CS 230 Short Assignment 13

(a) **Claim:** \( (g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0)) \).

**Proof:**

Since

\[
g^{-1}(C_0) = \{ b \in B | g(b) \in C_0 \}
\]
and

\[
f^{-1}(g^{-1}(C_0)) = \{ a \in A | f(a) \in g^{-1}(C_0) \},
\]

Substituting \( b = f(a) \), we have

\[
f^{-1}(g^{-1}(C_0)) = \{ a \in A | g(f(a)) \in C_0 \}.
\]

On the other hand,

\[
(g \circ f)^{-1}(C_0) = \{ a \in A | (g \circ f)(a) \in C_0 \}
\]

Therefore,

\[
(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))
\].

(b) **Claim:** If \( f \) and \( g \) are injective, then \( g \circ f \) is injective.

**Proof:**

Suppose \( \exists a_1, a_2 \in A \) such that

\[
(g \circ f)(a_1) = (g \circ f)(a_2).
\]

We want to prove \( a_1 = a_2 \).

Since \( g \) is injective and

\[
g(f(a_1)) = g(f(a_2)).
\]
We must have

\[
f(a_1) = f(a_2).
\]
Since \( f \) is also injective,

\[
a_1 = a_2.
\]
Therefore, \( g \circ f \) is injective.

(c) Since \( g \circ f \) is injective,

\[
(g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2.
\]

If \( f \) is not injective, \( \exists a_1, a_2 \in A \) such that \( f(a_1) = f(a_2) \) but \( a_1 \neq a_2 \).
Thus,
\[ g(f(a_1)) = g(f(a_2)) \]
\[ \Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2) \]
\[ \Rightarrow a_1 = a_2. \]

However, this contradicts \( a_1 \neq a_2 \).
Therefore, \( f \) is injective.

However, \( g \) may not be injective.

For example, if we have \( f: \mathbb{N} \to \mathbb{Z} \) and \( g: \mathbb{Z} \to \mathbb{Z} \) such that \( f(x) = x, g(x) = |x| \),

Then \( g \circ f \) is injective as its domain is \( \mathbb{N} \) and \( (g \circ f)(x) = x \).

However, \( g \) is not injective because \( g(-x) = g(x) \) but \(-x \neq x\).

(d) **Claim:** If \( f \) and \( g \) are surjective, then \( g \circ f \) is surjective.

**Proof:**

We want to prove that \( \forall c \in C, \exists a \in A \) such that \( (g \circ f)(a) = c \).

Since \( g \) is surjective, \( \exists b \in B \) such that \( g(b) = c \).

Since \( f \) is also surjective, \( \exists a \in A \) such that \( f(a) = b \),

\[ \Rightarrow (g \circ f)(a) = c. \]

Therefore, \( g \circ f \) is surjective.

(e) Since \( g \circ f \) is surjective, \( \forall c \in C, \exists a \in A \) such that
\[ (g \circ f)(a) = c. \]

Thus, \( \forall c \in C, \exists f(a) \in B \) such that
\[ g(f(a)) = c. \]

Therefore, \( g \) is surjective.

However, \( f \) may not be surjective.

For example, if we have \( f: \mathbb{Z} \to \mathbb{Z} \) and \( g: \mathbb{Z} \to \mathbb{N} \) such that \( f(x) = |x|, g(x) = |x| \),

Then \( g \circ f \) is surjective as its range is \( \mathbb{N} \) and \( \forall x \in \mathbb{N}, (g \circ f)(x) = x \).

However, \( f \) is not surjective because when \( a < 0 \), there does not exist \( x \in \mathbb{Z} \) such that \( f(x) = a \).
(f) Theorem:

Let $f: A \to B$ and $g: B \to C$. Then:

- If $f$ and $g$ are injective, then $g \circ f$ is injective;
- If $f$ and $g$ are surjective, then $g \circ f$ is surjective;
- If $g \circ f$ is injective, then $f$ is injective;
- If $g \circ f$ is surjective, then $g$ is surjective.

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  - I will act if the Standard is compromised.

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