f \) and \( g \) are injective (surjective), then \( g \circ f \) is injective (surjective)
2. If \( f \) and \( g \) are bijective, then \( g \circ f \) is bijective.

3. If \( g \circ f \) is injective, then \( f \) must be injective, but \( g \) need not be

4. If \( g \circ f \) is surjective, then \( g \) must be surjective, but \( f \) need not be. ■