

Bruce R. Donald

Computer Science Department
Cornell University
Ithaca, New York 14853

Dinesh K. Pai

Computer Science Department
University of British Columbia
Vancouver, British Columbia V6T 1Z2

The Motion of Planar, Compliantly Connected Rigid Bodies in Contact, With Applications to Automatic Fastening

Abstract

We consider the problem of planning and predicting the motion of a flexible object amid obstacles in the plane. We model the flexible object as a rigid "root" body attached to compliant members by torsional springs. The root's position may be controlled, but the compliant members move in response to forces from contact with the environment. Such a model encompasses several important and complicated mechanisms in mechanical design and automated assembly: snap-fasteners, latches, ratchet-and-pawl mechanisms, and escapements. The problem is to predict the motion of such a mechanism amid fixed obstacles. For example, our algorithm could be used to determine whether a snap-fastener design can be assembled with a certain plan.

In this article we analyze the physics of these flexible devices and develop combinatorially precise algorithms for predicting their movement under a motion plan. Our algorithms determine when and where the motion will terminate and also compute the time history of contacts and mating forces. In addition to providing the first known exact algorithm that addresses flexibility in motion planning, we also note that our approach to compliance permits an exact algorithm for predicting motions under rotational compliance, which was not possible in earlier work.

We discuss the following issues: the relevance of our approach to engineering (which we illustrate through examples we ran using our system), the computational methods employed, the algebraic techniques for predicting motions in contact with rotational compliance, and issues of robustness and stability of our geometric and algebraic algorithms. Our computational viewpoint lies in the interface between differential theories of mechanics and combinatorial collision detection algorithms. From this viewpoint, subtle mathematical difficulties arise in predicting motions under rotational compliance,

such as the forced nongenericity of the intersection problems encountered in configuration space. We discuss these problems and their solutions. Finally, we extend our work to predict the forces on the manipulated objects as a function of time and show how our algorithm can easily be extended to include uncertainty in control and initial conditions. With these extensions, we hope that our system could be used to analyze and design objects that are easy to assemble, even given control and sensing errors, and that require more force to disassemble than to mate.

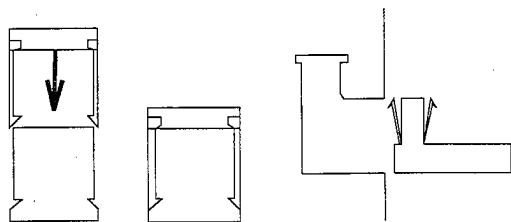
1. Introduction

We pursue an algorithmic theory of *design for assembly*. To this end we develop and implement algorithms that can analyze and generate designs for objects so that they will be easy to assemble. In particular, we observe that real objects that robots might assemble are typically not rigid. For example, a Sony Walkman is made of plastic parts that snap together. Significant advances were made in the design of the IBM ProPrinter by replacing traditional fasteners such as screws with plastic parts that simply snap together. The reason these plastic parts snap together is that they are *flexible*: more precisely, they are *passively compliant*. This means that when the parts are brought together and an external force is applied, the parts deform in a prescribed way. More interestingly, the force required to mate two parts may be much less than the force required to take them apart.

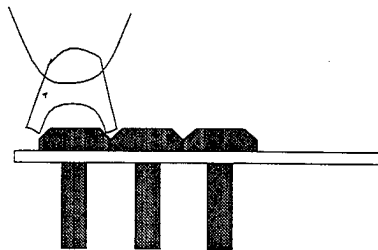
Because we wish to be able to design and have our robots assemble such objects given task-level descriptions, we must have a systematic program for reasoning about and predicting their motions in contact. To this end we make precise a sufficiently powerful notion of flexibility to model the objects above that encompasses several important and complicated mechanisms in mechanical design and automated assembly: snap-fasteners, latches, ratchet-and-pawl mechanisms, and escapements (Fig. 1). We model the physics of interaction of the flexible parts

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Snap fasteners



Escapements

Fig. 1. Examples of compliantly connected rigid bodies.

and the obstacles using generalized springs, quasistatic analysis, and Coulomb friction. Using these tools, we can proceed to develop combinatorially precise geometric algorithms for predicting the motion of a flexible object near and in contact with its mating part. A system based on the results of this article is also described, as is the relevance to the design of such mechanisms.

From a complexity-theoretic standpoint, our result may also be viewed as follows. Earlier work on compliant motion planning (Lozano-Pérez et al. 1984; Erdmann 1984; Donald 1988a) has employed a dynamic model called the *generalized damper* (Whitney 1976). While this model has led to exact algorithms in the case of pure translations (Donald 1988a; Canny 1989), so far we have not been able to provide combinatorially precise algorithms once rotations are permitted. This is true even if the commanded motion is a pure translation and the moving object (e.g., the peg) is merely rotationally compliant. One reason for this difficulty seems to be that under the damper model, the resulting motions are not obviously algebraic; even more disturbing, the outer envelope (or *forward projection*) of these motions cannot yet be algebraically described. (This is also a problem in many more complicated dynamical systems.) However, under the model in this article, rotationally compliant subparts are permitted, and yet all resultant trajectories of the system are algebraic. Finally, we believe that our algorithm is sufficiently simple that it could be implemented and used to plan and verify the design of flexible mechanisms for ease of assembly.

We have implemented our algorithm and in this article describe the details of the algorithm, implementation, and experiments. We have built a system for predicting and analyzing the motion of snap-fastener type devices and we describe experiments we have run to analyze and design particular objects.

A major impediment to developing systems such as ours has been the apparent necessity to integrate the differential mechanics to determine the long-term behavior of the system. This problem is exacerbated by the fact that in many models of rotational compliance, such as the generalized damper (Lozano-Pérez et al. 1984; Erdmann 1986; Donald 1988a,b; Canny 1989), the resulting trajectories are not known to be algebraic; neither do we have ways of computing algebraic bounding approximations (or forward projections). We begin by discussing in quite general terms how our model of compliance permits us to obtain algebraic, closed-form solutions to motion prediction problems for a rotationally compliant object and how this leads to *exact* algorithms for analyzing designs for assembly.

We continue by discussing the relevance of our approach to engineering. To this end, we give several examples of fastener type objects. We illustrate our discussion with experiments performed with our system to automatically analyze and predict the motion during execution of the assembly plan.

Our algorithm is algebraic and exact in principle. However, unlike many theoretical algebraic algorithms, it is also implementable. A chief goal of this article is to show how we implemented it. Next, we discuss our algorithm for motion analysis in detail. Our work is interdisciplinary in that it is situated at the interface between differential theories of mechanics and combinatorially precise computational approaches to collision detection and compliant motion prediction. We employ some simple tools from computational algebra and discuss their application in our algorithm in some detail. Although these tools often seem straightforward from a theoretical viewpoint, there is a host of practical and implementational problems in trying to build a system and reduce them to practice. Many of these issues focus on the problems of robustness, nongenericity, and stability. We discuss these problems in some detail. For example, while many algebraic and computational-geometric algorithms can assume genericity (e.g., general position), we can show that in the case of predicting rotational compliance, one is, in effect, forced to solve nongeneric intersection problems in configuration space. Careful thought is required to make such algorithms robust.

Our algorithm predicts where the motion will terminate and the configuration and contact history as functions of time. We also show how our algorithm can predict the forces experienced by the manipulated parts as a func-

tion of time. Thus, we can avoid designs that require excessive forces to assemble and can analyze how a design can be easier to mate than to disassemble. This force-history extension requires the introduction of transcendental functions and so results in a numerical (not an exact) algorithm; however, any desired precision for the forces may be obtained.¹

Previous work on algorithmic motion planning has largely concentrated on the movement problem for rigid bodies (Lozano-Pérez 1983; Donald 1988b; Latombe 1991). Work in compliant motion planning under uncertainty (Lozano-Pérez et al. 1984; Erdmann 1986; Canny and Reif 1989; Donald 1988a,b; Canny 1989; Briggs 1989; Friedman et al. 1989; Brost 1989, 1991) has focused on the problem of moving rigid objects (e.g., pegs) in contact (i.e., compliantly) with obstacles (such as holes) under force control, subject to bounded uncertainty and error. The only combinatorially precise results that have been obtained for compliant motion have been for pure translations. Hence they are essentially inapplicable for any real systems, which typically can rotate. Indeed, planning and simulation for systems with rotational bodies and rotational compliance have resisted solution in the sense that only approximate, heuristic, or numerical algorithms are known. We consider a mechanical system of bodies that can compliantly slide on obstacles while they translate and rotate in the plane. We provide an exact, efficient algorithm for predicting the motion of these rather interesting devices, which, as we discuss later, are important in manufacturing, design, and factory assembly. The key ideas we use are red-blue merge algorithms, a simple dynamical systems model, and local dynamic constraints. These tools permit us to reduce the simulation to a plane sweep of a “dynamically annotated” slice of configuration space. We hope that the paradigm of “simulation as sweep” may be useful in other domains.

We view the simulation problem, even with rotational compliance and quasistatic mechanics, as a problem that can be solved by careful reduction to a plane sweep. In particular, for each pawl, we reduce to sweeping a planar arrangement of algebraic curves of low degree. The connected component of free space defined in this planar arrangement has complexity $O(\lambda_r(n))$ and can be constructed in time $O(\lambda_r(n) \log^2 n)$ using a red-blue merge algorithm (Guibas, et al. 1988). Here $\lambda_r(n)$ is the (almost linear) maximum length of (n, r) -Davenport-Schinzle sequences (Guibas, et al. 1988). r is a small constant related to the number of times two cubic² configuration

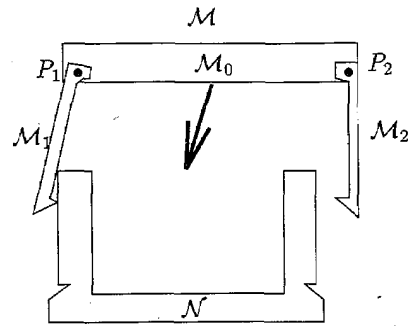


Fig. 2. Linked body \mathcal{M} moving among \mathcal{N} .

space constraint curves can intersect. Our approach to modeling rotational compliance and incorporating frictional constraints leads to the first formulation of the simulation prediction problem that permits a reduction of motion prediction to plane sweep. Our solution differs from previous work on predicting, bounding, and planning rotationally compliant motions with quasistatic mechanics in that it is (1) *purely algebraic*, and hence exact; (2) *combinatorially precise*, in that the computational complexity is exactly known, and (3) requires no integration.³

2. Problem Statement

We consider the problem of moving a flexible, linked body \mathcal{M} in the plane in a polygonal environment \mathcal{N} . The flexible body \mathcal{M} is modeled as follows: polygons \mathcal{M}_h , $h = 1, \dots, k$, called “pawls,” are attached to a root polygon \mathcal{M}_0 at hinge points P_h . Each hinge is coupled with a spring of stiffness κ_h . (Fig. 2). We refer to the static environment, \mathcal{N} , as “obstacles,” although in fact, it will probably consist of fixtures and/or the mating half of the manipulated part.

The motion of the body \mathcal{M} consists of rigid translation of the root polygon \mathcal{M}_0 . The pawls \mathcal{M}_h are free to move compliantly as dictated by interactions with the environment and the spring.

Time T will be represented by the nonnegative real numbers. A *solution trajectory* Φ for the system $(\mathcal{M}, \mathcal{N})$ is a family of maps $(\phi_0, \phi_1, \dots, \phi_k)$ where $\phi_0 : T \rightarrow \mathbb{R}^2$ specifies the configuration of the root and each $\phi_h : T \rightarrow S^1$ specifies the orientation of pawl polygon \mathcal{M}_h , $h = 1, \dots, k$. Hence the global configuration of pawl \mathcal{M}_h at time t is given by $(\mathbf{p}(t), \phi_h(t))$, where $\mathbf{p}(t)$ differs by a constant offset from $\phi_0(t)$. We see that the configuration

1. By substituting a polynomial approximation for the transcendental functions, the algorithm could be made an ϵ -approximation scheme.

2. By “cubic” we mean the total degree of the defining multinomial is 3. Our curves have additional structure, such as low-degree parameterizations, as well.

3. Note that Donald (1988a), Canny (1989), Briggs (1989), and Friedman et al. (1989) address geometric reachability issues for translationally compliant objects, but these objects cannot rotate.

space of the system $(\mathcal{M}, \mathcal{N})$ is⁴ $C = \mathbb{R}^2 \times \overbrace{S^1 \times \dots \times S^1}^{k \text{ } S^1\text{'s}}$.

DEFINITION 2.1. The *Simulation Problem* is to determine:

1. The solution trajectory $\Phi : T \rightarrow C$ of the system.
2. The time of termination of the motion, the cause of termination (such as sticking caused by friction, sticking caused by kinematic constraints, etc.), and the configuration of \mathcal{M} at termination.
3. The time history of contacts between \mathcal{M} and \mathcal{N} .

We also address some extensions of the simulation problem in Appendices A and B, including the determination of the time history of forces during the motion and the effect of uncertainty.

We make the following assumptions about the physics of object interactions and the motion of \mathcal{M} :

- Object interactions are restricted to those between \mathcal{M}_h and \mathcal{N} . In other words, pawls do not collide with each other but may collide with the environment \mathcal{N} . The effect of this assumption is to make the motion of each pawl independent of the motion of other pawls. Henceforth we shall consider the motion of a single pawl \mathcal{M}_h .
- Because the root \mathcal{M}_0 is undergoing a rigid translation, so does the hinge point P_h . We shall assume that this translation is a straight-line motion given by

$$\mathbf{p}(t) = \mathbf{p}_0 + \dot{\mathbf{p}}t, \quad (1)$$

where t is the time, \mathbf{p}_0 is the initial position (at $t = 0$) of P_h , and $\dot{\mathbf{p}}$ ($= \dot{\phi}_0$) is its velocity.

- Stable contact: Suppose the pawl is in contact with a feature (edge or vertex) of the environment (for example, during sliding). We assume that if we perform a small displacement of the pawl away from the environment, the torque on the pawl caused by the spring is such that the contact will be restored. This assumption is not very restrictive at all; in fact, in the face of even the smallest uncertainty, stable contacts are the only ones one can hope to observe in practice.
- Quasistatic motion: The motion is assumed to be slow enough that inertial effects are not significant. This corresponds to assuming that there is no acceleration of the pawl, and hence the forces on the pawl are balanced. The quasistatic assumption is

4. This is the configuration space with no springs, only kinematic constraints. The introduction of springs means that it is not possible to identify a rotation of 2π with 0, since at 2π the pawl is "cocked." Hence the introduction of dynamics forces us to pass to the covering space \mathbb{R}^{k+2} . Our analysis goes through *mutatis mutandis* for the covering space.

reasonable at small speeds and is widely used (see, for example, Whitney 1982; Mason 1982; Donald 1988b; Pai 1988).

- If the pawl slides off \mathcal{N} into free space, it may have a residual torque as a result of the spring being cocked. We shall assume that the pawl rotates back toward its rest orientation at such great speed that \mathbf{p} does not change significantly during the rotation and can be taken to be constant. This, incidentally, is the "snap" in the snap-fasteners that we wish to model. This assumption may appear to contradict the assumption of quasistatic motion, but is in fact practically a consequence. Quasistatic motion implies that the root is moving "slowly enough" for the forces to be balanced. Hence, when a pawl is in free space and has a residual torque, its motion can be fast compared with that of the root, resulting in a "snap." This assumption can be relaxed by assuming a linear relationship between the translation \mathbf{p} and the rotation (Canny 1986), but we do not deal with it here.
- The forces of friction arising from contact obey Coulomb's Law. We further assume that there is a single coefficient of friction. This assumption is also widely used.

These assumptions define a simple but adequate dynamical system. We will exploit the geometry of this system extensively to obtain our results.

3. Examples

3.1. The Two-Pawl Example

Figure 3 shows a flexible device that is to move and snap around a mating part (a black, T-shaped object). Now, for a real device, the pawls will be tiny in comparison to the root body and mating parts, but we have made them very large here so as to illustrate the geometric interaction. In this example, the coefficient of friction is very small. This enables the reverse motion to "pull the pawl back up" out of its mating part.

The "assembly plan" for the parts consists of a straight-line translation of the root body; this motion is parameterized by time and is specified by an initial configuration and a velocity vector. (Throughout this article, we will use the term "assembly/disassembly plan" to mean such a straight-line translation.) As the pawls contact the environment, they can move in a prescribed way from the contact forces. In particular, each pawl is attached to the root by a torsional spring. The pawls can also "snap" off the obstacle edge back to their zero position; this motion is modeled using a pure rotation. As the motion proceeds, the pawls deform (deflect) around the obstacles

in response to the kinematic constraints and the contact forces (Fig. 4).

When a root collision is detected, the system attempts to reverse the assembly plan and pull straight up. Because there is no friction, the pawl can be removed. If the coefficient of friction is increased, then the left pawl will stick on left of the T environment, and the flexible object cannot be pulled back up. This example shows that although it is relatively clear intuitively that during assembly the pawls may snap around the black T-shaped object, even this simple problem holds some surprises. First, the right-hand pawl cannot actually reach around the base of the T before the root collision occurs. Second, it is not a priori clear that the left pawl will not stick during the reverse (disassembly) plan. In fact, it will not stick on the left of the T merely because of kinematic constraints; friction is required.

Figure 4 was generated by our algorithm. Kinematic constraints are modeled using configuration space C surfaces, as in Lozano-Pérez (1983), Donald (1987), and Brost (1991). Reaction forces are modeled using Coulomb friction. The dynamics of the system are assumed to be quasistatic (Whitney 1982; Mason 1986); we regard this

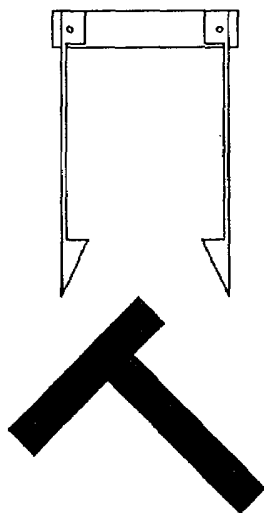


Fig. 3. The root body is rectangular, and the motion plan is straight down. The two pawls are hook-shaped and are attached with torsional springs at the hinges to the root body. The environment ("obstacles") are the two filled-in rectangles meeting at right angles; they form a T-shaped black body. In this example, it seems that the pawls will clearly comply to the T and snap around it; presumably this device cannot be disassembled by reversing the assembly plan. In fact, our algorithm shows that (1) the right-hand pawl cannot reach the base of the T, and (2) with low friction, there is no obstacle to disassembling the device.

assumption as a 0th-order approximation of the real dynamical system.

3.2. "Motion Diode" Example

In design for assembly, we often desire "locking" parts that, when mated, cannot be disassembled by motion plans in a particular family of directions. More generally, we may require interlocking parts that cannot be disassembled at all for any translational motion plan. Most generally, one might want parts that cannot be disassembled without exerting large forces (see Appendix A). Following a suggestion of Mason (personal communication, 1984), we call such objects "motion diodes." The term is motivated by the fact that motion is possible in certain directions, but not in others. Our usage differs from Mason's in that his motion diodes are geometries from which a robot cannot be *guaranteed* to emerge. Our motion diodes are (flexible object, environment) pairs such that for some family of controls (or perhaps all controls), no change in the sign of the controls can reverse the motion to reach the start position. In our simple case a "plan" is a translation given as a straight-line motion

$$\mathbf{p} = \mathbf{p}_0 + \dot{\mathbf{p}}t, \quad (2)$$

where t is the time, \mathbf{p}_0 is the initial position (at $t = 0$), and $\dot{\mathbf{p}}$ is the root velocity. A family of controls corresponds to a set of velocities $\{\dot{\mathbf{p}}\}$, and changing the sign amounts to specifying $-\dot{\mathbf{p}}$.

If motion diodes can be designed, analyzed, and verified, then they can be rigidly attached as "fasteners" to bodies that we wish to mate, but not disassemble. For example, if the triangular pawl in Figure 5 is attached in the z -axis (perpendicular to the figure) to a root body in a parallel x - y plane to the figure, then the root body can be fastened irreversibly to its mating part.

Our algorithm can analyze designs for these kinds of diodes. In Figure 4, the two-pawl device and the T-shaped object would form a motion diode with respect to pure translation in y , for sufficiently high coefficients of friction.

Now see Figure 5. In this example, there is no root body. The one triangular pawl can pivot about its center; a torsional spring is attached at the pivot. The pawl is moved down in a pure $-y$ translation, and in response to the reaction forces from the environment, it rotates compliantly. Let us label the black obstacles, starting with the uppermost one, in clockwise order: A, B, and C. The pawl contacts A and rotates counterclockwise while sliding along A's upper left corner. Eventually, the pawl breaks contact with A and snaps off, only to hit the rightmost vertex of C. It briefly slides (while rotating

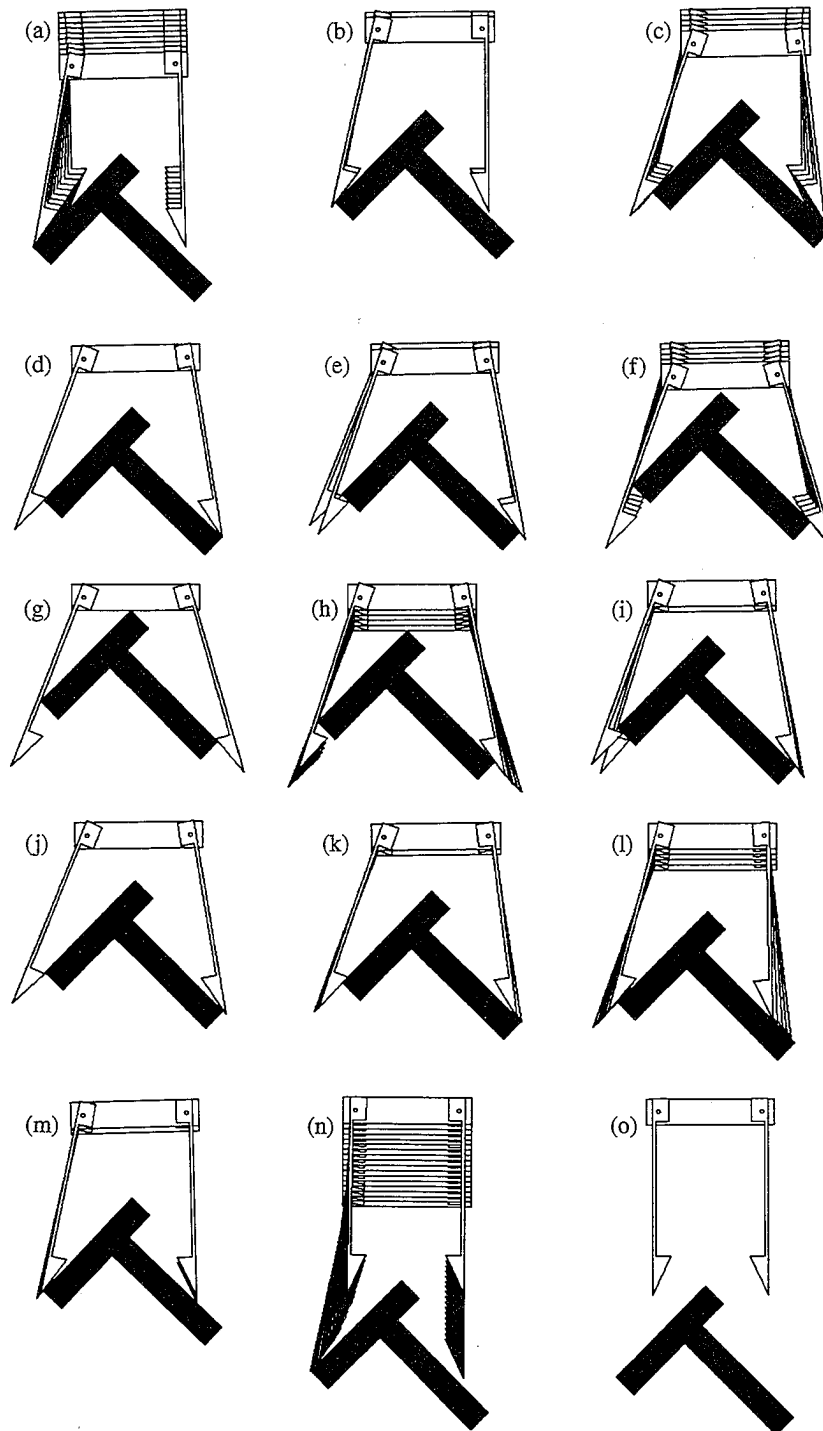


Fig. 4. The result of running our algorithm on the two-pawl example. Notice how the pawls comply around the obstacles as they deform in response to the kinematic constraints and the contact forces. Figs. A–G show the motion downward. At G, the root collides with the T, and the motion is reversed. Figs. H–O show the reverse motion upward.

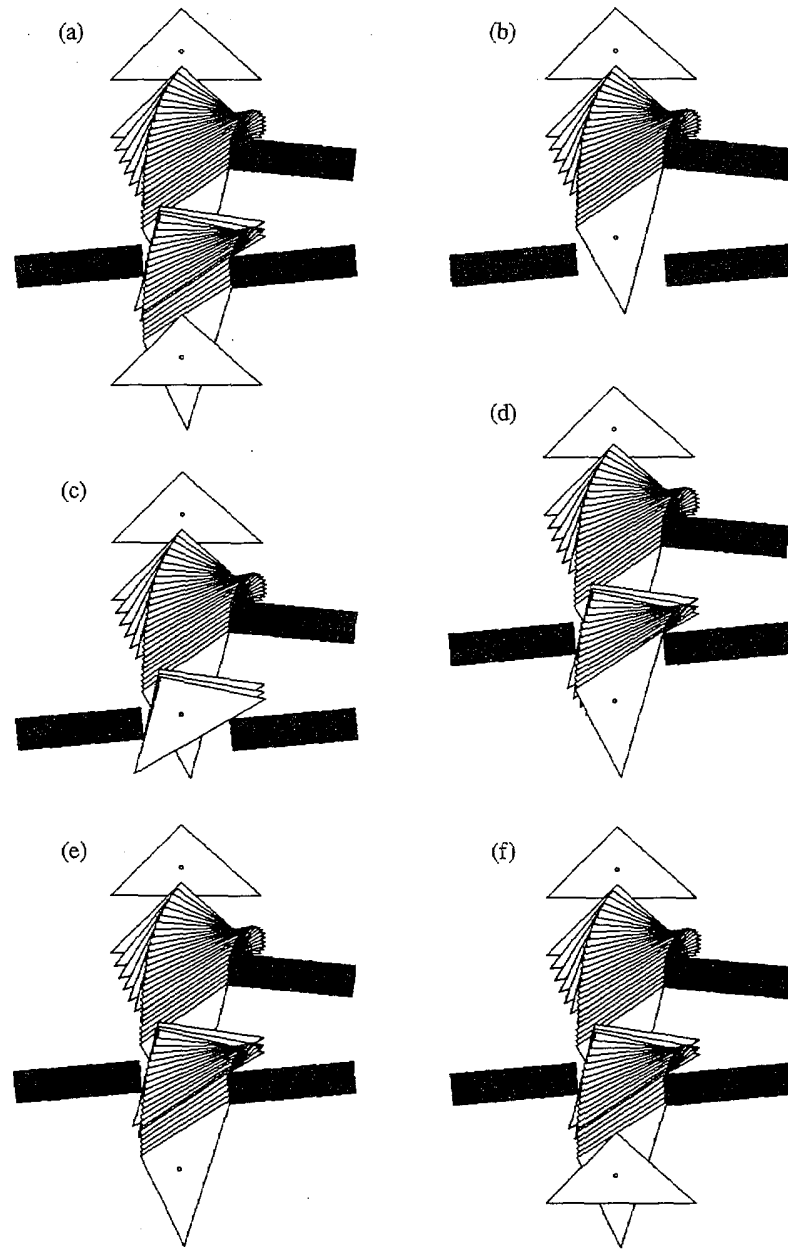


Fig. 5. "Motion diode" example. There is no root body and only one triangular pawl that can pivot about a torsional spring at its center. Pure translations of the diode can be commanded, and the triangular pawl deflects in response to contact forces from the environment. These figures were generated by running our analysis algorithm on the data shown. In C, the pawl snaps off the upper right obstacle and continues downward. Fig. 6 shows the reverse motion, during which the pawl gets stuck coming back up.

compliantly) along C until it hits B . The tighter constraint from B takes over, and the pawl is again “cocked” counterclockwise until it breaks contact at the lower left vertex of B . Finally, the pawl snaps off B to its rest position.

This mechanical system is a “diode” with respect to pure y -translation (see Figures 5 and 6)—i.e., when the pawl is moved back up in the $+y$ direction, it jams as a result of incompatible kinematic constraints. More interestingly, if B and C are extended to the right and left, respectively, the system is a diode with respect to *all* translational motions. That is, no commanded translation can bring the flexible body back out of the hole between B and C . Our algorithm can decide that for a particular motion plan, a system is a diode. There also exists a theoretical extension of our algorithm, using the theory of real closed fields, that can decide whether the system is a diode with respect to *every* disassembly plan, but this algorithm is not practical. In practice, applying the algorithm to a discretization of the control set would be more practical, although not exact.

4. Overview of Results

4.1. Differential Theories of Mechanics: An Algebraic Approach

We view the motion prediction problem, even with rotational compliance and quasistatic mechanics, as a problem that can be solved by careful reduction to intersection (or

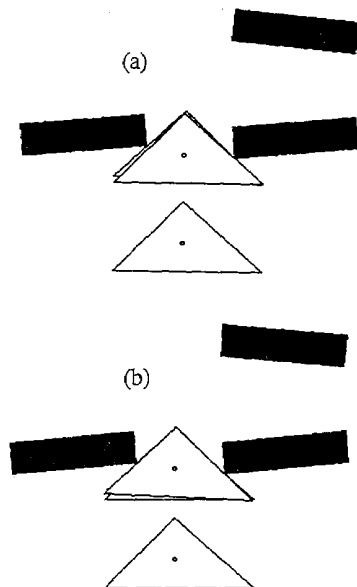


Fig. 6. “Motion diode” example. When we try to pull the flexible object back up (move in direction $+y$), it gets stuck because of kinematic constraints.

collision detection) problems (Canny 1986; Donald 1984, 1987). Our approach to modeling rotational compliance and to incorporating frictional constraints leads to the first formulation of the motion prediction problem that permits a *reduction* of motion prediction to collision detection. Our solution differs from previous work on predicting, bounding, and planning rotationally compliant motions with quasistatic mechanics in that it is (1) *purely algebraic*, and hence exact; (2) *combinatorially precise*, in that the computational complexity is exactly known; (3) practical and implementable; and (4) requires no integration. In this subsection we elaborate somewhat on these characteristics.

To predict the motion of objects in a mechanical system, we require a computational theory of mechanics. A *differential* theory of mechanics takes the instantaneous state and forces and computes the resultant instantaneous motion.

Quasistatic analysis is another differential theory of mechanics, in that it predicts the instantaneous motion given state and friction forces.⁵ In general, differential theories can be integrated, in the sense that the long-term behavior can be integrated out, given the differential mechanics. We say that a theory is *closed-form integrable* when we can solve for the resultant curves in state-space in closed form. When a theory is closed-form integrable, we can develop exact, combinatorially precise algorithms for predicting and planning the motion of objects (Donald 1988a,b; Canny 1989; Briggs 1989). When a closed-form solution is not known, numerical methods can be used to integrate out solutions in some cases.

Quasistatic mechanics and generalized damper dynamics are indeed closed-form theories in the case of pure translation. However, once rotational compliance—the tendency of a manipulated object to change its orientation in response to reaction forces from the environment—is introduced, we do not have closed-form integrable theories of mechanics. We do have precise, algorithmic theories of the differential mechanics, such as Erdmann’s generalized friction cone (Erdmann 1984, 1991) and the acceleration center force–dual model of Brost and Mason (1989), but these theories must be numerically integrated to obtain solutions. The best algorithms for this process are not combinatorially precise and do not have solution accuracy bounds. This problem is exacerbated by uncertainty in sensing and control; in this case we are left with the very different problem of trying to integrate out a stochastic vector field on a holonomic constraint, subject to a family of initial conditions.

In contrast, we use a well-known theory of mechanics with rotational compliance that is not only closed-form

5. In the case of ambiguity or nondeterminism, we view such a theory as a relation.

integrable, but closed-form *algebraic*. By this we mean the following: Our theory is based on the differential mechanics, but can be integrated to produce solutions that are algebraic curves in the configuration space. In fact, the solutions paths are parameterized by time and are piecewise quadratic or linear. Because we can represent configuration space C surfaces as quadric surfaces, all of the resulting intersection problems (Section 4.2) are no harder than intersecting a quadratic path with a quadric surface. Somewhat surprisingly, even the functions that determine at what times sliding and sticking will occur on a surface are also quadratic in the time parameter.

Closed-form algebraic solutions permit us to construct exact, combinatorially precise algorithms for the motion prediction problem. Furthermore, they allow us to straightforwardly generalize our techniques to encompass certain simple types of uncertainty in control and initial conditions.

In principle, our algorithms can be implemented using exact-precision, algebraic numbers. In practice, we use finite-precision approximation techniques. Robustness is a key issue, and algorithms that are theoretically correct with exact precision are often numerically unstable. A major component of our research consists of building a system that can strengthen the theoretical algorithms (e.g., by adding consistency checks) to make them practical.

4.2. Computing Motions and Intersection Problems

We now provide a somewhat informal view of our results. Here are the qualitative states of the root body: it is either undergoing a pure translation or is stuck as a result of incompatible kinematic constraints. It is clear that for the root body, all we need to compute is the time at which this sticking occurs. This time is an upper bound on the simulation.

The state of a pawl is more complicated. When a pawl is in free space undergoing a pure translation, it can strike a surface. This requires a translational collision detection algorithm. When the pawl is in contact with a surface, as time increases, its configuration (orientation in this case) must change so as to comply with this constraint. (We use the term “constraint” as in Lozano-Pérez [1983] and Donald [1987] to refer to the kinematic constraint that (1) an edge of the pawl touch a vertex of the environment or (2) a vertex of the pawl touch an edge of the environment. These constraints form type “(A)” and “(B)” configuration space surfaces, respectively.) See, for example, Figure 4. In this case, the configuration space of the flexible object is $\mathbb{R}^2 \times S^1 \times S^1$; a point in \mathbb{R}^2 specifies the configuration of the root body, whereas each angle in S^1 specifies the orientation of the right and left pawls. In this case, we require an algorithm that can return the mapping

from time to configuration and the curve in configuration space that is the image of this mapping.

Now, as the pawl traces out this curve, three things can happen. First, the pawl may break contact with the surface constraint because of incompatible kinematics. Second, the pawl may stick on the surface, due to force-balance from the reaction forces. (More generally, for every constraint, there may be at most two disjoint time intervals during which sliding (or sticking) occurs, separated by an interval during which sticking (or sliding) occurs.) Third, the pawl may strike another constraint. In Sections 6, 7, and 8, we perform the kinematic and physical analysis required to compute the times at which these events occur, and we describe the algorithms. In particular, we show how, given a constraint, to compute a quadratic function of time whose zeros define the end points of these intervals of sliding and sticking behavior.

Finally, when a pawl breaks contact and “snaps off,” we must perform a pure rotational intersection test to determine where it snaps to. When a pawl lies on the intersection of two constraints, we must determine whether sticking occurs there because of incompatible constraints, or which constraint takes precedence and becomes a new constraint at that time.

4.3. Exact Solutions for Mechanical Simulations

A major impediment to developing simulation systems has been the apparent necessity to integrate out the differential mechanics to determine the long-term behavior of the system. This problem is exacerbated by the fact that in many models of rotational compliance, such as the generalized damper (Lozano-Pérez et al. 1984; Erdmann 1986; Donald 1988a,b; Canny 1989), the resulting trajectories are not known to be algebraic; neither do we have ways of computing algebraic bounding approximations (or forward projections). Hence the traditional numerical approach to simulating such systems has been the following:

Typical Simulation Algorithm

1. Given a state x of the system, numerically integrate the differential equation governing motion of the system. Step forward in time to obtain (approximately) new state x' .
2. Perform collision detection either at x' or along the path from x to x' .
3. If the constraints have changed, reformulate the differential equation.
4. Repeat.

Numerical simulation of mechanical systems is fraught with error, special cases, and numerical problems. They are rarely combinatorially precise and almost never come

with guarantees of accuracy. In this article, we show how our simple model of compliance permits us to obtain algebraic, closed-form solutions to simulation problems for a rotationally compliant object and how this leads to *exact* algorithms for analyzing designs for assembly. In particular, the linear map $\mathbf{p}(t)$ is given by eq. (1), and once we “rationalize” rotations via the standard substitution $u = \tan \theta/2$, each map ϕ_h is piecewise cubic. Thus numerical simulation can, in principle, be avoided, and exact solutions can be obtained. We cannot claim that this can be done in general. However, our method yields, in this case, computationally efficient, exact solutions and may possibly be useful in other domains.

In Section 9, we give a new simulation algorithm that is exact and runs in time $O(k\lambda_r(n)\log^2 n)$. Here, we assume that the obstacles have m_b vertices and the moving object (root and one pawl) has m_a vertices; we define $n = m_a m_b$ to be the measure of the geometric complexity. $\lambda_r(n)$ is the (almost linear) maximum length of (n, r) Davenport-Schinzel sequences (Guibas et al. 1988), and r is a small constant. Our method reduces the simulation problem to a plane sweep of an arrangement of algebraic curves in configuration space.

4.4. A Classification of Motion Diodes

Consider Figure 7, which is a *gedankenexperiment* illustrating different types of diodes. The flexible mechanism, and in particular the pawl structure, is very similar to Figure 3. The commanded motion is in the $-y$ direction. Now, for geometry (b), the pawl deflects clockwise and snaps onto the sharp bump at the lower left of (b). The motion is not reversible, caused by kinematic constraints alone. Now, unless constraint (c) is present, there may exist other translations that will disassemble the mechanism. The addition of constraint (c) means that no translation can disassemble the mechanism (so long as (b) is “tall enough” (Fig. 7)). Now consider Figure 7D. This geometry also prevents $+y$ motion, but the sticking is caused by friction, not kinematics. Finally, consider Figure 7E. Here, relatively small forces are required on the root in the $-y$ direction to assemble the object; however, large forces are required on the root in the $+y$ direction to disassemble the object. (E) is an object that it is easier to assemble than it is to disassemble.

One application of our algorithm has been to determine whether a system is a motion diode and what kind of diode it is. Flexible mechanisms that function as motion diodes can be used to fasten one part to another and to make the mechanical connection robust with respect to attempted relative motion of the two parts. This kind of analysis could be very useful in design for assembly. Using our algorithm, we can form the following *classification* of flexible mechanism motion diodes. The

classification is “two dimensional” in the sense that one “axis” is relative vs. total motion, and the other axis is kinematic vs. friction vs. force diodes.

More specifically, the first classification is:

1. A *relative motion diode* is a (flexible object, environment) pair such that for some family of controls, called the *relative motions*, no change in the sign of the controls can reverse the motion to reach the start position.
2. A *total motion diode* is a relative motion diode for all relative motions.

The second classification is:

- A. A *kinematic diode* prevents reverse motion caused by purely kinematic constraints.

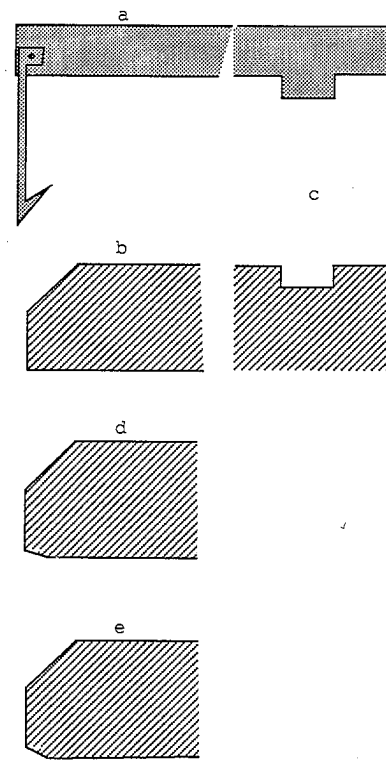


Fig. 7. In (A), we see a pawl-type mechanism affixed to a root body that is translating down in the $-y$ direction. The pawl is very similar to Fig. 3. (B) forms a relative motion kinematic diode. Without constraint (C), there exist other translations that disassemble the “snapped onto” final configuration. If we make the distance top long enough so that the root collides with the top of B, then the addition of constraint (C) turns (B) into a total-motion kinematic diode. Now, (D) forms a friction diode, and (E) forms a force diode. The presence or absence of (C) determines whether (D) and (E) are total or relative motion diodes.

- B. A *friction diode* prevents reverse motion caused by a combination of kinematic constraints and coulomb friction. It depends on the coefficient of friction.
- C. A *force diode* is described in Appendix A. Essentially, one imagines replacing the control of a root body by generalized spring position control, thus equating displacements of and forces on the root body in a first-order relation. This permits one to ask: What forces are experienced by the pawls as a function of time? and What force is exerted on the root to cause the motion? A force diode, roughly speaking, is a friction diode for all control forces below a certain modulus bound.

Hence, we see that without friction, Figures 3 and 4 are not a diode. With sufficiently high friction, it is a relative motion friction diode. Figure 5 is a total motion kinematic diode. Our algorithm can decide all these classifications automatically.

In Figure 7, we see that (B) is a kinematic diode, (D) is a friction diode, and (E) is a force diode. If we add constraint 7C and make (B), (D), and (E) “tall enough,” then these are total motion diodes; otherwise, they are relative.

One goal of our algorithm has been to provide an algorithmic means of classifying flexible objects such as snap-fasteners by diode type. Obviously, this is just a start, and other, finer types of classifications are possible. Ours seems useful in design and is efficiently computable by implemented, precise algorithms.

5. Simulation and Algebraic Intersection Problems

From an algorithmic point of view, our work can be thought of as reducing the flexible object motion prediction problem to intersection problems (like collision detection) in C space. There are, of course, additional complications, such as the presence of friction, that need to be addressed. In this section, we show how to reduce geometric intersection problems to problems in elementary elimination theory (particularly, the simultaneous solution of polynomial equations and inequalities).

As the linked body \mathcal{M} is moved in the environment \mathcal{N} (see Figure 2), the pawl \mathcal{M}_h may collide with an obstacle. We need robust algorithms to detect and report the collision. While a pawl is sliding on a feature of the environment, it may arrive at configuration at which it is kinematically impossible to continue to maintain contact—the pawl will either snap off the environment or jam on it. We need a way to determine when this occurs. These seemingly unrelated problems are in fact reducible to intersection problems in configuration space.

Three types of motion are possible for the pawl in our problem: pure translation without rotation; pure rotation without translation; and, most importantly, the motion of the pawl sliding while maintaining contact with a given feature of the obstacle. The collision detection problem is to find the times at which the pawl collides with \mathcal{N} and report which features are in contact during collision.

We are able to perform this analysis efficiently by developing a simple dynamical systems model using local dynamic constraints. These tools permit us to reduce the simulation to a plane sweep of a “dynamically annotated” slice of configuration space.

More specifically, two types of contact are possible between the pawl \mathcal{M}_h and a polygon in \mathcal{N} . Following the convention of Lozano-Pérez (1983), we say that type A contact occurs when a vertex of \mathcal{N} touches an edge of the pawl; type B contact occurs when a vertex of the pawl touches an edge of \mathcal{N} .

We can now write the contact constraint equations for the two types of contact, as in Canny (1986). We will index features (vertices and edges) of the moving pawl by the subscript i and features of the polygonal environment by the subscript j . Let an edge of \mathcal{M}_h be represented by its outward normal, \mathbf{n}_i , and its distance to the hinge point along the normal, d_i . Let \mathbf{p}_j be a vector to the contact vertex of \mathcal{N} . Let R_θ be the linear transformation that rotates a vector by an angle θ . Then the type A constraint can be written as

$$(\mathbf{p}_j - \mathbf{p}) \cdot R_\theta \mathbf{n}_i - d_i = 0. \quad (3)$$

Similarly, the type B constraint can be written as

$$(R_\theta \mathbf{p}_i + \mathbf{p}) \cdot \mathbf{n}_j - d_j = 0. \quad (4)$$

Where \mathbf{p}_i is the vector from the hinge point P_h to the contact vertex of \mathcal{M}_h .

In each of the cases, the obstacle constraint equations ((3) and (4)) can be reduced to algebraic equations. This is possible because the trigonometric functions in the rotation matrix R_θ can be replaced by new indeterminate quantities (e.g., $\sin \theta = s$, $\cos \theta = c$), with the addition of a new constraint ($s^2 + c^2 - 1 = 0$). More practically, the substitution $u = \tan \theta/2$ yields constraint equations that are quadratic in u , with coefficients that are affine in x and y . Because x and y are affinely parametric in time t , the coefficients are also affine in t . Intersecting two of these constraints requires intersecting two quadratics. Pure rotational intersection detection (the “snap”) requires solving a quadratic in u . Pure translational intersections require solving an affine equation. Hence there exists a closed-form, purely algebraic solution to these intersection problems. The collision detection for each obstacle constraint can be done in constant time, because the degree, size, and number of variables in the constraint

polynomials are fixed. If m_a is the geometric complexity of the pawl and m_b is the geometric complexity of the size of the environment, there are $m_a m_b$ such constraints.

Because the constraint equations are quadratic in u , it is possible that in some interval of time the discriminant Δ of the constraint equation (treated as a quadratic in u) becomes negative, yielding no solution for u . This corresponds to the pawl either having snapped off the constraint or having jammed on it. By solving for the critical times t_z at which the discriminant changes sign, we can determine, by careful analysis, which case occurs. The details of this analysis are found in Section 8.3, since it is related to the problem of choosing the "correct branch" for collision detection.

The geometric intersection algorithms for our problem, however, involve several subtle and important issues to make them robust in practice. We go into some detail about how to solve such systems, as specialized to our particular application, and how to implement the solution. We have two goals. The first is to elucidate a theoretical, exact, algebraic algorithm that is correct. However, this algorithm is numerically unstable when implemented with finite precision arithmetic. Hence robustness is a key issue. We emphasize the steps we have taken to enhance robustness and, in particular, to strengthen the theoretical algorithm by adding consistency checks.

Our approach to implementing a robust algorithm is somewhat experimental. The first step is noticing what robustness difficulties arise. The second is analyzing their causes and placing this analysis on a firm mathematical footing. Where possible, we propose solutions for some of the cases in which we believe we have found a robust strengthening of the underlying theoretical algorithms.

We implemented a specialization of Donald's (1984, 1987) representation for 6-DOF ($\mathbb{R}^3 \times SO(3)$) configuration space obstacles to the case of $\mathbb{R}^2 \times S^1$. We quickly review that representation. Let C denote the configuration space $\mathbb{R}^2 \times S^1$, and let (\mathbf{x}, θ) be a typical configuration. A C-space obstacle CO is defined by a predicate on configurations (as in Donald [1987]).

$$\bigwedge_{i \in C \text{ family}(A, B)} \left(\theta \in \mathcal{A}_i \implies f_i(\mathbf{x}, \theta) \leq 0 \right). \quad (5)$$

Here, the functions $f_i : C \rightarrow \mathbb{R}$ are called C functions; their negative conjunction in effect defines the C-space obstacle. Each f_i is restricted by an applicability region \mathcal{A}_i , which is a sector (angular interval $[\theta_{\min}, \theta_{\max}]$) of the unit circle S^1 . The reason for this is that each f_i is generated by considering the interaction of a feature (edge or vertex) of a pawl polygon A with a feature (vertex or edge) of an obstacle polygon B . Contact, or interaction between generating features, is only possible for a connected angular interval of orientations; this interval

is precisely the applicability interval \mathcal{A}_i . The set of all C functions generated by A and B is called the "C family" $c \text{ family}(A, B)$. See Brooks and Lozano-Pérez (1983) and Donald (1987) for details on computing the C functions and the applicability constraints.

A C-surface $\ker f_i$ is defined as the applicable zero-set (kernel) of a C function f_i . It contains a patch of configurations where the two generating features of f_i can be placed in contact. The key step in detecting a collision of a path with a C-space obstacle CO defined by (5) is to simultaneously solve for the path's parameter subject to the C-surface constraint⁶ $f_i = 0$.

6. Sticking Caused by Friction

A careful analysis of the physics of compliant sliding is necessary to formulate a precise, combinatorial version of this apparently continuous problem. When the pawl is in contact with the environment, it is possible for the motion to stop, because the contact forces are adequate to balance the applied forces. This is called "sticking caused by friction," to distinguish it from other kinds of sticking. The algorithm must determine if this can happen during motion. The work of Mason (1982) and Erdmann (1984) addresses this issue in considerable detail. In this section, we show that for our problem this determination can be made by a simple geometric test.

Coulomb's law of friction states that the tangential force caused by friction, \mathbf{f}_t , on a point that is sliding on an edge in a specified direction is given by $\mathbf{f}_t = \mu \mathbf{f}_n$, where \mathbf{f}_n is the force normal to the edge and μ is the coefficient of friction. In addition, the direction of the friction force is opposite that of the motion.

Coulomb's Law has a geometric interpretation that is frequently more useful (see, for example, Mason 1982). We provide here a slightly different interpretation which is more applicable to our problem. Suppose point \mathbf{p}_i is in contact with edge e_j . The outward pointing unit normal vector to the edge is denoted \mathbf{n}_j . We are also given the direction of sliding by specifying \mathbf{v} , the unit vector tangent to the edge in the direction of sliding (Fig. 8).

Coulomb's Law defines a half-cone of forces acting on the point \mathbf{p}_i . The cone is the convex combination⁷ of the vectors \mathbf{n}_j and the vector $\mathbf{f}_e = \mathbf{n}_j - \mu \mathbf{v}$. Any force in the interior of this cone and along \mathbf{n}_j can be resisted, and the point will stick. A force along \mathbf{f}_e will result in \mathbf{p}_i sliding in equilibrium in the direction of \mathbf{v} . A force outside the cone results in the point sliding along \mathbf{v} , out

6. We will often use the term "constraint" to blur the distinction between a C function and its kernel.

7. The convex combination of two vectors \mathbf{x} and \mathbf{y} is the set of all vectors of the form $\zeta \mathbf{x} + \eta \mathbf{y}$, where $\zeta, \eta \geq 0$.

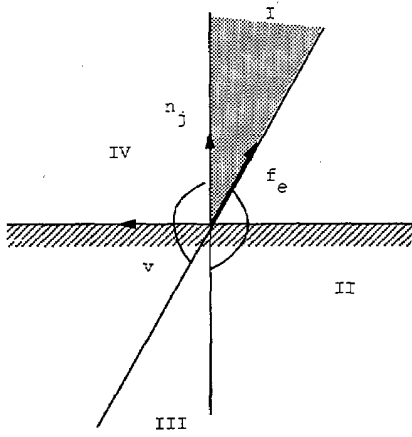


Fig. 8. Geometric interpretation of Coulomb's Law.

of equilibrium, which is precluded by our assumption of quasistatic motion.

We first consider the type B contact: the point is the vertex i of the pawl, and the edge is the edge e_j of the environment. Let $\mathbf{r} = R_{\theta}\mathbf{p}_i$. The assumption that stable contact is maintained (see Section 2) between the point and the edge restricts the possible values of the force \mathbf{f} on the pawl and the torque $\mathbf{r} \times \mathbf{f}$ ⁸ caused by the contact force. First, $\mathbf{f} \cdot \mathbf{n}_j$ must be positive; second, if the hinge is in the right half-plane, we require $\mathbf{r} \times \mathbf{f} < 0$ for stability, and if it is in the left half-plane, we require that $\mathbf{r} \times \mathbf{f} > 0$. For quasistatic sliding, the total force is given by $\mathbf{f} = \alpha \mathbf{f}_e$, $\alpha > 0$. Therefore \mathbf{f} trivially satisfies the condition $\mathbf{f} \cdot \mathbf{n}_j > 0$. The condition on the torque $\mathbf{r} \times \mathbf{f}$, however, depends on the location of the hinge point. Let the possible locations of the hinge point with respect to the contact point, $-\mathbf{r}$, be divided into four sectors as follows: if $-\mathbf{r} = \beta \mathbf{n}_j + \gamma \mathbf{f}_e$, the hinge point is in

- Sector I if $\beta > 0, \gamma > 0$,
- Sector II if $\beta < 0, \gamma > 0$,
- Sector III if $\beta < 0, \gamma < 0$,
- Sector IV if $\beta > 0, \gamma < 0$.

Figure 8 also indicates the sectors.

PROPOSITION 6.1. For type B contact, the contact point will slide when the hinge is in sectors II and IV. The contact point will not slide (will stick because of friction) when the hinge is in sectors I and III.

Proof. Suppose the pawl is sliding quasistatically. Then,

$$\begin{aligned} \mathbf{f} &= \alpha \mathbf{f}_e \\ &= \alpha(\mathbf{n}_j - \mu \mathbf{v}). \end{aligned}$$

8. Here $\mathbf{r} \times \mathbf{f}$ is the component of the usual cross product along the axis orthogonal to the plane (i.e., in coordinates, $\mathbf{r} \times \mathbf{f} = r_x f_y - r_y f_x$).

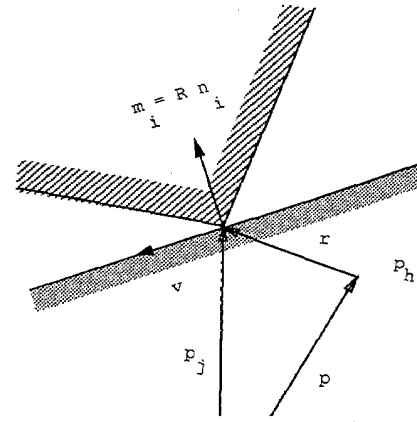


Fig. 9. Type A contact.

$$\begin{aligned} \mathbf{r} &= -[\beta \mathbf{n}_j + \gamma \mathbf{f}_e] \\ &= -[(\beta + \gamma) \mathbf{n}_j - \gamma \mu \mathbf{v}]. \\ \mathbf{r} \times \mathbf{f} &= -\alpha [0 - (\beta + \gamma) \mu + \gamma \mu + 0] \\ &= \alpha \beta \mu. \end{aligned} \quad (6)$$

Because $\alpha, \mu > 0$, the sign of $\mathbf{r} \times \mathbf{f}$ is just that of β . Now, in sectors I and II, the right-half plane, we require that $\mathbf{r} \times \mathbf{f} < 0$ to maintain stable contact. However, from equation (6), $\mathbf{r} \times \mathbf{f} > 0$ in sector I and $\mathbf{r} \times \mathbf{f} < 0$ in sector II. Hence the contact point will stick in sector I and slide in sector II. A similar argument shows that the contact point will stick in III and slide in IV. \square

A similar argument applies to type A contact if the sectors and related terms are defined appropriately (Fig. 9). Let \mathbf{f} be, as before, the force on the point of contact, which is now a vertex j of the environment, and let \mathbf{r} be the vector from the hinge point to the contact point, which is now given by $\mathbf{r} = \mathbf{p}_j - \mathbf{p}$. Let the outer normal of the edge e_i be denoted by $\mathbf{m}_i = R_{\theta}\mathbf{n}_i$, \mathbf{v} be the unit vector tangent to the edge in the direction in which the vertex slides with respect to the pawl's edge, and $\mathbf{f}_e = \mathbf{m}_i - \mu \mathbf{v}$. To maintain contact, $\mathbf{f} \cdot \mathbf{m}_i$ must be positive, and if the hinge is in the right half-plane, we require $\mathbf{r} \times -\mathbf{f} > 0$ (i.e., $\mathbf{r} \times \mathbf{f} < 0$); if it is in the left half-plane, we require that $\mathbf{r} \times \mathbf{f} > 0$.

With the terms defined as above, we can divide the possible locations of the hinge point with respect to the contact geometry into four sectors exactly as we did for type B contact, despite the fact that the interpretation of the sectors is different. Now it is easy to prove the corresponding proposition:

PROPOSITION 6.2. For type A contact, the contact point will slide when the hinge is in sectors II and IV. The contact point will not slide (will stick because of friction) when the hinge is in sectors I and III.

Proof. The proof is textually identical to the proof for type B contact, with \mathbf{n}_j replaced by \mathbf{m}_i . \square

7. The Sliding Direction

The analysis of Section 6 utilized the direction of motion to determine whether the pawl could stick as a result of friction. It turns out that the direction of the motion can be determined from the differential kinematics of the contact, given the motion of the hinge, \mathbf{p} .

Consider the motion for a type B contact. For convenience, assume that the y -axis of the reference coordinate frame is along the outward normal of the edge (i.e., $\mathbf{y} = \mathbf{n}_j$), and the x -axis is along the edge, oriented to form a right-handed coordinate system. Recall that $\mathbf{r} = R_\theta \mathbf{p}_i$. Define the *departure speed*, u , as the component of the velocity of the contact point along the edge normal \mathbf{n}_j . The velocity of the contact point is given by $\dot{\mathbf{r}} + \dot{\mathbf{p}}$. Hence

$$u = (\dot{\mathbf{r}} + \dot{\mathbf{p}}) \cdot \mathbf{n}_j = \omega r_x + \dot{p}_y. \quad (7)$$

The differential kinematic constraint for sliding corresponds to the requirement that the departure speed is zero.

$$\omega r_x + \dot{p}_y = 0. \quad (8)$$

The *sliding speed*, s , is defined as the component of the velocity of the contact point in the \mathbf{x} direction. (The vector \mathbf{v} of Section 6 is therefore $\text{sgn}(s)\mathbf{x}$.) Hence,

$$s = (\dot{\mathbf{r}} + \dot{\mathbf{p}}) \cdot \mathbf{x} \quad (9)$$

$$= -\omega r_y + \dot{p}_x. \quad (10)$$

Combining with equation (8),

$$s = \dot{p}_x + \frac{\dot{p}_y}{r_x} r_y. \quad (11)$$

Equation (11) has a simple geometric interpretation, shown in Figure 10. The circle represents the possible positions of the hinge point with respect to the contact point. The numbers on the circle indicate the sliding speed when $\dot{p}_y < 0$. (If $\dot{p}_y > 0$, the speeds shown have opposite sign; if $\dot{p}_y = 0$, s is constant.) Notice that the sliding speed is 0 when $\mathbf{r} \cdot \dot{\mathbf{p}} = 0$. Also, from equation (8), $\omega < 0$ when the hinge point ($-\mathbf{r}$) is in the right half-plane of the figure and $\omega > 0$ in the left half-plane.

We can now combine the above results with those of Section 6. Suppose, without loss of generality, that the hinge point is in the right half-plane. If the hinge is in the region corresponding to $s < 0$, the possible direction of slide is to the left. If, in addition, the hinge is in sector I, it will stick. If it is outside sector I, the location of the hinge will move in the direction of ω ; at some point (corresponding to $\mathbf{r} \cdot \dot{\mathbf{p}} = 0$), the direction of slide will

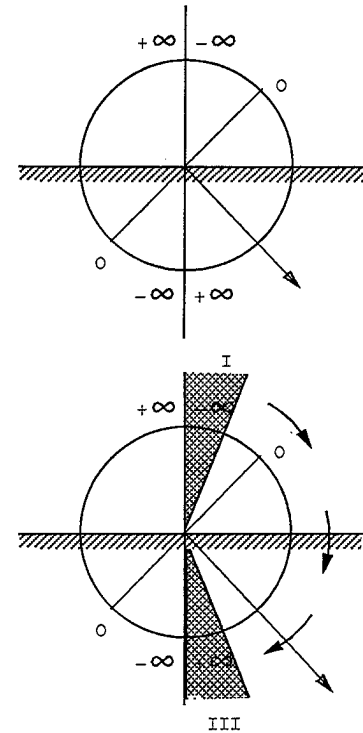


Fig. 10. The sliding direction for type B contact.

reverse to $s > 0$. This also brings sectors III and IV to the right half-plane. Eventually, the hinge will enter sector III for $s > 0$, and stick. Figure 10 also depicts the evolution of the hinge point.

The time at which the contact will stick in sector III is easily computed by letting

$$R_\theta \mathbf{p}_i = \frac{|\mathbf{p}_i|}{1 + \mu^2} (\mathbf{n}_j + \mu \mathbf{v})$$

in the constraint equation (4) and solving the resulting linear equation for t .

A similar but slightly more complicated analysis applies to type A contact. We will first derive an expression for the velocity \mathbf{w} of the contact point with respect to a frame fixed on the moving pawl and expressed in the world coordinate frame. We will assume, for simplicity of derivation, that the coordinate frame attached to the pawl has its y -axis along the edge normal $\mathbf{m}_i = R_\theta \mathbf{n}_j$, and the x -axis is along the edge so as to form a right-handed coordinate system. As in Section 6, for type A contact, $\mathbf{r} = \mathbf{p}_j - \mathbf{p}$, and let ${}^p \mathbf{r} = R_\theta^T \mathbf{r}$ be the corresponding vector in the coordinate frame fixed to the pawl. Let Ω be the skew-symmetric angular velocity matrix, given by

$$\Omega = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad (12)$$

where ω is the angular velocity of the pawl. From elementary kinematics (see, for example, Bottema and Roth 1979), $\Omega = \dot{R}_\theta R_\theta^T$.

The velocity in the frame attached to the pawl, $R_{\theta}^T \mathbf{w}$, can now be calculated by differentiation:

$$\begin{aligned} R_{\theta}^T \mathbf{w} &= \dot{p} \mathbf{r} \\ &= R_{\theta}^T \dot{\mathbf{r}} + R_{\theta}^T \dot{\theta} \mathbf{r} \\ &= R_{\theta}^T \dot{\mathbf{r}} + R_{\theta}^T \Omega^T \mathbf{r}. \end{aligned} \quad (13)$$

Therefore,

$$\mathbf{w} = \dot{\mathbf{r}} + \Omega^T \mathbf{r}. \quad (14)$$

The differential constraint corresponds to the departure speed, u , being zero. Therefore,

$$\begin{aligned} u &= \mathbf{w} \cdot \mathbf{m}_i \\ &= \dot{r}_y - \omega r_x \\ &= 0. \end{aligned} \quad (15)$$

Solving for ω ,

$$\begin{aligned} \omega &= \frac{\dot{r}_y}{r_x} \\ &= \frac{-\dot{p}_y}{r_x}. \end{aligned} \quad (16)$$

Hence, the sliding speed, s , is given by

$$\begin{aligned} s &= \mathbf{w} \cdot \mathbf{x} \\ &= \dot{r}_x + \omega r_y \\ &= \dot{r}_x + \frac{-\dot{p}_y}{r_x} r_y \\ &= -\dot{p}_x - \dot{p}_y \frac{r_y}{r_x}. \end{aligned} \quad (17)$$

The map of sliding speeds for type A contact is almost identical to Figure 10, except that the direction of the vector $\dot{\mathbf{p}}$ is reversed. However, the orbit of the hinge point during sliding is more complicated than in the type B case. The time at which the hinge point will enter sector III and stick can be computed as follows. At the time of entry into sector III, the \mathbf{r} is given by

$$\begin{aligned} \mathbf{r} &= \frac{|\mathbf{r}|}{\sqrt{1+\mu^2}} (\mathbf{m}_i - \mu \mathbf{v}) \\ &= \frac{|\mathbf{r}|}{\sqrt{1+\mu^2}} \left(\mathbf{m}_i + \mu \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m}_i \right) \\ &= \frac{|\mathbf{r}|}{\sqrt{1+\mu^2}} \begin{pmatrix} 1 & -\mu \\ \mu & 1 \end{pmatrix} \mathbf{m}_i. \end{aligned} \quad (18)$$

Hence,

$$\mathbf{m}_i = \frac{1}{|\mathbf{r}| \sqrt{1+\mu^2}} \begin{pmatrix} 1 & \mu \\ -\mu & 1 \end{pmatrix} \mathbf{r} \quad (19)$$

Plugging this back in equation (3), we get

$$\mathbf{r} \cdot \mathbf{m}_i - d_i = \frac{1}{|\mathbf{r}| \sqrt{1+\mu^2}} \mathbf{r}^T \begin{pmatrix} 1 & \mu \\ -\mu & 1 \end{pmatrix} \mathbf{r} - d_i \quad (20)$$

$$= \frac{|\mathbf{r}|}{\sqrt{1+\mu^2}} - d_i = 0. \quad (21)$$

Because \mathbf{r} is a linear function of t , this is a quadratic equation in time and can be solved for t .

8. Collision Detection in Configuration Space

We now describe the collision detection algorithms we implemented. These algorithms are essentially a specialization to $\mathcal{R}^2 \times S^1$ of the 6-DOF collision avoidance algorithms of Donald (1984, 1987). The basics of this geometric representation may be traced to Lozano-Pérez (1983) and Brooks and Lozano-Pérez (1983); in particular, see Brooks and Lozano-Pérez (1983) and Donald (1987) for a discussion of applicability constraints.

The special structure of the C functions f_i permit us to define combinatorially precise algorithms for collision detection and for solving general intersection problems. In particular see, for example (Brooks and Lozano-Pérez 1983; see Section 5), the general form of a C function in our application is

$$\begin{aligned} A_1 x S + A_2 x C + A_3 y S + A_4 y C \\ + A_5 S + A_6 C + A_7 x + A_8 y + A_9 \end{aligned} \quad (22)$$

where $\mathbf{x} = (x, y)$, $C = \cos \theta$, and $S = \sin \theta$.

Now, as is well known, we can make the substitution $u = \tan \theta/2$ in (22). Because $S = (2u/1+u^2)$ and $C = (1-u^2/1+u^2)$, we obtain a form of the C surface (22), which is quadratic in u :

$$B_1 u^2 + B_2 u + B_3 = 0, \quad (23)$$

where the coefficients B_i are all affine in x and y . When the root position \mathbf{x} is parameterized by eq. (2), then note that B_i are all affine in the time t in the initial position of the root \mathbf{p}_0 and in the root velocity vector $\dot{\mathbf{p}}$.

Pure translational collision detection is straightforward, see, for example (Lozano-Pérez 1983; Donald 1987).

8.1. Computing the "Snap": Pure Rotational Collision Detection

For pure rotational collision detection at a fixed translation \mathbf{x} , we implement the following algorithm. First, note that we must solve (23) for its u -zeros; these zeros determine the orientation at which the pawl can cross the boundary of the C-space obstacle CO defined by (5). Each u -zero can be found by solving a quadratic (23). For each of these events, we construct an *intersection* that is a tuple $(\theta, f_i, A_i, c \text{ family}(A, B))$. These intersections are sorted around the unit circle (by θ). The intersections are traversed in order, and the first valid one is returned. A valid intersection is one where:

Intersection Validity Tests

1. The constraint is zero: $f_i(\mathbf{x}, \theta) = 0$. (This is true by root finding.)
2. The constraint is applicable: $\theta \in \mathcal{A}_i$.
3. The intersection configuration is on the boundary of the CO 5. For all applicable f_j at orientation θ in c family (A, B) , $f_j(\mathbf{x}, \theta) \leq 0$. Thus, we must test the signs of all other applicable constraints f_j in f_i 's family.

If A and B have size m_a and m_b , respectively, this algorithm runs in time $O(m_a^2 m_b^2)$. We can achieve a better bound of $O(m_a m_b)$ by employing the algorithm of⁹ (Donald 1987). Letting $n = m_a m_b$, we can thereby achieve an overall exact simulation algorithm that runs in time $O(n^2 \log n)$. We provide a much faster algorithm in Section 9 that asymptotically beats this approach by almost a linear factor of n .

Finally, since one C family is generated for each convex-convex pawl-obstacle pair, this rotational intersection test must be iterated for each C family, and the minimum valid intersection returned.

8.2. Collision Detection Subject to a Holonomic Constraint

A more complicated intersection problem arises when the pawl is sliding on a surface (23), subject to (2) (which specifies the motion of the root). Hence, we view the situation as follows. Suppose the pawl is sliding subject to a type B (vertex-edge) or type A (edge-vertex) constraint. This corresponds exactly to the reference point sliding on a quadric C-space surface $\ker f_i$, which is generated by those contact features. The constraint on position (2) defines a "plane" in configuration space. The intersection of this plane and $\ker f_i$ form a quadratic curve in C space, parameterized by time. Using this parameterization, we can generate the configuration of the pawl at any time; that is, we have solved for how the pawl moves subject to the root motion (2) and subject to the C surface (23). We must do this for each pawl (since, as in Figure 4, it is possible for both pawls to be in simultaneous contact with different features of the environment).

The first problem we must solve is how to detect when the quadratic path above intersects another C surface. At that point, the new constraint may take over, or the pawl may snap off because of incompatible kinematic constraints.

9. We have given the slower algorithm here, because we have found that while it has higher asymptotic complexity, it is more robust geometrically and stable numerically. This is probably because of the redundant information in the representation of a C family.

Each C surface has form (23), where the coefficients B_i are affine in \mathbf{x} , and hence affine in t , \mathbf{p}_0 , and \mathbf{p} (see (2)). We call this a *trigonometric quadratic form (TQF)* (Donald 1984, 1987). To solve for the simultaneous intersection of two C surfaces in TQF, we view them as simultaneously quadratic in u and affine in t . We treat the variable t as indeterminate and use the resultant to obtain a quartic in t . We solve for the t -roots, and back-substitute for the u -roots. Naturally, we must check for the degenerate case where the leading coefficients of the TQFs are zero (however, see Section 8.4.1). Finally, given the (t, u) -roots, we must then perform the intersection validity tests described above in Section 8.1 (or the faster test in Donald [1987]). While there exist closed-form solutions to quartics, we can also solve them numerically using Ferrari's method.

In general, it is very difficult to find exact algebraic procedures for predicting motion subject to a holonomic constraint. This is because most dynamical systems do not lend themselves to such solutions. Our formulation of rotational compliance enjoys purely algebraic solution trajectories that can be computed using simple algorithms and that require no integration. Of course, this is largely because of the simplicity of the dynamic model. Nevertheless, from the examples in Section 3, it is clear that it is capable of producing complex behaviors.

8.3. Choosing the Correct Branch

This section deals with the following difficulties. When we reduce motion prediction problems to the existential theory of the real numbers, we obtain our answers by solving polynomial equations. These equations may have multiple roots. However, our system only has one evolution. We must have a method for choosing which root is correct. Conversely, situations will arise where the algebraic equations fail to have roots. We must handle these cases also.

The situation of multiple roots corresponds precisely to the existence of several "statically" or "kinematically feasible" solutions to the dynamical system, only one of which is feasible (or reachable) from the initial conditions. We present a method for choosing the correct root. When the correct root is parameterized by time, it sweeps out a connected component called a branch. This occurs in our case, where the formal coefficients are parameterized by t .

The absence of roots corresponds precisely to the pawl breaking contact with a C surface or jamming on the C surface so that no further motion is possible. We show how to determine which of these events has occurred. These cases are important to handle in practice, and our methods are both theoretically well founded and implementable.

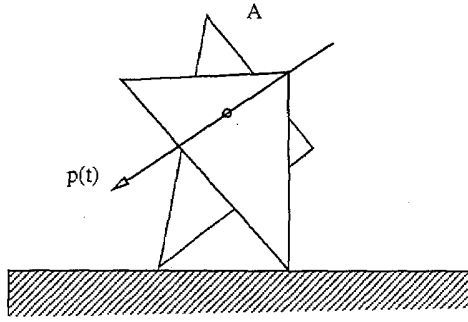


Fig. 11. There are two orientations (u -roots) at which the C function f_i generated by (v_i, e) is zero. Which of these two solutions should we choose?

Consider Figure 11. Clearly, when we solve a TQF such as (23) for its u -roots, we are solving a quadratic; hence we obtain two roots. Now, often one root will be inapplicable, or not on the boundary of the C space obstacle CO (eq. (5)). However, in Figure 11, both are good roots. Which one should we pick? This is an important question for an implementation. First, we note that even though both are good roots, there is no realizable path between them (and so they are disconnected). Consider the problem of generating a path in C space subject to C surface constraint (23) and to the straight-line motion (2). (This corresponds to intersecting a quadric C surface with a plane to obtain a quadratic curve, parameterized by time.) The path is constructed using elimination methods; consider the problem of “following” this path. One reason we need a systematic way of choosing the correct root is that, because of numerical errors, often one root will appear to become inapplicable “too early” (see, for example, Section 8.4.2) or else a root appears to just slide off the boundary of the C space obstacle. At this point the implementation finds the other root (as in Figure 11). This root is applicable and on the boundary, and so the predicted path “jumps” discontinuously from one root to the other, even though no such path is physically feasible.

This example illustrates pathologic behavior. Fortunately, we can give a principled way for choosing the correct root. Essentially, one is finding zeros of a simple algebraic variety in the plane; the problem is equivalent to choosing the correct “branch” of the variety. We show how to do this algorithmically. In the process, we also derive a method for determining when a pawl “jams” against a constraint and when a pawl “snaps off” a constraint as a result of incompatible differential kinematics. The two prove to be related problems.

Let us assume that the C surface is in the form (from (23))

$$S(u, t) = A(t)u^2 + B(t)u + C(t) = 0, \quad (24)$$

where

$$\begin{aligned} A(t) &= A_0 + A_1t \\ B(t) &= B_0 + B_1t \\ C(t) &= C_0 + C_1t \end{aligned} \quad (25)$$

and $u = \tan(\theta/2)$ as usual. Then the branch problem is essentially the problem of choosing the correct sign of the radical in

$$u = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \quad (26)$$

The problem of determining when the pawl snaps off or jams is essentially that of determining if the discriminant $\Delta(t) = B^2 - 4AC$ is becoming negative and hence failing to have a real solution. Let us discuss this second case first, and then we will return to discuss the branch problem (how to choose the sign of $\sqrt{B^2 - 4AC}$) afterward.

8.3.1. Application: Snapping Off and Jamming

In Figure 12, we plot the discriminant $\Delta(t) = B^2 - 4AC$ vs. time, for type (B) and (A) constraints. We illustrate the qualitative contact regions in the space. In the type (B) case, the “pawl” here is drawn as a straight-line segment pivoting about its end point. (The pivot is the center of rotation of the pawl—where it would be attached to a root body.) The other end point is in contact with an edge. A type (B) constraint is a C surface generated by a vertex of the moving pawl and an edge of the obstacle. A type (A) constraint is generated by an edge of the moving pawl and a vertex of the obstacle. Hence in the type (A) figure, the edge moves with the pivot p , and the vertex v is stationary, whereas in the type (B) figure, the edge is stationary, and the vertex at the end of the line segment pawl moves.

Now, the discriminant $\Delta(t)$ is non-negative if and only if eq. (24) has a real solution. Hence we need the following test for the zero crossings of the discriminant $\Delta(t)$, which is obtained by solving the quadratic $\Delta(t) = 0$ for its root t_z . That is,

$$\begin{aligned} \Delta(t) &= B^2 - 4AC \\ &= (B_1^2 - 4A_1C_1)t^2 \\ &\quad + 2(B_0B_1 - 2A_0C_1 - 2A_1C_0)t + B_0^2 - 4A_0C_0. \end{aligned} \quad (27)$$

This will produce two times, t_{z_1} and t_{z_2} , with $t_{z_1} \leq t_{z_2}$. Clearly, the time of the first collision with this surface, t_c , must satisfy $t_{z_1} \leq t_c \leq t_{z_2}$ for type (B) C surfaces, and $t_{z_1} \geq t_c$ or $t_c \geq t_{z_2}$ for type (A). Therefore, the next critical time t_{z_n} is $t_{z_n} = t_{z_2}$ for type (B) C surfaces. For type (A) C surfaces, $t_{z_n} = t_{z_1}$ if $t_c \leq t_{z_1}$. Otherwise, $t_{z_n} = \infty$ if $t_c \geq t_{z_2}$ (i.e., there is no critical time).

If t_{z_n} is finite, we must now check the behavior at this time. Two things can happen as t is increased a small amount from t_{z_n} (Fig. 13). The pawl-obstacle contact is

represented as in Figure 12. For either type (B) or (A) contact, we can either (i) break contact or (ii) jam. The figure shows these possibilities.

Now, which behavior occurs can be determined from the C surface equation. We assume that $S(u, t) > 0$ means that the vertex is outside the half-space bounded by the edge.

Equation (24) for $S(u, t)$ can be rewritten as

$$S(u, t) = (A_1u^2 + B_1u + C_1)t + (A_0u^2 + B_0u + C_0). \quad (28)$$

First, we solve for u_z at time t_{z_n} from eq. (24). There should be only one solution:

$$u_z = \frac{-B(t_{z_n})}{2A(t_{z_n})} = \frac{-(B_0 + B_1t_{z_n})}{2(A_0 + A_1t_{z_n})}. \quad (29)$$

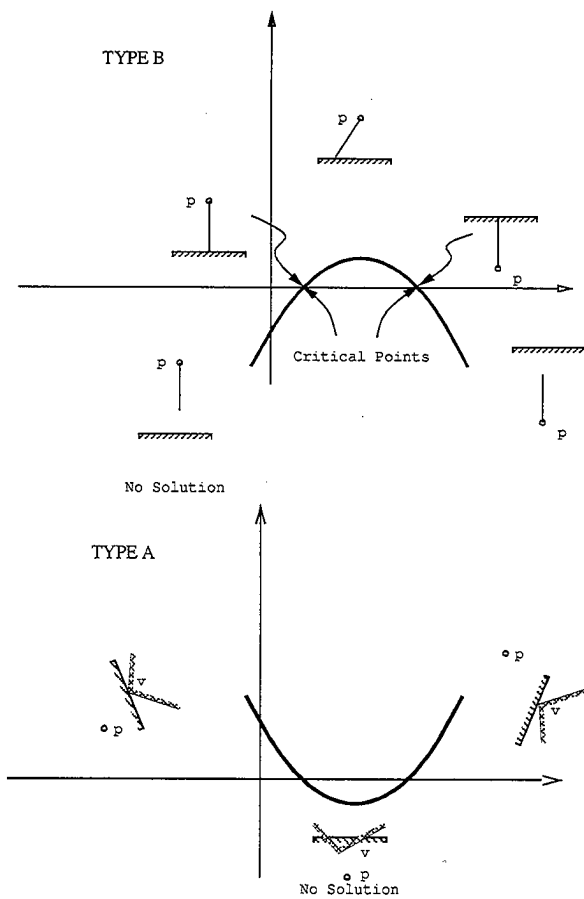


Fig. 12. Plotting the discriminant $\Delta(t) = B^2 - 4AC$ vs. time, for type B and A constraints. We illustrate the qualitative regions in the space. The “pawl” here is represented as follows. For a type B constraint, the pawl is a straight line connecting the pivot \mathbf{p} with the contact vertex. The obstacle edge is stationary. For a type A constraint, the edge moves rigidly with \mathbf{p} , and v is a stationary obstacle vertex.

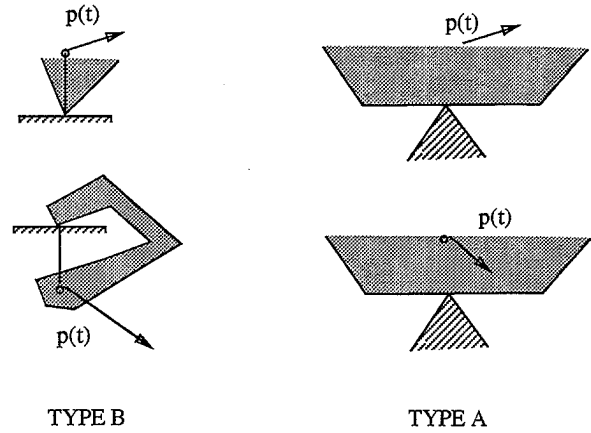


Fig. 13. If t_{z_n} is finite, we must now check the behavior at this time. Two things can happen as t is increased a small amount from t_{z_n} . We can break contact (1) or jam (2). See Fig. 12. \mathbf{p} is the pivot point, and $\dot{\mathbf{p}}$ the pivot velocity. Type B and A contacts are shown. Case B (break contact) shows the friction cone. Case A is impossible and cannot occur unless we started from this configuration.

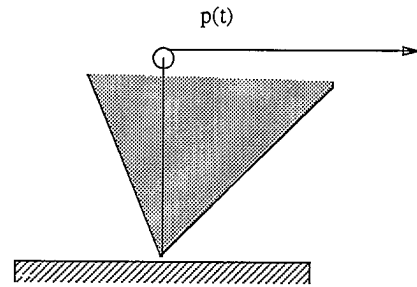


Fig. 14. Nongeneric case of translation parallel to the edge.

Next, we examine the sign of the coefficient of t in eq. (24) for this value of u_z . Let $\sigma = A_1u_z^2 + B_1u_z + C_1$.

1. If $\sigma \geq 0$, then the pawl breaks free. It performs a “snap” (pure rotation) toward the zero position.
2. Else, $\sigma < 0$. The pawl is jammed.
3. Note: $\sigma = 0$ is the nongeneric case of translating parallel to the edge (Fig. 14). This should probably be considered a “snap”, but the exact semantics are somewhat unclear.

8.3.2. The Branch Problem

We now return to the branch problem. This is essentially the problem of choosing the correct sign of the radical in eq. (26). This may be done as follows. First we define the following algorithm, which computes the branch sign for a C surface at time t and orientation u .

Algorithm Branch-Sign(u, t, C surface)

1. Let $A(t), B(t), C(t)$ be the coefficients of the C surface.
2. $x \leftarrow -\frac{B(t)}{2A(t)}$.
3. $y \leftarrow \frac{1}{2A(t)}\sqrt{B^2(t) - 4A(t)C(t)}$.
4. If $u = x + y$, then return Plus.¹⁰
5. If $u = x - y$, then return Minus.

Now, every time we move onto a new C surface, we record its sign $Branch-Sign(u_c, t_c, CsurfaceI)$. When new candidate intersections are produced when sliding along $csurfaceI$, then we check that the corresponding u_{new} and t_{new} for the intersection fall on the branch of $csurfaceI$ with the same branch-sign. That is, we check whether or not $Branch-Sign(u_c, t_c, CsurfaceI) = Branch-Sign(u_{new}, t_{new}, CsurfaceI)$.

8.4. The Genericity of Intersection Problems

Many algebraic robotics and motion planning algorithms make certain “genericity” or general position assumptions about the systems of polynomials they manipulate. We now discuss how certain of these assumptions are not necessary in our algorithm, because (in one sense) our constraints are “never singular.” (We make this notion precise below.) On the other hand, there are other general position assumptions that would seem safe (and are, in fact, common in geometry) but, as it turns out, are *not* valid in our application. We call this situation “forced nongenericity.” Intuitively, this occurs with rotational compliance because of the tendency of the manipulated parts to rotate until many features are simultaneously aligned. In such cases, special techniques are needed to solve the nongeneric intersection problems.

8.4.1. The Inherent Genericity of C functions in Trigonometric Quadratic Form

In essence, our model permits a reduction of the motion prediction problem to solving polynomial systems of equations. Now, for arbitrary polynomials, the leading coefficients of these systems can vanish. This singular situation is undesirable, since in these cases the resultant gives us no information, thereby necessitating special checks. However, in the special case of TQFs that arise from C functions, it turns out that we do not have to check explicitly for this case. More specifically, the degenerate case of the formal leading coefficients vanishing does occur. However, if this happens, it means that in the case of TQFs, $\theta = \pi \pmod{2\pi}$ is a solution for the trigonometric equations (Pai 1988). This means that for

C surfaces in TQF, we can blindly apply elimination theory (and, especially, the theory of resultants) to solve the polynomial systems. This fortuitous circumstance is only true in our special application, where the quadric surfaces come from TQFs such as (23). Thus, while many algebraic algorithms require the system of polynomials to be in general position (this is called a *genericity assumption*), our method has no such genericity requirement.

8.4.2. On the Forced Nongenericity of Intersection Problems Subject to a Holonomic Constraint

Whereas many algebraic motion planning algorithms can make assumptions about general position (or genericity) of the polynomial constraints, when we predict motions subject to a holonomic constraint, we find that these assumptions may not hold and that we are forced to solve nongeneric intersection problems.

In Figure 15, the geometry of the flexible object and the environment are the same as in Figure 4. However, the assembly plan is to move diagonally in direction $(1, -\frac{4}{10})$ instead of straight down. (See Figure 16 for a detail of the bottom of the pawl.) The bottom vertex v_i of the left pawl hits the obstacle edge e first. The next segment of the motion is subject to this constraint. That is, the motion is compliant subject to the C surface $\ker f_i$ generated by (v_i, e) . Now, refer to Figure 16 again, and consider an *arbitrary* (generic) motion of the triangle (a path in C space). Obviously, this path can cause either constraint (v_i, e) or (v_j, e) to be violated. However, we say that *generically*, both will not be violated at the same time. That is, for a path $\phi : [0, 1] \rightarrow C$, it will be generically true that $f_i(\phi(t))$ or $f_j(\phi(t))$ is nonzero. (f_j is generated by (v_j, e)). This would be a good general position assumption, but unfortunately, it is simply not true for collision detection subject to a rotationally compliant motion constraint. In Figure 15, we see that as the motion evolves, A rotates about v_i as v_i slides on e . Eventually, this rotation brings v_j down on e . In fact, constraint f_i expires (that is, we pass out of its applicability region) at the precise time that f_j is activated (we pass into its applicability region); at the same time, f_j changes sign from positive to zero.¹¹ More precisely, at this time t , the orientation $\theta(t)$ crosses the boundary of \mathcal{A}_i and \mathcal{A}_j (which share an end point), and $\phi(t)$ hits the zero set of f_j (Fig. 17).

The nongenericity illustrated in Figures 15 through 17 in fact occurs for a large (generic) class of paths and initial conditions. This is not surprising, since one of the outcomes of rotational compliance is to align parts. Second, it is not surprising that when one constraint (C function) expires, it is “replaced” by a constraint with

10. With finite precision arithmetic, these equality checks cannot be exact.

11. We use “expire” and “become active” following Donald (1987).

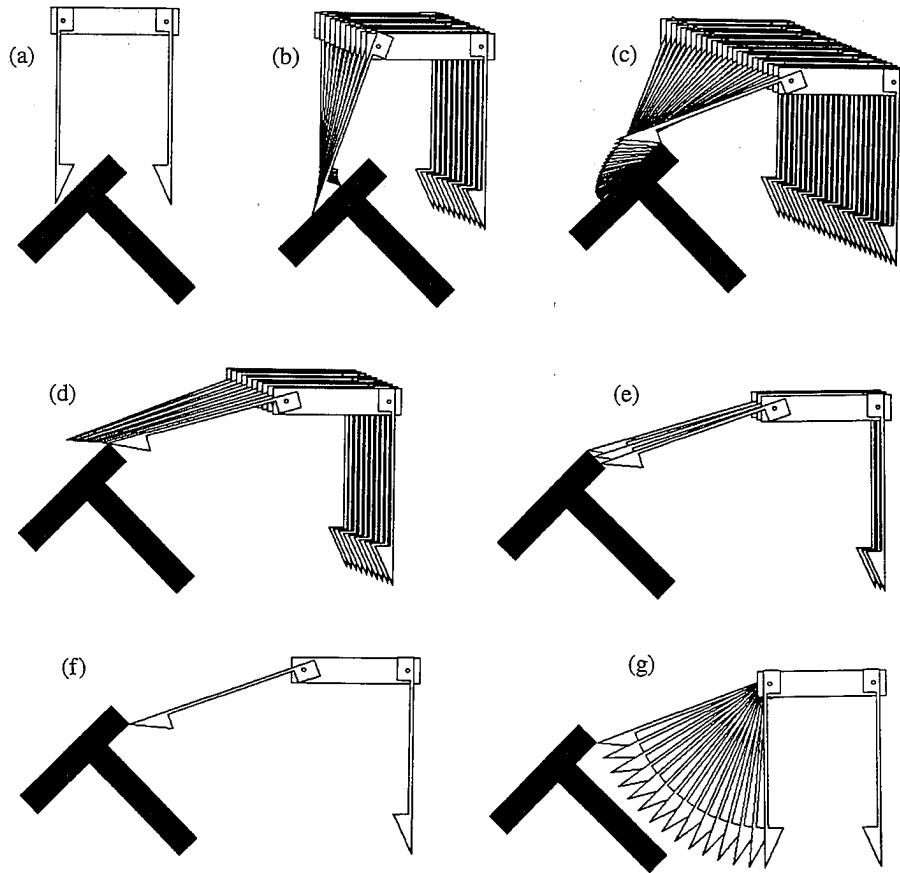


Fig. 15. The geometry of the flexible object and the environment are the same as in Fig. 4. However, the assembly plan is to move diagonally in direction $(1, -\frac{4}{10})$ instead of straight down. The bottom vertex of the left pawl hits first. The next segment of the motion is subject to this constraint, in 15B. At the end of 15B, both pawl vertices lie on the edge. In 15C–D, the pawl is dragged over the top of the T. In 15F–G, it snaps off to the rest position.

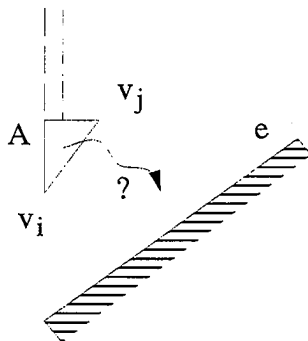


Fig. 16. A detail of the bottom of the pawl from Fig. 15.

“neighboring generators.” This observation has been formalized by Donald (1984, 1987) and also exploited by Erdmann and Lozano-Pérez (1987). For these reasons, naturally, it would seem as if nongeneric examples such as Figure 15 would happen all the time with our system. In fact, this has been our experience. It is unfor-

tunate in that it places high demands on the robustness of our algebra system. In particular, the applicability constraint boundaries of \mathcal{A}_i and \mathcal{A}_j are computed as the zeros of polynomials also. Hence, in observing that three events occur simultaneously (Fig. 17), we are effectively saying that three polynomials (in t) have simultaneous zeros. When these zeros appear to occur at different times because of numerical errors, then consistency is not maintained, and errors can occur. For example, if we find that ϕ crosses $\ker f_j$ before it crosses the applicability boundary $\mathcal{A}_i \cap \mathcal{A}_j$, then that intersection will be judged nonapplicable and discarded. If the events are ordered the other way, the intersection is detected.

We can attempt to maintain consistency by introducing additional tests into the algorithm based on topological information. For example, (Donald 1987), the polynomial g_{ij} whose zeros define the applicability boundary $\mathcal{A}_i \cap \mathcal{A}_j$ is essentially $f_i - f_j$. Hence, at any given configuration, there are consistency constraints between the signs (and values) of f_i , f_j , and $g_{ij} = f_i - f_j$. Second, we could em-

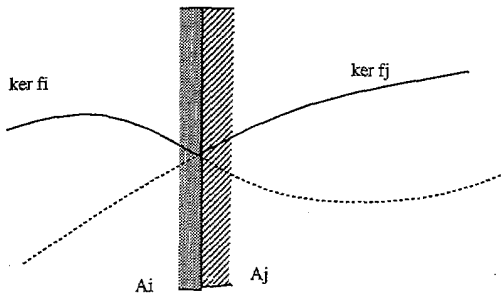


Fig. 17. $\ker f_i$ is the C surface for contact (v_i, e) (see Fig. 16). $\ker f_j$ is the C surface for contact (v_j, e) . The rotationally compliant path ϕ is following $\ker f_i$, when it simultaneously (1) strikes the boundary of f_i 's applicability region A_i , (2) pierces the boundary of f_j 's applicability region A_j , and (3) hits the zero-set of f_j . ϕ is shown schematically in solid lines.

ploy the techniques of Donald (1987) and Erdmann and Lozano-Pérez (1987), which compute, when a constraint “expires,” which constraints with neighboring generators become applicable. Thus, when f_i expires at time t , we have $\phi(t) = A_i \cap A_j$. We can look at $\dot{\phi}(t)$ and the neighboring generators of v_i and e to determine that f_j must “replace” f_i in the applicability set (see Fig. 16).¹² Finally, we must perform these constraint replacement computations “subject to a holonomic constraint.” That is, if f_j replaces f_i in free space, it suffices to simply substitute f_j for f_i in the applicability set. However, in contact, subject to the holonomic constraint

$$f_i(\mathbf{x}, \theta) = 0, \quad (30)$$

we must not only update the applicability set, but we must also update the constraint (30). That is, we must simultaneously replace f_i by f_j in the applicability set, and replace (30) by the constraint $f_j = 0$. This constraint replacement mandates that $f_j = 0$. (This is equivalent to saying that we maintain contact.) Hence, we know that at time t , we should simultaneously have f_i, f_j , and g_{ij} vanishing. If, because of numerical errors, these zeros occur at different times, then we should in effect “identify” these times into one canonical, simultaneous zero-crossing time for topological consistency.

Finally, we note that the issue of forced nongenericity is mathematically subtle, and has connections to other branches of singularity theory. For example, consider Figure 18. In general, we have a configuration space C containing a lower dimensional singular set S that is algebraic. (In our case, S is the boundaries in Fig. 16.) Now, if we choose two points z_1 and z_2 randomly in C ,

12. The applicability set is the set of all constraints (C functions) that are applicable at a configuration.

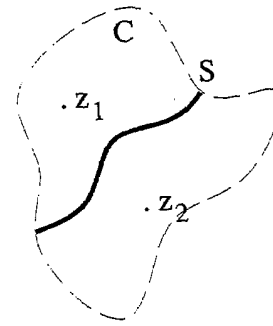


Fig. 18. Abstract depiction of forced nongenericity. In a configuration space C , consider a lower dimensional algebraic singular set S of “bad” points that we wish to avoid. For example, S could be the set of singular configurations of the forward kinematic map.

the chance that one will lie in S is essentially zero (this is really the definition of measure zero). However, if we now fix z_1 and z_2 and consider all paths from one point to the other, it turns out that “many” of them—a measurable set—cross S . In particular, if S separates C , then one might expect this probability to be roughly proportional to the ratio of the measure of the two components of $C - S$. Finally,

FACT If z_1 and z_2 are disconnected by S in C , then any path from z_1 to z_2 will cross S .

This means that any such path must experience a singular configuration at some time. This situation is exactly parallel to ours, in which a singularity cannot be avoided. Note that this problem arises in other applications—for example, in Pai and Leu (1992), we observed S as the singular set of a forward kinematic map. By the above claim, this means (for example) that any path from z_1 to z_2 must pass through a singularity.

We have sketched an approach to handling forced nongenericity that exploits the specific characteristics of our domain. General techniques for handling these problems await further research.

9. A Sweep Algorithm for Simulation

The focus of our work has so far been on modeling and robotics issues. In this section we give a fast algorithm by mounting a computational geometric attack. In addition, the new algorithm sheds light on several interesting combinatorial and structural issues. Furthermore, it leads to a systematic classification of special cases and has connections to the qualitative analysis of dynamical systems.

We have shown how our model of compliance permits us to obtain algebraic, closed-form solutions to simulation

problems for a rotationally compliant object. In particular, the linear map $\mathbf{p}(t)$ is given by eq. (1), and once we “rationalize” rotations via the standard substitution $u = \tan \theta/2$, each map ϕ_h is piecewise-cubic. Based on these results, a simple algorithm for simulation is as follows.

Simple Algorithm

1. Given a state x of the system, possibly subject to a configuration space constraint f (f has form eq. (3) or (4)) calculate a local solution trajectory Φ respecting f . Φ will be a linear or cubic curve in the configuration space.
2. The configuration space constraints are algebraic surfaces of bounded degree. Intersect Φ with the surfaces to perform exact collision detection.
3. A collision results either in termination of the motion or in a change of constraint. Update f if necessary.
4. Repeat.

Suppose the obstacles have m_b vertices and the moving object (root and one pawl) has m_a vertices. Let $n = m_a m_b$ be the measure of the geometric complexity. In Section 8 we were able to show that this simple algorithm runs in time $O(n^2 \log n)$. For k pawls, the complexity of the simple algorithm was $O(kn^2 \log n)$.

In this section, we give a new simulation algorithm that is also exact and runs in time¹³ $O(k\lambda_r(n)\log^2 n)$. Our method reduces the simulation problem (Def. 2.1) to a plane sweep of an arrangement of algebraic curves in configuration space. To obtain this result, we prove the following points. For simplicity of presentation, assume below that $k = 1$, and hence configuration space $C = \mathbb{R}^2 \times S^1$.

1. The simulation never leaves a connected component F_0 of free configuration space.
2. The solution trajectory Φ of the system is piecewise cubic in the configuration space, and the dynamics may be reduced to erecting cubic “local” configuration space constraints.
3. Translational motion of the root polygon (eq. (1)) restricts the reachable configurations to a 2D cylinder $Y \subset C$.
4. Y embeds in C as follows:

$$\begin{aligned} Y &\hookrightarrow \mathbb{R}^2 \times S^1 \\ (t, \theta) &\mapsto (\mathbf{p}(t), \theta). \end{aligned} \quad (31)$$

13. Recall that $\lambda_r(n)$ is the (almost linear) maximum length of (n, r) Davenport-Schinzel sequences (Guibas et al. 1988) and r is a small constant. For practical purposes, $\lambda_r(n)$ should be regarded as just a tiny bit bigger than $O(n)$. For more on the theory of Davenport-Schinzel sequences and their applications in computational geometry, see Agarwal et al. (1987), Hart and Sharir (1987), and Kedem and Sharir (1985).

where t moves along the axis of the cylinder. Hence we view Y as $\mathbb{R} \times S^1$. The time evolution $t \in T$ of the system corresponds to the \mathbb{R} factor. Hence the solution “sweeps” along the cylinder Y in the direction of the axis.

5. The configuration space constraints are manifest as cubic curves on Y .
6. As we sweep the cylinder Y , a simple dynamical system models the motion of the configuration point on the configuration space constraints. The orbits of this system are piecewise algebraic of bounded degree.
7. Parameterize Y to the plane (this only takes two charts). Then $F = Y \cap F_0$ is defined by a planar arrangement of cubic curves.
8. F has size $O(\lambda_r(n))$ and can be constructed in time $O(\lambda_r(n)\log^2 n)$ using a red-blue merge algorithm, as described in Guibas et al. (1988).
9. We can compute the solution Φ by a plane-sweep of the slice F (i.e., by sweeping a planar arrangement of cubic curves). This allows us to solve the motion prediction problem exactly in $O(\lambda_r(n)\log^2 n)$ overall time.

Hence we prove

THEOREM 9.1. The *Simulation Problem* defined above (Def. 2.1) can be solved in time $O(\lambda_r(n)\log^2 n)$ and space $O(\lambda_r(n))$, where r is a small constant.

We outline the proof in the following sections.

9.1. Computing the Connected Component of Free Space

Consider the motion of a single pawl, \mathcal{M}_h , whose translation is governed by eq. (1). As is well known, if we “rationalize” rotations using the standard substitution $u = \tan(\theta/2)$, the n constraints given by eqs. (3) and (4) are manifest as algebraic ruled surfaces $\{f_1, \dots, f_n\}$ in a 3D configuration space with coordinates (x, y, u) . These constraints are simultaneously linear in the position parameters x and y and quadratic in the rotational parameter u . Each surface is only “applicable” for some range of orientations $[u_0, u_1]$, by which we effectively mean that the surface only “exists” for u in this range (Donald 1987). See Figure 19 (Brost 1989).

Let $\hat{\mathbf{u}}$ be a vector along the u -axis (think of $\hat{\mathbf{u}}$ as $(0, 0, 1)$). Now, the constraint of pure translation (eq. (1)) of the hinge point P_h of the pawl \mathcal{M}_h restricts any possible evolution of the system to lie in a “plane” (2D subspace) of (x, y, u) -space (see Fig. 19). We call this “plane” P_Y ; it has “normal” $\hat{\mathbf{u}} \times (\dot{\mathbf{p}}, 0)$, and it corresponds to a chart for the cylinder Y discussed earlier. We note

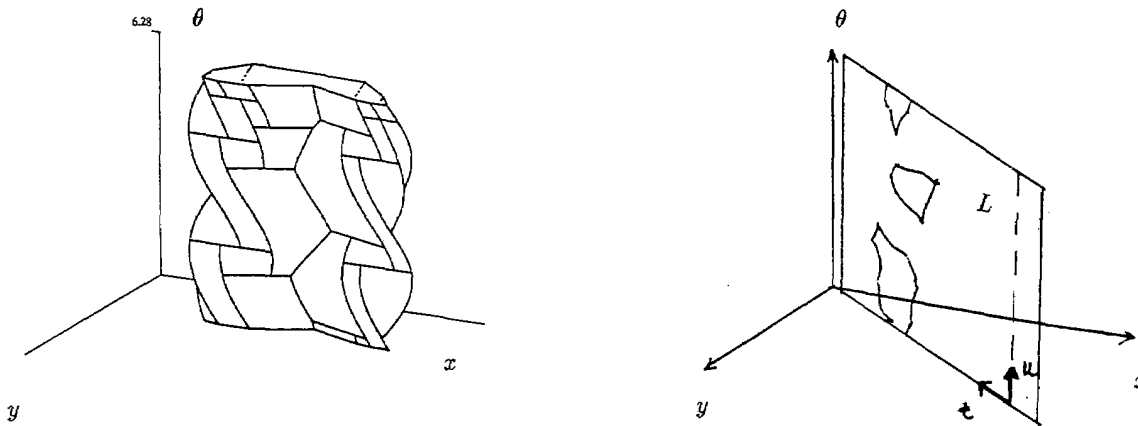


Fig. 19. Configuration space constraints for a moving pawl. The sweep plane $L(t)$ intersects the C space obstacles in a planar arrangement of cubic curves. (Reprinted courtesy of Randy Brost.)

that furthermore, at time t , the state of the system is constrained to lie on a line $L(t)$ parallel to \hat{u} in the plane P_Y . This is the line of points $(\mathbf{p}(t), u)$, for $u \in \mathbb{R}$. The line $L(t)$ sweeps across the plane in direction $(\hat{\mathbf{p}}, 0)$ as t increases, and thus we call $L(t)$ the *sweep line*.

Hence, a natural coordinate system for the plane P_Y is given by (t, u) . As time t increases, the vertical line $L(t)$ sweeps across P_Y . This line contains the state of the system. It is our task to calculate the u coordinate as t evolves (i.e., increases). Now, P_Y has degree 1 and hence, when intersected with a constraint f_i we obtain a cubic¹⁴ curve segment γ_i in the (t, u) plane. Thus all the configuration space constraints are manifest as an arrangement of cubic curve segments $\{\gamma_1, \dots, \gamma_m\}$ in this plane, where $\gamma_i = f_i \cap P_Y$ ($i = 1, \dots, m$). In our algorithm, the sweep line sweeps across this planar arrangement of curves, and as we sweep, we compute the trajectory of the system. “Events” caused by crossing the curves γ_i will modify the trajectory, as we discuss below.

A priori, the arrangement of curves can have complexity $O(n^2)$, and in fact, the set of free configurations in P_Y can also have size $\Omega(n^2)$ in the worst case (Kedem and Sharir 1985; Guibas et al. 1988). However, we can directly apply the results of Guibas et al. (1988) as follows. We note that the system begins at some configuration $\mathbf{z}_0 \in P_Y$. \mathbf{z}_0 lies in one connected component $F \subset P_Y$ of free space, and the resulting path can never leave F , since it corresponds to a physical simulation. Guibas et al. (1988) show that:

LEMMA 9.2. The combinatorial complexity of F is $O(\lambda_{s+2}(n))$, and we may precompute it (i.e., compute

14. In fact, each curve γ_i is simultaneously quadratic in u and linear in t .

it before our plane sweep) in time $O(\lambda_{s+2}(n) \log^2 n)$ (Guibas et al. 1988). Here s is a small constant bounding the number of times that two configuration space constraint curves γ_i and γ_j can intersect.

Recall $\lambda_r(n)$ is the (almost linear) maximum length of (n, r) Davenport-Schinzel sequences (Guibas et al. 1988; Hart and Sharir 1987; Agarwal et al. 1987). It is very likely that for a wide variety of situations encountered in practice (Brost 1989, 1991), $s \leq 1$. However, it is certain that a worst case bound is $s \leq d^2$ for curves of degree d . Note that $s \leq 1$ would tighten our bound to space $O(n\alpha(n))$ and time $O(n\alpha(n) \log^2 n)$ (Hart and Sharir 1987).

9.2. Sweep Events in Configuration Space Slice

9.2.1. Kinematics

We postpone our discussion of friction until Section 9.3. Here we treat the frictionless case first, defining six types of local geometric events, called “sweep events,” that will be detected and handled during the plane sweep. These events are purely kinematic—i.e., they do not depend on friction.

Having computed the connected component $F \subset P_Y$ containing the initial configuration, we now sweep F with the line $L(t)$ in the (t, u) -plane. By an abuse of notation, we will now let γ_i, γ_j , etc., denote the curves bounding F that we precomputed (along with their intersections) in Section 9.1. We now define the following sweep events: (1) *translational collision*, (2) *sliding collision*, and (3) *jamming due to incompatible kinematics*. The sweep algorithm will detect sweep events. Each event will be *handled*, by which we mean that the solution trajectory

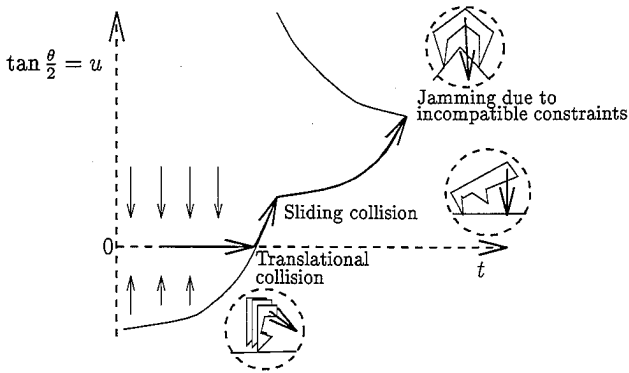


Fig. 20. Sweep events: translational collision, sliding collision, and jamming caused by incompatible kinematics.

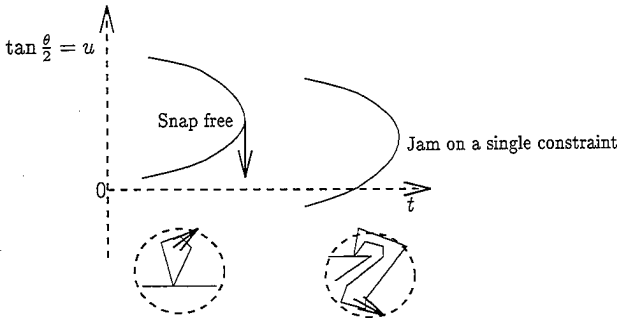


Fig. 21. Sweep events: snapping free and jamming on a single constraint.

we compute may be modified. Between events, the trajectory is piecewise algebraic.

First, suppose there are no obstacles. Then the trajectory of the system will stay at $u = 0$ in the (t, u) -plane (i.e., the orientation of the pawl will not change). Call the point $\mathbf{z}(t)$ on $L(t)$ representing the state of the system the *sweep point*. Thus $\mathbf{z}(t)$ is the solution trajectory we compute.

Now, in the presence of obstacles, sweep events occur as the sweep line crosses the curves $\{\gamma_i\}$. These events are enumerated in Figures 20 through 22. We can explain the trajectory computation algorithm like this: the dynamical system described in Section 2 has the following geometric interpretation in slice P_Y . As the sweep line $L(t)$ crosses the (t, u) -plane, the u -coordinate $u(t)$ of the sweep point $\mathbf{z}(t) \in L(t)$ moves. In the plane P_Y , the line $u = 0$ is an *attractor*, and we imagine a vector field on P_Y parallel to the u -axis and pointing toward the t -axis. Hence the attracting vectors are parallel to $-\hat{\mathbf{u}}$ for $u > 0$ and parallel to $+\hat{\mathbf{u}}$ for $u < 0$. The curves γ_i act as (holonomic) constraints. The sweep point cannot cross these curves, but it can follow them as $L(t)$ moves. They can prevent motion of the sweep point from attaining $u = 0$.

For example, see Figure 20. If the trajectory is at the $u = 0$ position and the sweep point $\mathbf{z}(t)$ encounters a

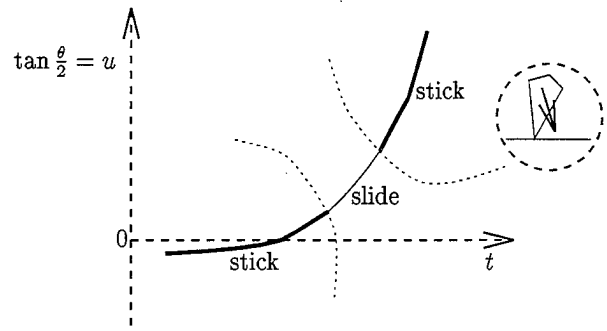


Fig. 22. Sticking events and qualitative dynamical regions.

constraint γ_i , then the sweep point complies to the constraint and is forced to move away from the zero line (u becomes positive here). This corresponds to a pure translational collision, followed by a continued motion of the root, which “cocks” the pawl against an obstacle. During this motion, the sweep point follows γ_i . If a new constraint γ_j is reached, then the sweep point slides along the curve γ_j in turn. This corresponds to a sliding collision: while sliding on constraint γ_i , the pawl hits constraint γ_j . The motion continues, following γ_j compliantly. Hence the sliding collision can result in a constraint change. Finally, if the sweep point is following a curve γ_i that crosses $u = 0$, the trajectory breaks contact there and continues along the t -axis. This event is a “dual” subcase of type (i).

As can be seen from Figure 20, some constraint changes result in jamming as a result of incompatible kinematics. This occurs as follows. Define the outward normal $\boldsymbol{\eta}_i$ of a curve γ_i to point into free space F . Let $\hat{\mathbf{t}}$ be a unit vector in the positive t direction. Jamming occurs at $\gamma_i \cap \gamma_j$ when both the inner products

$$\boldsymbol{\eta}_i \cdot \hat{\mathbf{t}} \quad \text{and} \quad \boldsymbol{\eta}_j \cdot \hat{\mathbf{t}} \quad (32)$$

are negative. At this point the simulation is terminated, because further motion is impossible.

Pure translational collision events can occur where a curve γ_i intersects the line $u = 0$. *Sliding collisions* can occur when two boundary curves of F intersect (i.e., at $\gamma_i \cap \gamma_j$). Jamming events can occur when both normals at $\gamma_i \cap \gamma_j$ point in the $(-t)$ direction. A nonjamming sliding collision causes a change of constraint (i.e., the sweep point now follows γ_j instead of γ_i). It is clear that sweep events of type (i), (ii), and (iii) are local geometric conditions and can be detected and handled while sweeping the line $L(t)$ over F . Similarly, it is clear that modifying the trajectory $\mathbf{z}(t)$ at a sweep event can be done in $O(1)$ time.

9.2.2. Snapping Free or Jamming on A Single Constraint

We now define the sweep events (iv) *snapping free from* and (v) *jamming on a single constraint*. Suppose the sweep point is following a constraint curve γ . A singularity occurs at vertical tangencies of γ (Fig. 21). Assume wlog that γ lies in the half-plane $u > 0$. There are two possibilities. If the F is concave at the singularity, then the sweep point has been following the “upper” branch of the curve. After the singularity, the sweep point follows the vector field attracting it toward $u = 0$. That is, the sweep point moves parallel to the u -axis toward the t -axis. It stops at the first new constraint curve it hits while moving away from the singularity toward the line $u = 0$. If no constraints are encountered, it stops at $u = 0$. This motion corresponds to the pawl “snapping free” from a single constraint edge. It executes an instantaneous pure rotation toward the zero position. If another constraint is in the way, then it stops there.

If F is convex at the singularity, then no further motion is possible, and the motion jams there on a single constraint. At this point the simulation is terminated.

Clearly, singularity (vertical tangency) is a local geometric condition that can be detected during the plane sweep of F , since each curve is algebraic.

There is one more kinematic sweep event that is “dual” to type (iii) jamming due to incompatible kinematics. It is type (vi), *snapping free from a vertex*. It occurs at a constraint change $\gamma_i \cap \gamma_j$ (i.e., the sweep point is following a curve γ_i , and it hits another curve γ_j). However, in this case, both outward normals η_i and η_j point in the positive t direction. That is, the dot products in eq. (32) are both positive. In this case, the sweep point “snaps free” from $\gamma_i \cap \gamma_j$ and moves vertically toward the attractor $u = 0$. The snapping free happens just as in event (iv) above. Snapping free from a vertex corresponds to the situation where suddenly there are no holonomic constraints on the pawl, so it can move toward its rest position $u = 0$. Conceptually, there is little difference from event (4) (snapping free from one constraint).

It is clear that sweep events of type (iv), (v), and (vi) are local geometric conditions and can be detected and handled while sweeping the line $L(t)$ over F . It is clear that modifying the trajectory $\mathbf{z}(t)$ at a sweep event can be done in $O(1)$ time. To see that six event types suffice, simply enumerate the ways γ_i can (a) intersect γ_j , (b) intersect $u = 0$, or (c) become vertical. Hence we have,

LEMMA 9.3. There are six types of kinematic sweep events, as described earlier. There are $O(\lambda_{s+2}(n))$ such events overall. Each event is a local geometric condition that can be detected and handled in $O(1)$ time. The output trajectory is piecewise algebraic with at most $O(\lambda_{s+2}(n))$ pieces and degree of at most 3.

COROLLARY 9.4. A plane sweep of F that handles all kinematic sweep events can be performed in time $O(\lambda_{s+2}(n) \log \lambda_{s+2}(n))$. This sweep solves the frictionless simulation problem (given F) for a single pawl.

9.3. Friction

We now briefly describe how friction is handled. From the analysis in Section 6, the following is clear: for each configuration space surface f_i we can define two constraints, g_i and h_i , that are also algebraic surfaces of the same degree as f_i . g_i and h_i depend on the direction of assembly \mathbf{p} in (1).

The surfaces g_i and h_i break up f_i into *sliding* and *sticking* regions. We call these *qualitative dynamical regions (QDRs)*, by analogy with Briggs and Donald (1990). In a sliding region, motion is possible as t increases. In a sticking region, equilibrium results, and no further motion is possible. (Compare work on translational compliant motion: e.g., Donald [1988a]; Briggs [1989].) Now, when we intersect f_i with the plane P_Y to obtain a curve γ_i (Fig. 22), we obtain a 1D slice of these qualitative dynamic regions (sliding and sticking). Now, the Bezout bound gives an a priori $O(1)$ bound on the number of QDRs per surface. However, in fact, the special structure of our constraints ensures that there will be at most three QDRs per connected curve γ_i on the boundary of F . Type B constraints have (at most) one sliding region surrounded by two sticking regions. Type A constraints have (at most) one sticking region surrounded by two sliding regions.

Now, we define a seventh type of sweep event, (vii) a *sticking event*, as follows. Suppose the sweep point is following a curve γ_i . If it enters a sticking region on the curve, then equilibrium is reached, and the simulation is terminated. Entry into the sticking region corresponds to crossing another algebraic curve h_i or g_i and hence is a local geometric event that can be detected and handled during the sweep. Note that g_i and h_i apply only to f_i and do not affect any other surface f_j ; hence we call them *local dynamic constraints*.

Finally, we must slightly modify our kinematic plane sweep. After a pure translational or pure rotational collision with a curve γ_i , we first check to see whether we are in a sticking or sliding region on that curve. If it is a sticking region, we terminate the simulation in equilibrium; otherwise, we proceed as above (Section 9). To summarize:

PROPOSITION 9.5. There are $O(1)$ sticking events per constraint curve γ_i . Each occurs at the intersection of γ_i with a local dynamic constraint (another algebraic curve) in P_Y .

This completes our proof of the main Theorem 9.1.

9.4. Summary

Red-blue merge algorithms allow us to construct a connected component of free space F containing the initial configuration in time $O(\lambda_{s+2}(n) \log^2 n)$. We desire to simulate a simple dynamical system within F . We compute the simulation trajectory using a plane sweep of F . To do this, we augment F with a vector field (defining an attractor at $u = 0$) and “annotate” each curve γ_i on the boundary of F with certain “markings” at which the dynamical behavior of a sweep point traversing γ_i can change. The markings break the curve into a finite number of subsegments. The markings are (1) sticking/sliding transition, (2) vertical tangency, and (3) intersection with the line $u = 0$. Each marking is determined by the intersection of γ_i with a line (such as $u = 0$) or a curve (g_i or h_i), or by vertical tangency. Hence each marking is algebraic, and there are $O(1)$ markings per curve. Finally, at an end point of γ_i we have an intersection with the next cubic curve γ_j on the boundary of F .

10. Conclusions

In this article we addressed the problem of predicting the motion of a flexible object amid a polygonal environment. To this end, we presented a careful analysis of the physics of the interaction of the flexible body with the environment. The analysis yielded several simple tests to determine the motion of the body that were then integrated into an algorithm for predicting the motion under a given motion plan.

This motion prediction problem is the first step in reasoning about such devices. We also described the implementation of the algorithm and related issues, as well as the significance of the algorithm for designing compliant parts for ease of assembly. In Appendix B we describe the effect of uncertainty.

Suppose we wish to build a system for reasoning about and analyzing the motion of a “flexible object” (i.e., a compliantly connected system of rigid bodies). To this end, we developed a basic theory on the motion prediction problem for such bodies, at a fairly abstract level, emphasizing connections to computational mechanics and the long-term behavior of dynamical systems. We discussed how our formulation of the problem led to theoretically precise algorithms for motion prediction with rotational compliance and showed how these algorithms could be implemented and used to analyze designs for assembly. We stressed the fact that while the theoretical algebraic algorithms we give are correct when exact arithmetic is employed, in practice we must strengthen the algorithms to make them robust when implemented using finite-precision arithmetic. We showed how this may be done in a systematic fashion. Issues of robustness and

numerical stability turned out to be related to problems of genericity and branch choice, and we described the theory behind these problems and their practical solution. We emphasized our computational approach while considering the pitfalls and subtleties that an implementation foregrounds.

10.1. Summary of Results

The research contributions of this article include:

1. New algebraic techniques for predicting the motion of objects in contact under our model of rotational compliance.
2. A systematic catalog of singular and nongeneric situations that must be handled and algorithms for dealing with these events.
3. An extension of our algorithm to compute mating force information. This facility can be used to design objects that are easier to mate than to disassemble.
4. An extension of our algorithm to efficiently predict motions given uncertainty in sensing and control (Appendix B).
5. A manifesto for the relevance of our approach to engineering, and particularly to design for assembly. As an application, we studied several classes of flexible devices that we term “motion diodes.” We developed a classification of these devices into total vs. relative motion diodes and described kinematic, frictional, and force diodes. Our algorithm can analyze designs and classify the diode type. We suggested how motion diodes could be useful in design for assembly.
6. We considered the problem of simulating the motion of compliantly connected rigid bodies in frictional contact with obstacles during an assembly motion. While compliant motion has been considered in a computational geometric setting for pure translations (Donald 1988a; Briggs 1989; Friedman et al. 1989), the problem for rotational compliance has proved resistant to solution. We showed that, unlike many simulation problems for rotational bodies, we can obtain exact solutions without integration. We first developed a naïve yet combinatorially precise $O(n^2 \log n)$ algorithm based on collision-detection techniques such as those in Donald (1987).
7. We improve on this algorithm by the introduction of several techniques. The key ideas we use are red-blue merge algorithms, a simple dynamical systems model, and local dynamic constraints. These tools permit us to reduce the simulation to a plane sweep of a “dynamically annotated” slice of configuration space. More specifically, first we precompute

the connected component of the simulation. This component of free space has low combinatorial complexity (by Davenport-Schinzel arguments) and can be computed efficiently using a red-blue merge algorithm (Guibas et al. 1988). Next, we reduce the simulation problem to a plane sweep of F . To do this, we first introduce additional local constraints ($O(1)$ per curve bounding F), an attractor at the rest orientation $u = 0$, and a corresponding attractive vector field on F . These constructions allow us to view the plane sweep as a simple dynamical system. This in turn permits us to bound the number of sweep events by $O(\lambda_{s+2}(n))$, which yields our main result. This is one of the first combinatorially efficient, exact solutions to any simulation problem for a rotational mechanical system or for rotational compliant motion.

10.2. Discussion and Future Work

There are many problems left for the future. First, we would like to extend our work to “trees” of compliantly connected bodies (articulated bodies). Our flexible objects are trees of depth 1; however, one could imagine many compliantly connected links in a chain. Future work includes the incorporation of more complicated models of the flexible objects (such as that of Trantina and Minnichelli [1987]) and the dynamics of interaction. For example, we hope to add more sophisticated dynamics to our work and to model continuously deformable bodies as well.

Second, we believe our work can be extended to incorporate uncertainty in the initial conditions and in the control. This result would be of considerable interest, since it would permit simulation of a differential inclusion. Exact simulation does not take into account the uncertainty (e.g., the impossibility of a comprehensive description of the dynamics of a system) or error in actuation. We view the introduction of a more realistic mechanical model (rotationally compliant bodies) as a step in this direction. We feel that the key property that allowed us to reduce the local dynamics to a plane sweep is a kind of “monotonicity” that is inherent in our system. It is our hope that other “monotonic” systems (and even differential inclusions) may be simulated using the concept of “simulation as sweep.”

We also wish to extend and test our system in analyzing real designs. We are also fabricating parts we have designed and testing them by assembling them with real robots (Böhringer 1992). Perhaps the most challenging area for the future involves the interaction of our analysis approach with uncertainty. We hope to extend the analysis and algorithms in Appendix B to model uncertainty

more precisely and to generate designs that can be assembled robustly. In particular, we hope to develop a precise notion of the “stability” of a motion with respect to a design. For example, an assembly plan might be “stable” if the qualitative outcomes of execution are equivalent.

We feel that algorithmic research on design for assembly represents a new direction in robotics that could have some impact on the field. In the future, one could imagine our techniques employed in an approach to “*design as search*,” in which we modify and improve an existing design by changing its geometry incrementally and applying our analysis algorithm. For example, suppose we are given geometric models of two parts to mate and an approximate “assembly plan” (direction of mating). We envision an algorithm that searches around the boundaries of the models and modifies the shape by introducing snap-fasteners and ratchet-and-pawl mechanisms. After each modification, the analysis algorithm described above could be run to determine how the parts will mate and the forces required to mate and disassemble them. Depending on the results of the analysis, the design change could be modified, retracted, or declared suitable. Of course, there is a host of conceptual and algorithmic problems that must be solved to make such an approach possible, but we believe that with the algorithmic underpinnings described in our analysis algorithm, a research program in this area is now viable. Such research could help to put design for assembly on a firm algorithmic footing, build software tools for generating good designs, and result in a unified framework for designing and assembling mechanisms.

Appendix A: Computing Mating Force Information

Consider Figure 7E once again. We wish to make precise the notion that this is a force diode.

Force diodes are interesting and useful in that we can use them to build objects that are easier to mate than to take apart. More generally, we wish to compute the forces required to assemble and disassemble our parts, because we must determine that excessive forces that could damage the parts are never exerted. Conversely, if an assembly we have designed can be taken apart with very small forces or by a wide range of motions, then the mechanical connection between the parts may be insufficient. By extending our algorithm to calculate the forces the robot is required to exert (external forces) and the forces the parts experience while mating, we can develop a general tool that performs all these computations.

The forces and torques experienced by the root as result of the interaction of the pawls with the obstacles can be computed by considering the force balances on the pawls. We shall show the computation of the force

and torque at the hinge point of a single pawl due to its contact. This information is useful to ascertain that the pawl does not experience excessive loading during assembly. It is straightforward to transform this force and torque into some other coordinate frame on the root body. The total force and torque on the root is found by summing the contributions of the individual pawls. The total force and torque information is important, as it has to be supplied by the robot or assembly machine.

First of all, we observe that if the pawl sticks because of friction (as in the case of a friction diode) or because of incompatible kinematic constraints (as in the case of a kinematic diode), the forces will be infinite as time is increased. This is due to the assumption that the root and pawl are rigid bodies, but even in practice, we expect the forces to get extremely large. Hence, we need only to examine the case of the pawl sliding on the obstacles. Second, our results provide us with a mapping from time to configurations (i.e., to the angles of the pawl). Hence we shall derive the dependence of force on the configuration θ instead of on time.

We first consider type B contact. The torque τ at the hinge point is only due to the displacement of the pawl from its resting configuration, θ_0 . Hence,

$$\tau = k(\theta - \theta_0), \quad (A1)$$

where k is the torsional stiffness of the spring connecting the pawl to the root. Now the force on the contact point during quasistatic sliding is some multiple of the vector \mathbf{f}_e (i.e., $\mathbf{f} = \alpha \mathbf{f}_e$). (\mathbf{f}_e is the vector along the friction cone edge in the opposite direction of sliding.) Therefore, balancing the torques on the pawl about the hinge point,

$$\tau = \mathbf{r} \times \mathbf{f} = k(\theta - \theta_0). \quad (A2)$$

Simplifying and rearranging terms, we can express α as a function of only θ :

$$\alpha = \frac{k(\theta - \theta_0)}{R_\theta \mathbf{p}_i \times (\mathbf{n}_j - \mu \mathbf{v})}. \quad (A3)$$

The type A case is similar. As in Section 6, we redefine \mathbf{r} to be the vector from the hinge point to the contact vertex, $\mathbf{r} = \mathbf{p}_j - \mathbf{p}$, and $\mathbf{f}_e = R_\theta \mathbf{n}_i - \mu \mathbf{v}_\theta$. We have written the vector in the sliding direction as \mathbf{v}_θ instead of the usual \mathbf{v} to emphasize that the direction depends on θ ; more precisely, $\mathbf{v}_\theta = R_{\theta \pm \frac{\pi}{2}} \mathbf{n}_i$. Then, performing a torque balance as in the type B case, we get

$$\tau = \mathbf{r} \times -\mathbf{f} = k(\theta - \theta_0). \quad (A4)$$

The negative sign appears in front of \mathbf{f} because it was defined as acting on the obstacle. Simplifying and rearranging,

$$\alpha = \frac{k(\theta - \theta_0)}{(R_\theta \mathbf{n}_i - \mu \mathbf{v}_\theta) \times (\mathbf{p}_j - \mathbf{p})}. \quad (A5)$$

The above expressions could be used directly to numerically compute the maximum force during sliding, using well-known one-dimensional optimization methods. However, the maximum force and torque computation for a *single* pawl turns out to be much simpler. We define a *sliding segment* as a connected interval of time in which the pawl is sliding along an obstacle feature, without changing the contact topology. Because the torque τ varies linearly with θ , its maximum clearly occurs at a boundary of a sliding segment. We shall show below that the force maximum in a sliding segment can also occur only at a boundary of the segment. Hence one need only check the force and torque at these transition points. We have thus reduced the apparently continuous problem of finding the maxima to a discrete one.

PROPOSITION A.1. Let

$$\alpha(\psi) = \frac{A\psi + B}{C \sin \psi}, \quad (A6)$$

Subject to

$$\alpha(\psi) > 0 \text{ for all } \psi \in I = [\psi_{\min}, \psi_{\max}]. \quad (A7)$$

Then α does not achieve a relative maximum (and hence a global maximum) in the interior of I .

Proof. We look at the relative extrema of α in I and show that they are minima. Differentiating with respect to ψ ,

$$\alpha' = \frac{\sin \psi A - (A\psi + B) \cos \psi}{C \sin^2 \psi}. \quad (A8)$$

At a relative extremum, $\alpha' = 0$; hence the numerator of equation (A8) must be zero. Computing the second derivative of α at the extremum and simplifying, we find

$$\alpha'' = \frac{A\psi + B}{C \sin \psi} = \alpha > 0. \quad (A9)$$

Hence, the extremum is a relative minimum. \square

Our claim follows by showing that our expressions for α are of the form required by the proposition. First, notice that for both equations (A3) and (A5), the denominators simplify to the form

$$C_1 \sin \theta + C_2 \cos \theta,$$

where C_1 and C_2 are independent of θ , which can be easily rewritten as

$$C \sin(\theta + \phi),$$

where

$$C = \sqrt{C_1^2 + C_2^2} \quad \text{and} \quad \phi = \tan^{-1} \frac{C_2}{C_1}.$$

Hence, by making the substitution $\psi = \theta + \phi$, the expressions for α have the desired form for both type B and type A contacts. Finally, the requirement that $\alpha > 0$ in the sliding segment is nothing but the requirement that to maintain contact, the force has to be a positive multiple of f_e (see Section 6). Hence the above proposition applies, and the maximum α , and hence the maximum force, can occur only at the boundaries of a sliding segment.

With this extension to our algorithm, our system can compute the force required in assembly and disassembly and also the forces experienced by the pawls during execution. However, the computation of forces is not exact, because they involve transcendental functions that cannot be “rationalized” as in Section 8 for pure kinematic constraints to make them algebraic; hence this extension makes our algorithm approximate. (The use of provably good polynomial approximations to the transcendental functions will make the algorithm a provably good approximation algorithm, however.) The computation of mating forces is perhaps one of the most important aspects of our algorithm for design for assembly.

Appendix B: Incorporating Uncertainty in Control and Initial Conditions

B.1. Assumptions

Real robots are subject to significant uncertainty and error in sensing and control. For this reason we would like to generalize our algorithm to deal with uncertainty in initial conditions and in control. In this section we show how this may be done using a simple model of uncertainty. We can easily extend the simple algorithm of Section 9 to handle a very simple form of uncertainty. We expect that the sophisticated sweep line algorithm could be similarly extended, but we have not done so.

The introduction of uncertainty changes the complexity of our naïve algorithm from $O(n^2 \log n)$ to $O(n^3 \log n)$; the algorithm remains combinatorially precise and algebraic.

First, we assume that the initial position \mathbf{p}_0 of the root body is known to start within some “start region” R . This region represents uncertainty in the initial conditions. We will still assume that the root body is controlled as a “moving constraint,” parameterized by time. However, we assume that it is controlled by something like a generalized spring controller with feedback position correction. This model is very similar to that analyzed by Buckley (1987) (the spring-damper model), and we adopt it because Buckley’s error analysis is convenient algorithmically. However, we will not use the damper component of the model to slide on surfaces. Under these assumptions, the set of possible positions that the root position \mathbf{p} can

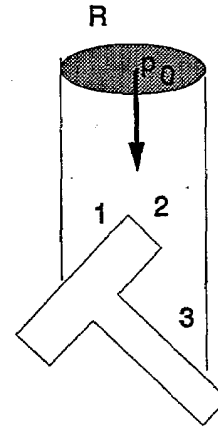


Fig. 23. With uncertainty, the root position starts out in some region R . The control system ensures that as time evolves, the root position stays in a cylinder $T_{\mathbf{p}}$.

reach (in free space, without obstacles) is bounded by a cylinder. In short, we incorporate sensing in control by assuming that with initial conditions \mathbf{p}_0 in R , the set of positions that the root position \mathbf{p} can reach is contained in a cylinder $T_{\mathbf{p}}$ starting at R . (Fig. 23).

The geometric formalization of our assumptions is that (i) $\mathbf{p}_0 \in R$, and (ii) for all times t , the root position $\mathbf{p}(t)$ lies in a cylinder $T_{\mathbf{p}}$ from R , as in Fig. 23 (see eq. (2)). The practical realization of these assumptions requires (i) a bound on sensing errors so that the initial position can be ascertained to some known accuracy, and (ii) a feedback control system with bounded error, such as that of Buckley (1987).

B.2. Algorithms

B.2.1. Motion Prediction Under Uncertainty Is Computable

It is immediately clear that with this formulation of uncertainty, the motion prediction problem (to predict all possible outcomes from R subject to $T_{\mathbf{p}}$) is decidable. We see this as follows. First, our algorithm for motion prediction from a single initial condition, given \mathbf{p} (as described above), defines an algebraic predicate $F_{\mathbf{p}}(\mathbf{p}_0, \mathbf{z})$ that decides whether a configuration \mathbf{z} is reachable from initial condition \mathbf{p}_0 , given commanded motion \mathbf{p} . In effect, our algorithm computes the entire set

$$\exists \mathbf{z} F_{\mathbf{p}}(\mathbf{p}_0, \mathbf{z}). \quad (\text{B1})$$

That is, our algorithm outputs the entire set $Z(\mathbf{p}_0, \mathbf{p})$ satisfying (B1). Thus we can define another predicate

$$\mathcal{F}(\mathbf{x}, \mathbf{z}) \iff (\exists \mathbf{z}) (\exists \mathbf{x} \in R) F_{\mathbf{p}}(\mathbf{x}, \mathbf{z}). \quad (\text{B2})$$

Correspondingly, predicates can be defined to decide all surfaces on which the pawls might stick, all surfaces

the pawls might slide on, etc. Because the predicates are algebraic, they are decidable.

B.2.2. A Practical Algorithm

More practically, we can define a direct algorithm with a computational geometric flavor. Assume that, generically, only one initial contact can occur at a time. Imagine that we “project” R onto the obstacle environment in “direction” \mathbf{p} . This is similar to “intersecting” $T_{\mathbf{p}}$ with the environment. That is, we sweep the flexible body along \mathbf{p} from R and find all edges of the environment it can hit. We find all initial contact tuples

$$(g_A, t_{c_{\min}}, t_{c_{\max}}, g_B, \mathbf{x}(t_c))$$

such that pawl feature g_A can strike obstacle feature g_B at times t_c , $t_{c_{\min}} \leq t_c \leq t_{c_{\max}}$. Note that at initial contact of (g_A, g_B) , the orientation of g_A and the other pawls is unchanged. $\mathbf{x}(t_c)$ is a function that maps initial contact times to root positions. Hence, the initial contact tuple represents the fact that for $t_{c_{\min}} \leq t_c \leq t_{c_{\max}}$, pawl feature g_A can strike obstacle feature g_B if the root position is at $\mathbf{x}(t_c)$.

For example, in Figure 23, assume the pawl is a line segment anchored at \mathbf{p} (as in Figure 12). There would be three initial contact tuples, one for each type (B) C surface of $\{v\} \times \{1, 2, 3\}$. We call the set of initial contact tuples the *projection* $\pi_{\mathbf{p}}(R)$ of R under $T_{\mathbf{p}}$.

Given the initial contact tuples, we chose a “sample point” for each one and run our standard, no-uncertainty algorithm from that sample point. The algorithm is given below.

Algorithm for Motion Prediction Under Uncertainty

1. Compute the initial contact tuples $\pi_{\mathbf{p}}(R)$.
2. For each initial motion tuple $(g_A, t_{c_{\min}}, t_{c_{\max}}, g_B, \mathbf{x}(t_c))$, do:
3. Assume type (B) contact, so that g_B is an obstacle edge.¹⁵ Segment g_B into three regions:
4. The sliding region e_1 ,
5. The sticking region e^*
6. and the portion e_0 of e that cannot be hit initially.
7. Output e^* .
8. Choose a sample point $\mathbf{x} \in e_1$.
9. Compute and output the reachable set $Z(\mathbf{x}, \mathbf{p})$ (eq. (B1)) using our motion prediction algorithm for perfect initial conditions and no uncertainty.

As usual, let n be the geometric complexity—that is, the product of the number of moving object features and the number of obstacle features. Then there are at most

$O(n)$ initial contact tuples. Thus the loop is executed $O(n)$ times. e_1 , e^* , and e_0 have size at most 2. Hence there are at most $O(n)$ sample points \mathbf{x} , and so the basic algorithm is called $O(n)$ times as a subroutine in the last step. Because the basic naïve algorithm runs in time $O(n^2 \log n)$, that means the algorithm for motion prediction under uncertainty takes time $O(n^3 \log n)$.

B.3. Extensions and Limitations

Ideally, one would like to extend our model of uncertainty to be more realistic. Ideally it should cover uncertainty in the direction of the moving constraint, and uncertainty in initial orientation. As of now, we hope it is rich enough to model interesting phenomena but still be computationally feasible. Much work remains to be done in reasoning about assemblies in the presence of significant uncertainty. Our initial algorithm is just a start.

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