all formulas we will obtain, the denominators consist only of $d_i(\alpha)$. For an arbitrary polygon, d_i is a linear function of α. If all a_i are parallel, then $d_i = 1$. If the polygon is rectilinear, i.e., all a_i are either parallel or perpendicular, then $d_i(\alpha) = 1$ if $a_i || a_k$, and $d_i(\alpha) = 1 + \alpha a_\perp$ if $a_i \perp a_k$, where $a_\perp$ is constant. So in this case, there are only two different constant denominators, one of which is 1.

Translating the Bisector. We now consider the case where l' shifts parallel (Fig. 11). Analogously to the previous paragraph, let $r_i' = s' + \rho_i' r'$, and $r_i'' = s'' + \rho_i'' r'$. Also, let the vector between s' and s'' be $s'' - s' = \beta a_2$. Then the polygon area between l' and l'' is

$$\begin{aligned} B &= \beta a_2 \times \frac{1}{2}\left(\left(r_2' + r_2''\right) - \left(r_1' + r_1''\right)\right), \\ &= \frac{\beta}{2}\left(\rho_2' + \rho_2'' - \rho_1' - \rho_1''\right)\left(a_2 \times (r + \alpha a_2)\right), \\ &= \frac{\beta}{2}\left(\rho_2' + \rho_2'' - \rho_1' - \rho_1''\right). \end{aligned} \tag{5}$$

In the general case, l' and l'' intersect multiple edges of some arbitrary polygon P at points $r_1', r_2', \ldots, r_k'$ and $r_1'', r_2'', \ldots, r_k''$. Now the ρ_i'' can be determined from the two vector equations $r_i'' = r_i' + \lambda a_i$, $\lambda \in \mathbb{R}$, and $r_i'' = s'' + \rho_i'' r'$:

$$\begin{aligned} \rho_i'' &= \rho_i' - \beta\frac{a_i \times a_k}{a_i \times r'}, \\ &= \rho_i' - \beta\frac{a_i \times a_k}{1 + \alpha(a_i \times a_k)}, \\ &= \frac{\rho_i - \beta(a_i \times a_k)}{1 + \alpha(a_i \times a_k}. \end{aligned} \tag{6}$$

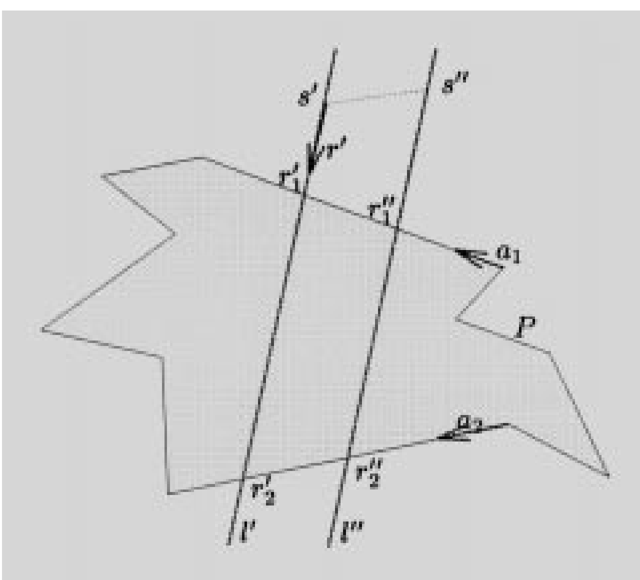

Fig. 11. Two parallel lines l' and l'' in combinatorially equivalent intersection with polygon P.

Then the polygon area between l' and l'' is

$$\begin{aligned} B &= \frac{\beta}{2}\sum_{i=1}^{k}(-1)^i(\rho_i' + \rho_i''), \\ &= \frac{\beta}{2}\sum_{i=1}^{k}(-1)^i\frac{(\rho_i - \beta(a_i \times a_k)}{1 + \alpha(a_i \times a_k)}. \end{aligned} \tag{7}$$

This is a quadratic polynomial in β (unless all a_i are parallel, in which case it simplifies to the linear equation $B = \beta\sum_{i=1}^{k}(-1)^i\rho_i$).

Maintaining the Bisector Property. From the above two paragraphs, we see that if the bisector l is rotated to l', then the left and right areas are changed by a value A ($\neq 0$ in general) as described in eq. (4). Hence, a subsequent shift of l' is necessary to restore the bisector property, by changing the areas by a value B, as described in eq. (7).

This implies the condition $A + B = 0$, with A and B given by eqs. (4) and (7):

$$\begin{aligned} A + B &= \frac{1}{2}\sum_{i=1}^{k}(-1)^i\frac{\alpha\rho_i^2 + 2\beta\rho_i - \beta^2(a_i \times a_k)}{1 + \alpha(a_i \times a_k)}. \\ &= 0. \end{aligned} \tag{8}$$

This equation ensures that l is a bisector of P. It is a necessary and sufficient condition for translation equilibrium in a unit-squeeze field. Equation (8) is a rational equation in α, and a quadratic polynomial equation in β. Hence for all combinatorially equivalent bisectors, we can obtain an explicit formula to describe β as a function of α.

In general, eq. (8) is equivalent to a polynomial in α and β whose degree depends on the number k of polygon edges intersected by the bisectors l, l', or l''. The degree of this polynomial is limited by k for α, and by 2 for β. In the rectilinear case, the degrees for α and β are limited by 2. In the case where all a_i are parallel, eq. (8) simplifies to a linear equation: $\sum_{i=1}^{k}(-1)^i(\alpha\frac{\rho_i}{2} + \beta)\rho_i = 0$.

Moment Equilibrium. After rotating (parameter α, obtain l') and translating (parameter β, obtain l'') the bisector l, its intersections with the polygon edges move from r_i to

$$\begin{aligned} r_i'' &= s + \rho_i'' r' + \beta a_k, \\ &= s + \frac{\rho_i - \beta(a_i \times a_k)}{1 + \alpha(a_i \times a_k)}(r + \alpha a_k) + \beta a_k. \end{aligned} \tag{9}$$

If all a_i are parallel, this simplifies to $r_i'' = s + \rho_i r + (\alpha\rho_i + \beta)a_k$.

Suppose that c_l and c_r, are the left and the right centers of area of P, and A_l and A_r are the respective area sections, so $A_l + A_r = A$. We are interested in how these points change when the bisector changes. Note that always $c = \frac{1}{A}(A_l c_l + A_r c_r)$, and if P is bisected (i.e., $A_l = A_r = \frac{1}{2}A$) then $c = \frac{1}{2}(c_l + c_r)$.

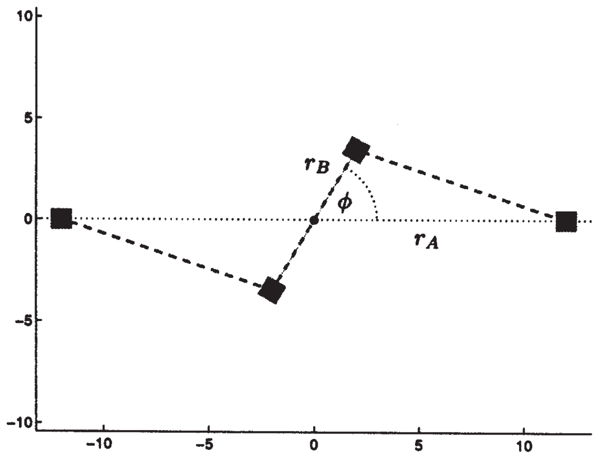

Fig. 18. An S-shaped part with four rigidly connected square "feet" in configuration $(x, y, \theta) = (0, 0, 0)$, $r_A = 12$, $r_B = 4$, and $\phi = 60°$.

60°, and the feet have area size 10. Notice that in poses with θ-angles corresponding to minima in the potential, the moment has a root with negative slope, which indicates a stable (orientation) equilibrium. Figures 20b and 20c show the (normalized) moments and potentials for parts with feet sizes 5 and 1, respectively. We observe that with decreasing contact areas, these functions become "less smooth," and the slope at the moment root increases. Figure 20d depicts moment and potential for a part with infinitesimally small feet. In this case, the moment function does not have a root at the minimum of the potential; rather, it exhibits a discontinuity at this orientation. This has the consequence that the part is not stable in this pose. In fact, for the moment function in Figure 20d, there exist no roots with negative slope, and hence there exists no stable equilibrium.

This observation can be made mathematically precise. The exact equations for the lifted potential and the moment of P_S are

$$U_{P_S} = 2r_A|\cos\theta| + 2r_B|\cos(\theta + \phi)|, \tag{17}$$
$$M_{P_S} = 2r_A S(\theta) + 2r_B S(\theta + \phi),$$

$$\text{with } S(\theta) = \begin{cases} \sin\theta & \text{if } 0 \le \theta < \pi/2 \text{ or } 3/2\pi < \theta < 2\pi, \\ -\sin\theta & \text{if } \pi/2 < \theta < 3/2\pi, \\ 0 & \text{if } \theta = \pi/2 \text{ or } \theta = 3/2\pi. \end{cases} \tag{18}$$

The potential minimum is reached at $\theta = \pi/2$ or $\theta = 3/2\pi$. However, we see that, for example, $M_{P_S}(\pi/2) = -2r_B S(\pi/2 + \phi) = -2r_B \cos\phi \neq 0$. Furthermore, $M_{P_S}(\pi/2-) > 0$, and $M(\pi/2+) < 0$. This implies that the part P_S will oscillate about $\theta = \pi/2$. Under first-order dynamics, this oscillation will be infinitesimally small, because any infinitesimal angular deflection of P_S results in a restoring moment with opposed orientation. Under second-order dynamics, the part may have a finite oscillation amplitude because of the inertia of the part. However, damping will reduce this amplitude over time.

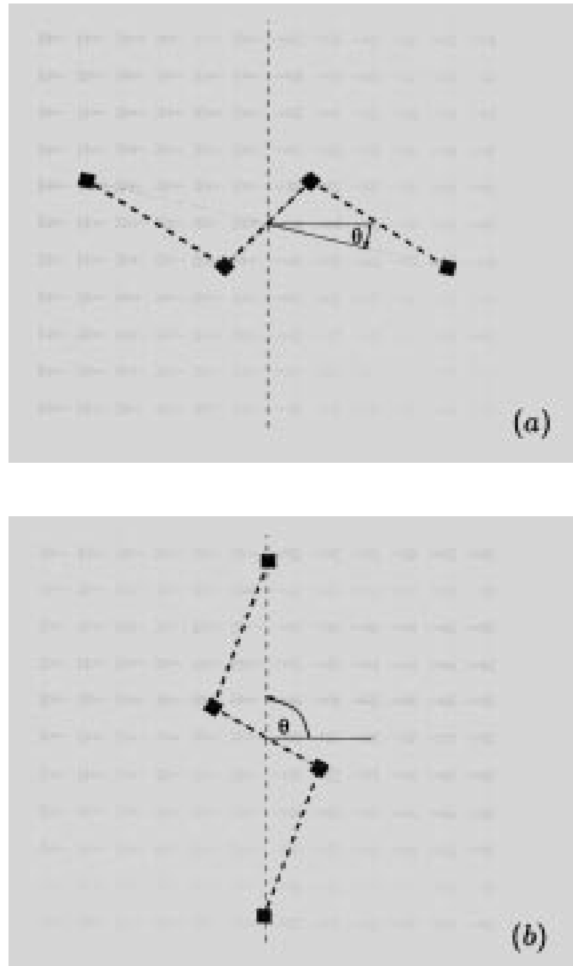

Fig. 19. Total equilibria of an S-shaped part with area contacts in a squeeze field. (a) Maximum potential, $\mathbf{z}_{\max} = (0, 0, \theta_{\max})$, such that $r_A \sin\theta_{\max} = -r_B \sin(\theta_{\max} + \phi)$; $\theta_{\max} \approx -0.24$. (b) Minimum potential, $\mathbf{z}_{\min} = (0, 0, \theta_{\min})$; $\theta_{\min} \approx \pi/2$.

We conclude that parts with point contacts can exhibit pathological behavior even in very simple and otherwise well-behaved potential fields: this example shows that for such parts, it is possible that the generalized force is not zero in a pose that minimizes the potential of the part.

This pathology cannot occur when only parts with finite area contact are allowed. From Corollary 3, we know that the (lifted) potential of a part with area contact is C^1; hence its gradient exists everywhere. In particular, the gradient is zero at the minimum of the potential. This means that in a pose with minimum potential, the generalized force must be zero. Let us summarize these results.

Table 1.

Task	Field(s)	Complexity: Fields	Planning	Plan Steps
Translate	Constant	Constant magnitude and direction	–	1
Center	Radial	Constant magnitude, continuous directions	–	1
	Orthogonal squeezes magnitude and direction	Piecewise constant	$O(1)$	$O(1)$
Uniquely orient	Sequence of squeezes	Piecewise constant magnitude and direction	$O(k^2n^4)$	$O(kn^2)$
	Inertial	Smooth magnitude piecewise-constant direction	$O(1)$	$O(1)$
Uniquely pose	Manipulation grammar	m arbitrary fields, at most E stable equations	$O(m^2 2^E)$	$O(m2^E)$ (not complete)
	Sequence of radial + squeeze	Piecewise-continuous magnitude and direction	$O(k^2n^2)$	$O(kn)$
	Elliptic	Smooth magnitude and direction	$O(1)$	$O(1)$
	UFO	Continuous magnitude and direction	––	1

- Magnitude control. Consider an array in which the *magnitude* of the actuator forces cannot be controlled. Does there exist an array with constant magnitude in which all parts reach one unique equilibrium? Or can one prove that, without magnitude control, the number of distinct equilibria is always greater than one?

- Geometric filters. This paper focuses mainly on sensorless manipulation strategies for *unique positioning* of parts. Another important application of programmable vector fields are *geometric filters*, which would be useful for the sorting and singulation of parts. Figure 1 shows a simple filter that separates smaller and larger parts. We are interested in the question, Given n parts, does there exist a vector field that will separate them into specific equivalence classes? For example, does there exist a field that moves small and large rectangles to the left, and triangles to the right? In particular, it would be interesting to know whether for any two different parts there exists a sequence of force fields that will separate them.

- Force-field computers. In this paper, we have demonstrated that even with a rather limited vocabulary of simple force fields, useful and quite complex tasks such as sensorless posing or sorting of parts can be performed. It might be possible that force fields could be used to solve certain classes of problems, by encoding them in particular force fields, part shapes, and initial and goal poses, resulting in a "force-field computer" that provides a physical implementation of the problem. Identifying the class of encodable problems might yield deeper insights into the complexity of parts manipulation with force-vector fields.

- Performance measures. Are there performance measures for how fast (in real time) an array will orient a part? In some sense, the actuators are fighting each other (as we have observed experimentally) when the part approaches equilibrium. For squeeze grasps, one measure of "efficiency," albeit crude, might be the integral of the magnitude of the moment function, i.e., $\int_0^{2\pi} |M(\theta)| d\theta$. The issue is that if, for many poses, $|M(\theta)|$ is very small, then the orientation process will be slow. Better measures are also desirable.

- Uncertainty. In practice, neither the force-vector field nor the part geometry will be exact, and both can only be characterized up to tolerances (Donald 1989). This is particularly important at the microscopic scale. Within the framework of potential fields, we can express this uncertainty by considering not one single potential function U_P, but rather *families of potentials* that correspond to different values within the uncertainty range. Bounds on part and force tolerances will correspond to limits on the variation within these function families. An investigation of these limits will allow us to obtain upper error bounds for manipulation tasks under which a specific strategy will still achieve its goal.

  A family of potential functions is a set $\{U_\alpha : \mathcal{C} \to \mathbb{R}\}_{\alpha \in J}$ where J is an index set. For example, we may start with a single potential function $U : \mathcal{C} \to \mathbb{R}$ and define a family of potential functions $\mathcal{F}(U, \epsilon, z)$ as $\{\{U_\alpha : \mathcal{C} \to \mathbb{R} | \ ||U_\alpha(p) - U(p)||_z < \epsilon\}$ for some ϵ and