1 Combinatorics

1.1 Permutations

Assume you have n objects. The number of ways to fill n ordered slots with them is

\[ n \cdot (n - 1) \cdot \ldots \cdot 2 \cdot 1 = n!, \]

while the number of ways to fill \( k \leq n \) ordered slots is

\[ n \cdot (n - 1) \cdot \ldots \cdot (n - k + 1) = \frac{n!}{(n - k)!} \]

1.2 Example 2.6

Problem:
1. \( P(\text{top card is an Ace}) \)
2. \( P(\text{all cards of the same suit end up next to each other}) \)
3. \( P(\text{hearts are together}) \)

Solution:
1. \( P(\text{top card is an Ace}) = \frac{4 \cdot 51!}{52!} \)

Remember, for equally likely outcomes, we can think of the probability of an event as a fraction where the numerator is the number of “good” or desirable outcomes corresponding to the event, and the denominator is the total number of possible outcomes.

So, since there are 4 aces in a deck of 52 cards, we can choose any 4 of them to be on the top of the deck. And then we must order the rest of the 51 cards after we’ve picked an ace. The denominator represents the total number of orderings for all 52 cards.
2. \( P(\text{all cards of the same suit end up next to each other}) = 4! \cdot (13!)^4 \frac{4!}{52!} \)

For the numerator, think about having 4 slots for each of the 4 suits. We can arrange these 4 suits in 4! ways. For each suit, we can arrange the 13 cards of that suit in 13! ways. Hence, \((13!)^4\). The story of the denominator is the same.

3. \( P(\text{hearts are together}) = \frac{40!13!}{52!} \)

For the numerator, think about grouping all the hearts into one clump. Since we have used up 13 cards to make this clump, we have 39 cards left. There are 40! ways to arrange 39 cards and the clump of hearts. We multiply by 13! for the same reason as above: we can arrange the 13 hearts in 13! ways. The story of the denominator is the same.

### 1.3 Combinations

Let \( \binom{n}{k} \) be the number of different subsets with k elements of a set with n elements. Then,

\[
\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!} = \frac{n!}{k!(n-k)!}
\]

Here are two ways to see why this makes sense:

1. First choose a subset of size k, then order its elements in a row to fill k ordered slots with elements from the set with n objects. Then, distinct choices of a subset and its ordering will end up as distinct orderings. Therefore,

\[
\binom{n}{k} k! = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)
\]

which gives back the formula for permutation.

2. Start with the permutation formula, with which we arranged n objects into k slots. Since we do not care about the ordering of the k slots we have to divide out by k!.

### 1.4 Example 2.11

Problem:

We have a bag that contains 100 balls, 50 of them red and 50 blue. Select 5 balls at random. What is the probability that 3 are blue and 2 are red?

Solution:
\[ P(3 \text{ are blue and 2 are red}) = \frac{\binom{50}{3} \binom{50}{2}}{\binom{100}{5}} \]

The number of outcomes is \( \binom{100}{5} \) and all of them are equally likely, which is a reasonable interpretation of “select 5 balls at random.” For the numerator, we want to choose any 3 of the 50 blue balls and any 2 of the 50 red balls. Be sure to check out the [hypergeometric distribution](#).

### 2 More Practice

#### 2.1 Example 2.7 (modified)

**Problem:**

A bag has 8 pieces of paper, each with one of the letters, N, T, G, W, N, I, I, and N, on it. Pull 8 pieces at random out of the bag (1) without, and (2) with replacement. What is the probability that these pieces, in order, spell TWINNING?

**Solution:**

**Without replacement:** \( P(\text{order in which the letters are drawn spells TWINNING}) = \frac{3! \cdot 2!}{8!} \).

There are a few ways to see this result. Since we have equally likely outcomes, the denominator represents the total number of arrangements of the letters, where all letters (including I’s and N’s) are distinguished from each other. TWINNING can be spelt several ways with the 8 letters (because we have repeat letters!), meaning two distinguishable I’s could be switched around, and still spell TWINNING. So in the numerator we enumerate the number of arrangements of I’s and N’s in the sequence of letters that spell TWINNING.

Another way of seeing this result is to compute the total number of unique spellings that can be made with the letters N, T, G, W, N, I, I, and N. To compute this we would take the total number of arrangements for the 8 letters, \( 8! \), and divide out arrangements for repeated letters, i.e. \( \frac{8!}{2!3!} \). The “good” outcome for us is the 1 unique sequence of letters that spells TWINNING, so we have \( \frac{1}{2!3!} \), which gives us back \( \frac{3! \cdot 2!}{8!} \).

**With replacement:** \( P(\text{order in which the 8 letters are drawn spells TWINNING}) = \frac{3^2 \cdot 2^2}{8^8} \).

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1 inspired by a most esteemed set of twins Hunter and Tyler Nisonoff

2 Twinning means a variety of things, but most commonly: dressing like a non-twin companion, typically without prior planning
Since we are replacing letters, the probability of drawing a letter does not change from draw to draw. For a non-repeated letter, the probability is 1/8. For a repeated letter, such as, N, the probability is 3/8. So, we can reconcile this formula in the following way:

\[
P(\text{order in which the 8 letters are drawn spells TWINNING}) = \frac{1}{8} \cdot \frac{1}{8} \cdot \frac{2}{8} \cdot \frac{3}{8} \cdot \frac{3}{8} \cdot \frac{2}{8} \cdot \frac{3}{8} \cdot \frac{1}{8}
\]

2.2 Example 2.15

Problem:

A middle row on a plane seats 7 people. Three of them order chicken (C) and the remaining four pasta (P). The flight attendant returns with the meals, but has forgotten who ordered what and discovers that they are all asleep, so she puts the meals in front of them at random. What is the probability that they all receive correct meals?

Solution:

A reformulation makes the problem clearer: we are interested in \( P(3 \text{ people who ordered C get C}) \). Let us label the people 1,...,7 and assume that 1, 2, and 3 ordered C. The outcome is a selection of 3 people from the 7 who receive C, the number of them is \( \binom{7}{3} \), and there is a single good outcome. So, the answer is \( \frac{1}{\binom{7}{3}} = 1/35 \). Similarly,

\[
P(\text{no one who ordered C gets C}) = \frac{\binom{4}{3}}{\binom{7}{3}} = \frac{4}{35}
\]

\[
P(\text{a single person who ordered C gets C}) = \frac{\binom{4}{1}\binom{3}{1}}{\binom{7}{3}} = \frac{18}{35}
\]

\[
P(\text{two persons who ordered C get C}) = \frac{\binom{4}{2}\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}
\]

Note: We can define other events too, e.g. \( P(\text{at least 2 who ordered C get C}) \). We could break this down \( P(2 \text{ who ordered C get C}) + P(3 \text{ who ordered C get C}) \). But do you see why it might be cumbersome and inefficient if there were n (large) orders and m (also large, but < n) were for chicken?

Instead, we apply a very useful trick to compute this probability. Note that \( P(\text{at least 2 who ordered C get C}) = 1 - P(1 \text{ or less people who ordered C get C}) = 1 - (P(\text{no one who ordered C gets C}) + P(\text{a single person who ordered C gets C})) \). Does this computation change as n and m grow? (It does not). Many times it’s much easier or computationally feasible to compute the complement!
3 Probability Theory

3.1 Conditional Probability

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

3.2 Law of Total Probability

\[ P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) \]

More generally, assume that \( F_1, \ldots, F_n \) are pairwise disjoint and that \( F_1 \cup \ldots \cup F_n = \Omega \), that is, exactly one of them always happens. Then, for an event \( A \):

\[ P(A) = P(F_1)P(A|F_1) + P(F_2)P(A|F_2) + \ldots + P(F_n)P(A|F_n) \]

3.3 Bayes’ Rule

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]

Conditional Probability helps us compute the numerator, and Law of Total Probability helps us compute the denominator. For example, the partition can be \( A \) and \( A^C \) or \((\Omega - A)\):

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B)P(B|A) + P(B|A^C)P(A^C)} \]