Introduction to Voting

4x 😞 : 🎅 > 👷 > 🐶

3x 😞 : 🐶 > 👷 > 🎅

2x 😞 : 🐶 > 🐶 > 🎅
Introduction to Voting

Pairwise comparisons:

- 4x 😞: 🎅gorm>😀>😄
- 3x 😞: 😄>😀> Blowjob
- 2x 😞: 😄>😀> Blowjob

Markus Brill: Strategic Voting and Strategic Candidacy
Introduction to Voting

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3x:\text{ 😞 } & : & \text{erman} > \text{plus} > \text{erman} \\
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\end{align*}
\]

Pairwise comparisons:

Majority tournament:

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\]
Introduction to Voting

• A candidate is a **Condorcet winner** if he wins all pairwise comparisons

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Majority tournament:
Introduction to Voting

- A candidate is a **Condorcet winner** if he wins all pairwise comparisons
  - a rule is **Condorcet-consistent** if it selects a Condorcet winner whenever one exists

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  ‣ a majority winner is also a Condorcet winner
Single-Peaked Preferences
Single-Peaked Preferences

- Let $L$ be a linear ordering of the candidates
Single-Peaked Preferences

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• A ranking is **single-peaked w.r.t. $L$** if its corresponding preference curve has a unique peak, and preference is declining if we move away from the peak.
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Single-Peaked Preferences

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Strategic Voting & Strategic Candidacy

- Standard assumption in voting theory: set of candidates is fixed
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![Bush](image1), ![Gore](image2), ![Nader](image3)

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- Most papers on strategic candidacy assume **truthful voting**
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• Question: What if **both** voters and candidates act strategically?
  › will this lead to “better” voting outcomes?
The Candidacy Game

- Finite set of candidates $C = \{\text{👮, 👷, 🙅, 🎅, \ldots}\}$
The Candidacy Game

- Finite set of candidates \( C = \{\hat{\text{man}}, \hat{\text{woman}}, \hat{\text{black}}, \hat{\text{red}}, \ldots \} \)
- Finite set of voters \( V = \{\hat{\text{man}}, \hat{\text{woman}}, \hat{\text{black}}, \ldots \} \)
  - we assume that \(|V|\) is odd
The Candidacy Game

• Finite set of candidates $C = \{\text{👮 }, \text{👷 }, \text{💂 }, \text{🎅 }, \ldots \}$
• Finite set of voters $V = \{\text{😐 }, \text{😐 }, \text{😐 }, \ldots \}$
  ‣ we assume that $|V|$ is odd
• Both voters and candidates have preferences over candidates

\[
\begin{align*}
\text{😐 } & : \text{mı} > \text{👷 } > \text{俐} > \text{,module} \\
\text{ kè } & : \text{mı} > \text{俐} > \text{سوف} > \text{سعد}
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  😐 : 👮 > 👷 > 👉 > 👽

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- Two-stage game
  - stage 1: candidates decide to run or not
  - stage 2: voters submit ranking of running candidates
The Candidacy Game

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  - $婵 : 갯 > BODY > 🤷 > 🤴$  
  - $婵 : 但不限 > 🤴 > 🤴 > 🤷$
- Two-stage game
  - Stage 1: candidates decide to run or not
  - Stage 2: voters submit ranking of running candidates
- What are the equilibrium outcomes of this game?
  - Setting 1: single-peaked preferences and majority-consistent voting rules
Related Work

- Dutta, Jackson, & Le Breton [Econometrica 2001]: **impossibility result**

- Dutta, Jackson, & Le Breton [JET 2002]: binary voting rules
  - characterization of equilibrium outcomes for **successive elimination**

- Samejima [Jap Econ Rev 2007]: **single-peaked preferences**
  - characterization of voting rules that never give candidates incentives not to run

- Lang, Maudet, & Polukarov [SAGT 2013]: existence of **pure equilibria**
Setting

- Preferences of voters and candidates are *single-peaked*
  - there exists a Condorcet winner, denoted 🧑‍🚀
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- **Majority-consistent** voting rules
  - examples: *plurality*, plurality with runoff, instant runoff (STV)
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  - **C-equilibrium** if no candidate wants to deviate
  - **strong C-equilibrium** if no coalition of candidates wants to deviate
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- Relationships
  - $(C\text{-eq.} \land V\text{-eq.}) \iff$ subgame-perfect equilibrium
  - $(\text{strong } C\text{-eq.} \land \text{strong } V\text{-eq.}) \iff$ subgame-perfect strong equilibrium
Assumptions: single-peaked preferences, majority-consistent voting rule

Question: Which combinations of equilibrium notions guarantee that the Condorcet winner is selected?
Results

**Assumptions:** single-peaked preferences, majority-consistent voting rule

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Assumption: single-peaked preferences, majority-consistent voting rule.

Question: Which combinations of equilibrium notions guarantee that the Condorcet winner is selected?

Results:
- **Truthful Voting** (Voting where voters vote honestly) guarantees that the Condorcet winner is selected when coupled with **V-eq.** (Voting equilibrium) and **naive candidacy** (Candidacy where candidates act naïvely).
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Results:

- (“everybody running”, “truthful voting”) is a C-equilibrium and a V-equilibrium

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Strong V-equilibria

- Consider a single-peaked preference profile with Condorcet winner 🤷‍♂️ and a majority-consistent voting rule.

**Theorem:** (i) There exists a subgame-perfect strong equilibrium. (ii) In every strong V-equilibrium in which 🤷‍♂️ runs, 🤷‍♂️ wins.

**Corollary:** In every strong V-equilibrium that is also a C-equilibrium (strong or not), 🤷‍♂️ wins.
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  (i) There exists a strong C-eq. where all voters vote truthfully.
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**Assumptions:** single-peaked preferences, majority-consistent voting rule

**Question:** Which combinations of equilibrium notions guarantee that the Condorcet winner is selected?

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<th>truthful voting</th>
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**Results**

**Assumptions:** single-peaked preferences, majority-consistent voting rule

**Question:** Which combinations of equilibrium notions guarantee that the Condorcet winner is selected?

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(strong C-eq. ∧ strong V-eq.), but not subgame-perfect strong equilibrium

Example 1. Consider a preference profile with candidates a, b, c and a single voter with preferences a ∨ b ∨ c. The preferences of candidate b are given by b ∨ b c ∨ b a. The voting rule f selects the candidate ranked first by the voter whenever all three candidates run; if, however, at most two candidates run, the lexicographically last one is chosen, ignoring the voter’s vote. Let s be the strategy profile in which a and c run and the voter votes truthfully. The outcome of s under f is o_f(s) = c. We claim that s is (1) a strong C-equilibrium and (2) a strong V-equilibrium, but (3) not a subgame-perfect strong equilibrium (in fact not even a strong equilibrium).

For (1), observe that c has no incentive to participate in any deviation. The same holds for a, because the outcome will still be c if a deviates (whether b runs or not). And when all three candidates run, the outcome is a, making candidate b—the only deviator—worse off. For (2), s is a strong V-equilibrium because the voter makes his favorite candidate win in the only case where his vote has any influence. For (3), consider the following deviation. Candidate b deviates to running and the voter deviates to ranking b first whenever b runs. The outcome will change to b, and both deviators (candidate b and the voter) prefer b to c.
Example 4. Let $R$ be a single-peaked preference profile with candidates $a < b < c$ and peak distribution $(5, 0, 4)$. If $f$ is Borda’s rule, there does not exist a strong V-equilibrium (and hence no subgame-perfect strong equilibrium). To see this, consider the case where all candidates run. Observe that in any strong V-equilibrium, the outcome would have to be $a$. (Suppose the outcome is not $a$. Then, the five voters in $V_R(a)$ can jointly deviate and change the outcome to $a$. They can do this by having one voter voting $a > b > c$, and the remaining four voters voting exactly the opposite rankings of the voters in $V_R(c)$.) However, there is no strong V-equilibrium that yields outcome $a$. This is because the voters in $V_R(c)$ prefer both other alternatives to $a$, and—no matter how the voters in $V_R(a)$ vote—the voters in $V_R(c)$ can jointly deviate and achieve an outcome other than $a$. (One of $b$ and $c$ will obtain a score of at least 3 from the voters in $V_R(a)$. Without loss of generality, suppose it is $b$. Then the voters in $V_R(c)$ can all vote $b > c > a$, making $b$ the winner.)
Example 5. Let $R$ be a single-peaked preference profile with candidates $a < b < c$ and five voters: three voters have preferences $a > b > c$ and two voters have preferences $b > c > a$. The Condorcet winner is $a$. Let $f$ be the voting rule veto$^8$ and let $s$ be the strategy profile where all candidates run and all voters vote truthfully. Then, $o_f(s) = b$. Moreover, $s$ is a strong $C$-equilibrium and a strong $V$-equilibrium. The former holds because any deviation involving $a$ does not change the outcome (provided $b$ still runs), and $c$ can only change the outcome to the less preferred alternative $a$. For the latter, the only interesting case is when all three candidates run. In this case, the two voters in $V_R(b)$ have no incentive to deviate from truthful voting (their favorite candidate is winning) and there is no way for the three voters in $V_R(a)$ to jointly deviate and achieve outcome $a$. (They can change the outcome to $c$ by voting $a > c > b$, but they prefer $b$ to $c$.) It can furthermore be shown that, when all candidates run, every strong $V$-equilibrium yields outcome $b$. 
strong C-eq., truthful voting (1)

Example 6. Consider a single-peaked preference profile with candidates $a \prec b \prec c$ and five voters: three voters have preferences $a \succ b \succ c$ and two voters have preferences $b \succ c \succ a$. The Condorcet winner is $a$. Let $s$ be the strategy profile where $s_a = s_b = s_c = 1$ and $s_v$ is “truthful voting” for all voters $v$. It is easily verified that $s$ is a strong C-equilibrium and $o_{Borda}(s) = b$. In fact, it can be checked that the Condorcet winner is not chosen in any strong C-equilibrium with truthful voting. (The only other strong C-equilibrium under truthful voting has candidates $b$ and $c$ running and also yields outcome $b$.)
strong C-eq., truthful voting (2)

Example 7. Consider the following preference profile with candidates \( a, b, c \) and 14 voters.

<table>
<thead>
<tr>
<th></th>
<th>4</th>
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<tbody>
<tr>
<td>a</td>
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<td>c</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td></td>
</tr>
</tbody>
</table>

The preferences of the candidates are such that \( a \) prefers \( c \) over \( b \) and \( b \) prefers \( c \) over \( a \). Whereas the preferences of the voters are single-peaked with respect to the ordering \( a < b < c \), this is not true for the preferences of the candidates. (Therefore, this profile is not single-peaked according to the definition in Section 3.1.) The Condorcet winner is \( b \) and the Condorcet loser is \( c \). Let \( s \) be the strategy profile where all candidates run and all voters vote truthfully. It is easily verified that \( s \) is a strong C-equilibrium and \( o_{\text{plurality}}(s) = c \). In fact, “everybody running” is the only strong C-equilibrium under truthful voting.