Bruce's Derivation of the Integral of the Second Legendre Polynomial Over S^2

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December 2024

We want the integral over S^2 of $(3(\mathbf{v} \cdot \mathbf{B}_0) - 1)d\mathbf{v}$
where $\mathbf{v} \in S^2$ is a unit vector, and $\mathbf{B}_0$ is a fixed unit vector.

Solution: We aim to evaluate the integral:

$$\int_{S^2} \left(3(\mathbf{v} \cdot \mathbf{B}_0)^2 - 1\right) \, d\mathbf{v},$$

where:
S^2 is the unit sphere in $\mathbb{R}^3$,
$\mathbf{v}$ is a unit vector on S^2,
$\mathbf{B}_0$ is a fixed unit vector,
$d\mathbf{v}$ is the surface area element on the sphere.

1 Decomposing the Integral

We break the integral into two terms:

$$\int_{S^2} \left(3(\mathbf{v} \cdot \mathbf{B}_0)^2 - 1\right) \, d\mathbf{v} = 3 \int_{S^2} (\mathbf{v} \cdot \mathbf{B}_0)^2 \, d\mathbf{v} - \int_{S^2} 1 \, d\mathbf{v}.$$

2 Integral of 1 over S^2

The surface area of the unit sphere S^2 is:

$$\int_{S^2} 1 \, d\mathbf{v} = 4\pi.$$

3 Integral of $(\mathbf{v} \cdot \mathbf{B}_0)^2$ over S^2

Let $\mathbf{v} \cdot \mathbf{B}_0 = \cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{B}_0$. The integral $\int_{S^2} (\mathbf{v} \cdot \mathbf{B}_0)^2 \, d\mathbf{v}$ is rotationally symmetric, so we can simplify using spherical coordinates.

3.1 Write v in spherical coordinates with $\mathbf{B}_0$ aligned along the z-axis:

$$\begin{aligned}
&\mathbf{v} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),\\
&d\mathbf{v} = \sin\theta\, d\theta\, d\phi,\\
&\cos\theta = \mathbf{v}\cdot\mathbf{B}_0.
\end{aligned}$$

3.2 Compute the integral of $(\cos\theta)^2$:

$$\int_{S^2} (\cos\theta)^2\, d\mathbf{v} = \int_0^{2\pi}\int_0^{\pi} (\cos\theta)^2 \sin\theta\, d\theta\, d\phi.$$

3.3 Integrate over ϕ:

$$\int_0^{2\pi} d\phi = 2\pi.$$

3.4 Integrate over θ:

$$\int_0^{\pi} (\cos\theta)^2 \sin\theta\, d\theta = \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta.$$

Use the substitution $u = \cos\theta$, where $du = -\sin\theta\, d\theta$, and the limits change from $u = 1$ to $u = -1$:

$$\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta = \int_1^{-1} u^2(-du) = \int_{-1}^{1} u^2\, du.$$

3.5 Evaluate $\int_{-1}^{1} u^2\, du$:

$$\int_{-1}^{1} u^2\, du = \left[\frac{u^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}.$$

3.6 Combine results:

$$\int_{S^2} (\cos\theta)^2\, d\mathbf{v} = 2\pi\cdot\frac{2}{3} = \frac{4\pi}{3}.$$

4 Combine Terms

Now substitute back into the original equation:

$$\int_{S^2} \left(3(\mathbf{v} \cdot \mathbf{B}_0)^2 - 1\right) \, d\mathbf{v} = 3 \int_{S^2} (\mathbf{v} \cdot \mathbf{B}_0)^2 \, d\mathbf{v} - \int_{S^2} 1 \, d\mathbf{v}.$$

Substitute the values:

$$\int_{S^2} (\mathbf{v} \cdot \mathbf{B}_0)^2 \, d\mathbf{v} = \frac{4\pi}{3}, \quad \int_{S^2} 1 \, d\mathbf{v} = 4\pi.$$

$$\int_{S^2} \left(3(\mathbf{v} \cdot \mathbf{B}_0)^2 - 1\right) \, d\mathbf{v} = 3 \cdot \frac{4\pi}{3} - 4\pi = 4\pi - 4\pi = 0.$$

5 Final Answer

$$\int_{S^2} \left(3(\mathbf{v} \cdot \mathbf{B}_0)^2 - 1\right) \, d\mathbf{v} = 0.$$