Questions

Here are the questions. Probability is something that feels easy, but problems often require a subtle way of thinking which makes the answer intuitive, yet tricky to find (at least for me). I’d recommend taking a hack at these, then comparing your answers to the solutions on the next page and practicing with the supplementary problems on the course site. That way, there should be no need to lose points on probability problems, and you’ll have developed a new logic to problem solving.

1

You flip two coins, blindfolded. Somebody tells you that at least one resulted in a Heads. What is the probability both were heads?

1a

Now, same situation, but someone said that the first coin came up heads. What is the probability both were heads?

2

You flip ten coins, and 7 are heads. What is the probability the first coin was heads?

2a

You flip ten coins, and at least 7 are heads. What is the probability the first coin was heads?
Solutions

1
We have some event A conditioned on B. What are B and A? A is the event that both coins are heads. B is the event at least one coin is heads. With two flips of a coin, our event space is \( \{H,H\}, \{T,H\}, \{H,T\}, \{T,T\} \). The probability of at least one heads is \( 1 - P(\text{no heads}) = 1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} \). A is \( \left(\frac{1}{2}\right)^2 = \frac{1}{4} \). From our definition of conditional probability,

\[
P(A|B) = \frac{P(A \cap B)}{P(B)}.
\]

Substituting, we see that \( P(2 \text{ Heads}) = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16} \).

1a
At first glance, knowing that the first coin is Heads feels like it should be the same as the first problem, but we see this eliminates from the event space the two outcomes where the first flip it tails, so this problem becomes as simple as the likelihood of the second flip being Heads, so \( P(A|B) = \frac{1}{2} \). This gives insight towards the intuition behind our relationship between the conditional probability and the intersection. Note \( P(A \cap B) = \frac{1}{4} \).

2
Event A is the first coin is Heads, and event B is that 7 coins are heads. We observe there are \( 2^{10} \) possible outcomes of flipping 10 coins. There are \( \binom{10}{7} \) ways to arrange 7 heads in the 10 coins. Thus, \( P(B) = \binom{10}{7}/2^{10} \). Now, what is \( P(A \cap B) \)? We expand \( \binom{10}{7} = \frac{10!}{7!3!} \) and wonder how many of these arrangements have a Heads in the first place? After fixing one Heads, we have 9 spots left to arrange 6 Heads, so \( P(A \cap B) = \frac{\binom{9}{6}}{2^{10}} \). Perhaps more convincingly, with the identity \( \binom{10}{7} = \binom{9}{6} + \binom{9}{7} \), we see that 7 Heads in 10 flips can be viewed as the union of six heads in 9 flips and 7 heads in 9 flips, where the latter is when we don’t have a heads on the first flip. Putting all of this together,

\[
P(A|B) = \frac{\binom{9}{6}/2^{10}}{\binom{10}{7}/2^{10}} = \frac{\binom{9}{6}}{\binom{10}{7}} = \frac{7}{10}.
\]

This is logical, as there should be an equal probability of any one of the coins being Heads.
2a

Very similar problem, except we see B as the union of the events of flipping exactly 7,8,9,10 heads, and denote these as $B_7, B_8, \ldots$, respectively. As these are independent,

$$P(B) = \sum_{i=7}^{10} B_i = \sum_{k=7}^{10} \binom{10}{k} \frac{1}{2^{10}}.$$ 

Likewise, we distribute to see

$$P(A \cap B) = P(A \cap B_7) + P(A \cap B_8) + P(A \cap B_9) + P(A \cap B_{10}).$$ 

Applying the same logic from the problem above, this gives

$$P(A \cap B) = \sum_{k=7}^{10} \binom{9}{k-1} \frac{1}{2^{10}}.$$ 

Finally,

$$P(A|B) = \frac{\sum_{k=7}^{10} \binom{9}{k-1}}{\sum_{k=7}^{10} \binom{10}{k}} = \frac{130}{176} \approx 0.74.$$ 

This means that it is more likely than not that the first coin was a heads which intuitively makes sense.