Dynamic Proportional Sharing: A Game-Theoretic Approach

ABSTRACT

Ensuring user entitlements in computing systems with scarce and commonly preferred resources requires time sharing. We allocate resources using a novel token mechanism that supports game-theoretic desiderata such as sharing incentives and strategy-proofness. The mechanism frames the allocation problem as a repeated game. In each round of the game, users report their demands for resources and spend a token if their reported demands are granted. We finish by evaluating our mechanism on real and synthetic data.

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1 INTRODUCTION

Shared systems are defined by the competition for resources between strategic users. In this paper, we consider a community of users who share a non-profit system and its capital and operating costs. Sharing increases system utilization and amortizes its costs over more computation [11]. Examples include supercomputers for scientific computing [26], datacenters for Internet services [13, 32], and clusters for academic research [8, 9]. Note, however, that our focus excludes systems in which users explicitly pay for time on shared computational resources (i.e., infrastructure-as-a-service).

Shared systems ensure fairness by allocating resources according to entitlements, which specify each user's minimum share of system resources relative to others [23, 25, 33]. Entitlements are dictated by exogenous factors such as users' contributions to the shared system or priorities within the organization. Entitlement-based allocations are deterministic with respect to the minimum share, unlike priority-based allocations, which vary based on other activity in the system. Moreover, allocations are efficient as excess, under-utilized resources are redistributed to users in proportion to their entitlements.

In this paper, we focus on dynamic proportional sharing between users. Dynamic sharing is required because user utility varies over time. We consider utilities that, in each time period, can be modeled as a step function—users derive high utility up until some resource allocation and derives low utility beyond that allocation. The high-low formulation is appropriate for varied resources such as processor cores, cache and memory capacity, or virtual machines in a datacenter.

For example, suppose computation is structured as a multi-stage job with different degrees of task parallelism within each stage (e.g., Apache Spark). In each stage, the user derives high utility when additional processors permit her to dequeue more tasks and increase throughput. And she derives low utility when the queue is empty and parallelism is exhausted. Low utility could be zero or some small, marginal value, perhaps because the user can deploy extra resources to clone tasks and guard against stragglers or failures. An allocation mechanism should ensure users' entitlements across time while assigning resources to computational stages that benefit most.

Guaranteeing entitlements and redistributing under-utilized resources are difficult when users are strategic. The allocation mechanism does not know and must extract users' utilities, which are private information. Strategic users act selfishly to pursue their own objectives given their limited resources. Users will determine whether misreporting demands can improve their performance even at the expense of others in the system. For example, an agent is likely to over-report her demand in the current time period to obtain more resources, unless doing so leads to a reduction in the resources allocated to her in later periods. Although rigidly enforcing entitlements in every time period is strategy-proof, its efficiency is poor and ignores the advantages of dynamic sharing across time.

We seek allocation mechanisms that satisfy strategy-proofness (SP), which ensures that no user benefits by misreporting her demand for resources. Strategy-proofness enhances efficiency by allocating resources according to each user's true utility from them. It also reduces the cognitive load on users by eliminating the need to optimally fabricate resource demands or preemptively respond to misreports by other participants in the system.

Strategy-proofness complements sharing incentives (SI), which ensures that users perform at least as well as they would have without participating in the allocation mechanism (i.e., using their own resources as a smaller, private system). With sharing incentives, users would willingly federate their resources and manage them according to the commonly agreed upon policy; moreover, strategy-proofness would ensure that users have no incentive to circumvent that policy. In some situations, resources can become federated even if sharing incentives fail to hold. For example, an organization could force users to share resources by withholding funding otherwise. However, strategy-proofness remains important for enhancing system efficiency and reducing users' reporting burdens.

We propose allocation mechanisms for dynamic proportional sharing to address limitations in existing approaches. We begin by proving that policies used in state-of-the-art schedulers [1–3] fail to satisfy SP or SI. We then propose two alternative mechanisms. First, the T-period mechanism satisfies SP and SI but with low efficiency. Second, the token mechanism satisfies only SP but guarantees at least 50% of SI's performance. Tokens enable theoretical guarantees on SP and SI. Performance is comparable to that of state-of-the-art mechanisms, and sharing incentives, in practice, is much better than the lower bound (e.g., 98% of SI's performance, on average).

2 PRELIMINARIES

Consider a dynamic system with n agents and R discrete rounds. Agent i contributes $e_i > 0$ units of a resource at each round, which we refer to as her *endowment*. That is, e_i is agent i's contribution

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to the federated system and this does not vary over time. Let $[n] = \{1, \ldots, n\}$. Let $E = \sum_{i \in [n]} e_i$ denote the total number of units to be allocated at each round. At round r, agent i has a true demand of $d_{i,r} \geq 0$ units and reports a demand of $d'_{i,r} \geq 0$. Let $\mathbf{d}'_i = (d'_{i,1}, \ldots, d'_{i,R})$ denote the vector of agent i's reports, and $\mathbf{d'}_{-i}$ denote the reports of all agents other than i. A *Repeated Allocation Mechanism M* assigns each agent an allocation $a^M_{i,r}(\mathbf{d}'_i, \mathbf{d'}_{-i})$ using only information from the first r entries in the demand vectors. We will often write simply $a_{i,r}$ when the exact mechanism and the demands are clear from context. Let $\mathbf{a^M_i}(\mathbf{d}'_i, \mathbf{d'}_{-i})$ (often simply $\mathbf{a_i}$) denote the vector of agent i's allocations. Agents have high (H) utility per resource up to their demand, and low (L) utility per resource that exceeds their demand. Formally, the utility of agent i at round r for $a_{i,r}$ units is denoted by $u_{i,r}(a_{i,r})$ and modeled as the following.

$$u_{i,r}(a_{i,r}) = \begin{cases} a_{i,r}H & \text{if } a_{i,r} \leq d_{i,r}, \\ d_{i,r}H + (a_{i,r} - d_{i,r})L & \text{if } a_{i,r} > d_{i,r}. \end{cases}$$

For simplicity, we assume H and L are the same for all agents, but all our results extend to the case where agents have different values of H and L (with the exception of Section 5.3). While resources and demands are discrete, we allow the allocations $a_{i,r}$ to be real-valued. The interpretation of a real-valued allocation can be thought of as probabilistic – the realized allocation is a random allocation where agent i is allocated $a_{i,r}$ resources in expectation (this is always possible as a result of the Birkhoff-von Neumann theorem [12]) Agent i's overall utility after q rounds for allocation $\mathbf{a_i}$ is calculated additively as follows.

$$u_{i,q}(\mathbf{a_i}) = \sum_{r=1}^{q} u_{i,r}(a_{i,r}).$$

We do not consider discounting for simplicity of presentation, however our mechanisms readily extend to the case where agents discount their utilities over time.

In this paper we focus on three fundamental game-theoretic desiderata: strategy-proofness, sharing incentives, and efficiency. First, strategy-proofness says that agents never benefit from lying about their demands. In other words, agent i's utility decreases if she reports any demand, d'_i, other than her true demand, d_i.

Definition 2.1. Mechanism M satisfies *strategy-proofness (SP)* if

$$u_{i,R}(\mathbf{a^M_i}(\mathbf{d_i}, \mathbf{d'_{-i}})) \geq u_{i,R}(\mathbf{a^M_i}(\mathbf{d'_i}, \mathbf{d'_{-i}})) \qquad \forall i, \forall \mathbf{d_i}, \forall \mathbf{d'_i}, \text{ and } \forall \mathbf{d'_{-i}}.$$

Next, sharing incentives says that by participating in the mechanism, agents receive at least the utility they would have received by not participating.

Definition 2.2. Mechanism M satisfies *sharing incentives (SI)* if

$$u_{i,R}(\mathbf{a^M_i}(\mathbf{d_i}, \mathbf{d'_{-i}})) \geq u_{i,R}(\mathbf{e_i}) \qquad \forall i, \forall \mathbf{d_i}, \text{ and } \forall \mathbf{d'_{-i}}.$$

We also define a relaxed notion α-sharing incentives, which says that every agent gets at least an α fraction of the utility that she would have received without taking part in the mechanism.

Definition 2.3. Mechanism M satisfies *α-SI* if

$$u_{i,R}(\mathbf{a^M_i}(\mathbf{d_i}, \mathbf{d'_{-i}})) \geq \alpha\, u_{i,R}(\mathbf{e_i}) \qquad \forall i, \forall \mathbf{d_i}, \text{ and } \forall \mathbf{d'_{-i}}.$$

Finally, efficiency says that all resources should be allocated, and an agent with L valuation should never receive a resource while there are agents with H valuation for that resource.

Definition 2.4. Mechanism M satisfies *efficiency (E)* if

$$\sum_{i \in [n]} a^M_{i,r} = E,$$

and if $a^M_{i,r} > d'_{i,r}$ for some agent i, then $a^M_{j,r} \geq d'_{j,r}$ for all agents.

Note that efficiency is relative to the agents' *reports*, not their actual valuations, which are hidden from the mechanism. Therefore, in situations where agents lie about their valuations, it is possible that even an efficient mechanism allocates a unit inefficiently with respect to the actual valuations. With this in mind, there is little value in a mechanism that is efficient but not strategy-proof. Similarly, if a mechanism does not satisfy SI, then agents may not want to participate in it. So an efficient mechanism that does not satisfy SI may not actually exhibit efficiency gains in practice because agents choose not to participate. We note that in some contexts, SI may not be of concern because agents are forced to participate, or be willing to take a chance if it looks like they may gain a lot and are not likely to lose much from participation.

Due to space constraints, many proofs are omitted and appear in the appendix.

3 EXISTING POLICIES AND THEIR PROPERTIES

In this section we examine properties of some existing dynamic allocation policies. The first policy that we consider in this paper is the static allocation policy which statically allocates to agents their endowments. The static allocations are identical to no sharing of resources. We use the static allocation policy as a benchmark to evaluate other policies.

The second policy that we consider is the (weighted) max-min fairness policy. Max-min is one of the most widely used policies in computing systems. It is deployed in many state-of-the-art datacenter schedulers, such as Hadoop Fair Scheduler [2], Hadoop Capacity Scheduler [1] and Spark Dynamic Allocator [3]. And, it has been extensively studied in the literature [17, 18, 30].

A dynamic allocation mechanism could deploy max-min policy for two different objectives: maximizing the minimum accumulated allocations up to a round, or maximizing the minimum allocation at each round, independent of previous rounds. We call the first mechanism *Dynamic Max-Min (DMM)* and the second mechanism *Static Max-Min (SMM).*

- **Dynamic Max-Min (DMM).** At each round, resources are allocated to maximize the minimum weighted cumulative allocations in lexicographical order, subject to the constraint that no resource is allocated to an agent with low valuation as long as there are agents with high valuation. That is, at round r, DMM maximizes the minimum weighted cumulative allocation, $\min_{i \in [n]} \sum_{q=1}^{r} a_{i,q}/e_i$; subject to this constraint, it maximizes the second lowest weighted cumulative allocation, and so on.
- **Static Max-Min (SMM).** At each round, resources are allocated to maximize the minimum weighted allocations

at that round in a lexicographical order, subject to the constraint that no resource is allocated to an agent with low valuation as long as there are agents with high valuation. That is, at round r, resources are allocated to maximize the minimum weighted allocations, $a_{i,r}/e_i$, in lexicographical order, such that all agents' demands are met before any agent is allocated a resource above their demand. Note that in SMM, agents are guaranteed to receive their demands as long as it is less than or equal to their endowment. Agents with demands more than their endowments receive extra resources from agents with demands less than their endowments. Note also that unlike DMM, SMM allocates resources locally at round r, regardless of agents' allocations prior to round r.

3.1 Special Case: L=0

When $L = 0$, one might think that agents do not have any incentive to misreport their demands. However, we show that DMM fails to satisfy SI and SP.

Theorem 3.1. *DMM violates SI, even when $L = 0$.*

Proof. Suppose that $R = 10$ and there are three agents, each with $e_i = 3$. For all rounds $r \neq 10$, the demands are $d_{1,r} = 1, d_{2,r} = 2, d_{3,r} = 6$. For rounds $r = 1, \dots, 9$, each agent gets allocated exactly their demand. After round 9, utilities for agents 1, 2 and 3 are $9H$, $18H$ and $54H$, respectively. At round 10, demands are $d_{1,10} = 9, d_{2,10} = 9, d_{3,10} = 6$. DMM allocates all 9 units to agent 1, which maximizes the minimum weighted cumulative allocation. Consider agent 2. Under DMM, agent 2's allocation is $a_{2,r} = 2$ for all $r \neq 10$ and $a_{2,10} = 0$. If she had not participated in the mechanism, then she would have obtained the same utility in each round $r \neq 10$, but a strictly higher utility in round $r = 10$. □

Theorem 3.2. *DMM violates SP, even when $L = 0$ [6].*

Proof. Consider three agents with equal endowments $m_1 = m_2 = m_3 = 1$ sharing three units of a resource for three rounds. The demand of agent 1 is 3 for all three rounds. Agent 2's demand is 3 for rounds 1 and 3 and 0 for round 2. And agent 3 has a demand of 3 for round 2 and 0 for rounds 1 and 3. Agent 1 achieves utility of $3.375H$ by truthful reporting. If agent 1 misreports 0 for round 1, her utility would increase to $3.75H$. □

Next, we show that SMM satisfies SI, SP, and efficiency.

Theorem 3.3. *SMM satisfies SI, SP, and E when $L = 0$.*

Proof. SMM satisfies E by definition, since it either completely fulfills all demands, or allocates all resources to agents that value them highly. Next, we prove that SMM satisfies SP. First, note that allocations at round r are independent of allocations at previous rounds. Next, suppose that agent i reports $d'_{i,r} > d_{i,r}$ at round r. Let $a'_{i,r}$ and $a_{i,r}$ denote the allocations i receives at round r for reporting $d'_{i,r}$ and $d_{i,r}$, respectively. If $a_{i,r} \geq d_{i,r}$, then agent i already receives her highest utility possible, $d_{i,r}H$ (because $L = 0$), and she cannot benefit from misreporting. If $a_{i,r} < d_{i,r}$, then by definition $a_{j,r} \leq d_{j,r}$ for all agents $j \neq i$. Also, $a_{i,r}/e_i$ is greater than or equal to $a_{j,r}/e_j$ for all $j \neq i$. This is true because if there was an agent j with $a_{j,r}/e_j$ greater than $a_{i,r}/e_i$, then SMM would decrease $a_{j,r}$ and increase $a_{i,r}$. Now, suppose for a contradiction that $a'_{i,r} > a_{i,r}$. Then, there should be an agent ℓ with $a'_{\ell,r} < a_{\ell,r} \leq d_{\ell,r}$. Therefore, we have:

$$a'_{\ell,r}/e_\ell < a_{\ell,r}/e_\ell \leq a_{i,r}/e_i < a'_{i,r}/e_i$$

This is contradiction because SMM could decrease $a'_{i,r}$ and increase $a'_{\ell,r}$. A similar argument shows that reporting $d'_{i,r} < d_{i,r}$ could only decrease agent i's utility. Finally, to see that SMM satisfies SI, note that an agent can guarantee herself at least e_i resources (her utility for not participating) at each round by reporting $d'_{i,r} = e_i$ for all r. By SP, truthful reporting achieves at least this utility. Therefore, truthful reporting achieves at least as much utility as not participating in SMM, which proves sharing incentives. □

3.2 General Case: L>0

We now consider the general setting where an agent's low valuation is still positive. Unfortunately, SMM no longer retains its good properties from the $L = 0$ case. Agents are no longer indifferent to giving up low-valued resources, and may lie to receive them.

Theorem 3.4. *When $L > 0$, SMM violates SP and SI.*

Proof. Consider an instance with 2 agents with $e_i = 1$ and 1 round. Agent 1 has demand 2 and agent 2 has demand 0. SMM allocates both resources to agent 1 and zero resources to agent 2. However, had agent 2 not participated in the mechanism, she would have received one resource and got utility $L > 0$. Similarly, had she misreported her demand to be 1, she would have received one resource and got utility $L > 0$. □

Indeed, in this general setting, no mechanism can simultaneously satisfy efficiency and either of the two other desired properties.

Theorem 3.5. *When $L > 0$, there is no dynamic mechanism that satisfies α-SI and E, for any $\alpha > 0$.*

Proof. Consider an instance with two agents with $e_i = 1$ and 1 round. Agent 1 has demand 2 and agent 2 has demand 0. Efficiency dictates that we allocate both resources to agent 1, which would violate α-SI for agent 2 for any $\alpha > 0$. □

Theorem 3.6. *When $L > 0$, there is no dynamic mechanism that satisfies SP and E.*

Proof. Consider an instance with two agents, each of whom has endowment $e_i = 1$, and a single round. Both agents have demand 0. By efficiency, the mechanism must allocate all the resources, so at least one agent receives $a_{i,1} > 0$. Suppose without loss of generality that $a_{1,1} > 0$. Therefore, $a_{2,1} < 2$. But suppose that agent 2 misreports her demand $d'_{2,1} = 2$. Now, by efficiency, the mechanism must allocate both resources to agent 2, which is an improvement over her utility from reporting truthfully. □

Note that SP and SI are compatible. The static mechanism satisfies SP and SI, since agents have no incentives to misreport (since the allocation does not depend on the reports) and they receive their fair share of resources. Unfortunately, this mechanism does not extract any benefit from sharing. A natural direction is to search for a different mechanism that satisfies SP and SI, that, while it can not satisfy efficiency, does exhibit at least some gains from sharing.

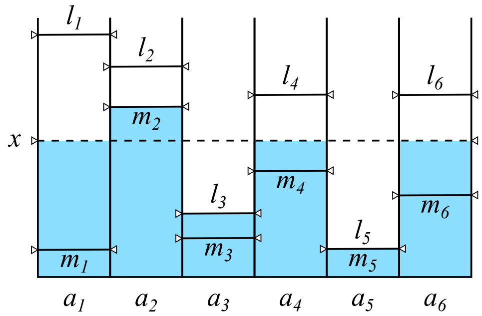

Figure 1: Proportional Sharing With Constraints. There are six agents with equal weights. The allocations are represented by the height of the corresponding blue vertical bar. The allocations can be thought of as the 'most equal' allocations, subject to no agent receiving less than her minimum constraint or more than her maximum.

4 ACHIEVING SHARING INCENTIVES AND STRATEGY PROOFNESS

4.1 The Proportional Sharing With Constraints Procedure

All of the mechanisms that we present in the remainder of this paper have at their core a procedure that we call *Proportional Sharing With Constraints (PSWC)*. The idea is to allocate some amount of resources among agents proportionally to their (exogenous) weights, subject to (agent-dependent) minimum and limit constraints: (1) each agent has to receive at least her minimum allocation, and (2) each agent should not receive more than her limit allocation.

Formally, PSWC takes as input an amount to allocate A, limit allocations $\mathbf{l} = (l_1, \dots, l_n)$, minimum allocations $\mathbf{m} = (m_1, \dots, m_n)$, and weights $\mathbf{w} = (w_1, \dots, w_n)$. It outputs a vector of allocations $\mathbf{a} = (a_1, \dots, a_n)$ defined as the solutions to the following program.

$$
\begin{aligned}
&\text{Minimize } x, \\
&\qquad \text{s.t. } a_i/w_i \le x && \text{if } m_i < a_i \le l_i, \\
&\qquad\qquad a_i \le l_i && \forall i, \\
&\qquad\qquad a_i \ge m_i && \forall i, \\
&\qquad\qquad \sum_{i\in[n]} a_i = A.
\end{aligned}
$$

PSWC is illustrated in Figure 1. The program can be solved in $O(n \log(n))$ time by the Divvy algorithm [21].

Lemma 4.1. *For every agent i, $a_i = \max(m_i, \min(l_i, xw_i))$ under PSWC.*

Proof. First, we show that if $m_i < xw_i$, then $a_i = \min(l_i, xw_i)$. To see this, note that if $a_i > \min(l_i, xw_i)$, then at least one constraint is violated. If $a_i < \min(l_i, xw_i)$, then there exists at least an ℓ such that $a_\ell = xw_\ell$, because otherwise, x is not optimal. In this case, a_i can be increased while a_ℓ for all ℓ with $a_\ell = xw_\ell$ decreases. This allows for a smaller value of x, which This contradicts the optimality of x.

Next, we show that if $m_i \ge xw_i$, then $a_i = m_i$. Since a_i cannot be less than m_i, if a_i is not equal to m_i, then $a_i > m_i$, which means $a_i > xw_i$. However, since $a_i > m_i$, the first constraint dictates that $a_i \le xw_i$, a contradiction. Combining these two cases gives the desired result. □

Our main mechanisms all have similar forms. Firstly, agents will always be allocated exactly the same number of resources as they contributed to the system (over the entire R rounds). This can be seen as a fairness primitive in its own right, but is primarily a design feature that will help us obtain the properties that we want.

When allocating resources at any given round r, there are two cases. The first case is when the total demand for resources is at least as high as the total supply. In this case, resources should be allocated proportionally among all agents who want them.

The second case is when the total demand for resources is lower than the total supply. In this case, we can give each agent their full demand, and remaining resources will be allocated to some agents who have low valuation for them. In both cases, we run PSWC with some mechanism-dependent minimum and limit allocation constraints to allocate the supply. Our mechanisms are determined primarily by how we set the minimum and limit allocation constraints.

4.2 T-Period Mechanism

The T-period mechanism splits the rounds into periods of length $2T$.[1] For the first T rounds of each period, we allow the agents to 'borrow' unwanted resources from others. In the last T rounds of each period, the agents 'pay back' the resources, so that their cumulative allocation across the entire period is equal to their endowment, $2Te_i$.

The allocations in the second set of T rounds are independent of reports and determined completely by the allocations in the first set of T rounds. Note that because the number of resources that an agent i can pay back over T rounds is bounded by Te_i, we allow an agent to borrow at most Te_i resources (*i.e.*, receive at most $2Te_i$ resources) over the first T rounds of a period.

In Algorithm 1, each agent i has a borrowing limit, b_i which is defined to be the maximum amount of resources that agent i could borrow in whatever remains of the first T rounds of each period. For our analysis, we denote the value of b_i at the start of round r by $b_{i,r}$. At the beginning of each period, we set $b_{i,r}$ to be Te_i, because agent i can at most pay back her whole endowment, e_i, at every T 'payback' rounds. We also define $\bar{d}_i$ to be the allocatable demand of agent i at each round of the first T rounds (and refer to $\bar{d}'_{i,r}$ in our analysis as agent i's allocatable demand at round r). At each round r, the allocatable demand of agent i is the minimum of her reported demand $d'_{i,r}$, and her endowment plus her borrowing limit, $e_i + b_{i,r}$.

If the total allocatable demand is at least as high as the total supply, then we run PSWC with the minimum allocation for each agent set to 0, and the limit allocation set to $\bar{d}_i$. This way, resources are allocated proportionally among all agents that want them. If the total allocatable demand is less than the total supply, then we

[1] For convenience, we suppose that R is a multiple of $2T$. If this is not the case, we can adapt the mechanism by returning each agent their endowment for any leftover rounds.

Algorithm 1 T-Period Mechanism

Input. Agents' reported demands, $\mathbf{d'}$, and their endowments, $\mathbf{e}$
Output. Agents' allocations, $\mathbf{a}$

for $r \in \{1, \ldots, R\}$ **do**
  if $(r \bmod 2T) = 1$ **then**
    $\mathbf{b} \leftarrow T\mathbf{e}$ ▷ b_i is the amount that i is able to borrow
    $\mathbf{y} \leftarrow \mathbf{0}$ ▷ resources received so far this period.
  end if
  if $1 \leq (r \bmod 2T) \leq T$ **then**
    $\bar{\mathbf{d}} \leftarrow \min(\mathbf{d'_{\cdot,r}}, \mathbf{e} + \mathbf{b})$ ▷ $\bar{d}_i$ is i's allocatable demand
    $D \leftarrow \sum_{i \in [n]} \bar{d}_i$
    if $D \geq E$ **then**
      $\mathbf{a}_{\cdot,\mathbf{r}} \leftarrow \text{PSWC}(A = E, \mathbf{l} = \bar{\mathbf{d}}, \mathbf{m} = \mathbf{0}, \mathbf{w} = \mathbf{e})$
    else
      $\mathbf{a}_{\cdot,\mathbf{r}} \leftarrow \text{PSWC}(A = E, \mathbf{l} = \mathbf{e} + \mathbf{b}, \mathbf{m} = \bar{\mathbf{d}}, \mathbf{w} = \mathbf{e})$
    end if
    $\mathbf{y} \leftarrow \mathbf{y} + \mathbf{a}_{\cdot,\mathbf{r}}$
    $\mathbf{b} \leftarrow \mathbf{b} - \max(\mathbf{0}, \mathbf{a}_{\cdot,\mathbf{r}} - \mathbf{e})$
  else
    $\mathbf{a}_{\cdot,\mathbf{r}} \leftarrow \frac{1}{T}(2T\mathbf{e} - \mathbf{y})$
  end if
end for

can give each agent their full allocatable demand, and we also need to allocate extra resources to some agents. To do this, we run PSWC with the minimum allocation for each agent set to be their allocatable demand, and their limit allocation set to their endowment plus their borrowing limit. The idea is that all agents should get their allocatable demand, while agents with low demand should also be assigned some low-valued resources (since no agent has high valuation for these extra resources).

We illustrate the T-period mechanism with an example.

Example 4.2. Consider an instance with three agents and four rounds. Each agent has endowment $e_i = 1$. Suppose that agents' (truthful) reports are given by the following table:

	$d_{i,1}$	$d_{i,2}$	$d_{i,3}$	$d_{i,4}$
$i = 1$	3	1	1	0
$i = 2$	0	2	1	2
$i = 3$	0	0	0	4

When $T = 1$, agents can 'borrow' resources at odd rounds, and 'pay back' those resources at even rounds. Therefore, maximum allocatable demand for each agent and at each round is 2, because the 'payback' period only has one round. The *1-period (1P)* mechanism allocates resources as follows.

	$a^{1P}_{i,1}$	$a^{1P}_{i,2}$	$a^{1P}_{i,3}$	$a^{1P}_{i,4}$
$i = 1$	2	0	1	1
$i = 2$	0.5	1.5	1	1
$i = 3$	0.5	1.5	1	1

At round 1, agent 1 wants 2 extra resources in addition to her endowment. However, under 1P, she only can afford 1 extra resource. In this case, she borrows 0.5 resources from agent 2 and 0.5 resources from agent 3. At round 2, agent 1 pays back agent 1 and 2 and receives zero resources. As can be seen, with $T = 1$, the mechanism rigidly forces agents to pay back resources right after they borrow them. Agent 1 would prefer to get her high-valued resource at round 2 and delay paying back agent 2 and 3 to the last round where her true demand is zero (and agent 3 would prefer to be paid back then, also, since that is the only round in which she has non-zero demand).

To see how increasing T makes the mechanism more flexible, consider $T = 2$ for the same example. The *2-period (2P)* mechanism allocates resources as follows.

	$a^{2P}_{i,1}$	$a^{2P}_{i,2}$	$a^{2P}_{i,3}$	$a^{2P}_{i,4}$
$i = 1$	3	1	0	0
$i = 2$	0	2	1	1
$i = 3$	0	0	2	2

In this case, agent 2 is allowed to borrow 2 extra resources at first two rounds. She borrows these two resources at the first round from agent 2 and 3, and pays them back at round 3 and 4.

Since the T-period mechanism increases flexibility over the static mechanism, it provides some gains from sharing. We would expect that increasing T, in general, will improve efficiency as it allows for 'borrowed' resources to be spent more flexibly. In the following subsection, we show that these efficiency gains do not come at the expense of SI or SP, but only for $T \leq 2$.

4.3 Axiomatic Properties of the T-Period Mechanism

We first prove a lemma characterizing the allocations of the T-Period mechanism that we will use throughout this section.

LEMMA 4.3. *Let x denote the objective value of a call to PSWC. Suppose that $1 \leq (r \mod 2T) \leq T$. If $D \geq E$, then $a_{i,r} = \min(e_i + b_i, d'_{i,r}, xe_i)$. If $D < E$, then $a_{i,r} = \min(e_i + b_i, \max(d'_{i,r}, xe_i))$.*

PROOF. If $D \geq E$, substituting the relevant terms into Lemma 4.1 gives us the following.

$$a_{i,r} = \max(0, \min(\min(d'_{i,r}, e_i + b_i), xe_i)) = \min(e_i + b_i, d'_{i,r}, xe_i).$$

If $D < E$, then again by substituting into Lemma 4.1 we have the following.

$$\begin{aligned} a_{i,r} &= \max(\min(e_i + b_i, d'_{i,r}), \min(e_i + b_i, xe_i)) \\ &= \min(e_i + b_i, \max(d'_{i,r}, xe_i)). \end{aligned}$$

The final equality, $\max(\min(A, B), \min(A, C)) = \min(A, \max(B, C))$ can easily be checked to hold case by case for any relative ordering of A, B, and C. □

To prove strategy-proofness of the 1-Period and 2-Period mechanisms, we will show that no agent has an incentive to report $d'_{i,r} \neq d_{i,r}$ for any round r. We will often be considering parallel universes: one in which agent i misreports $d'_{i,r}$ and one in which she truthfully reports $d_{i,r}$. Allocations and borrowing limits in the former case will be denoted by a' and b' respectively, and by a and b in the latter case.

Since the T-Period mechanism resets every $2T$ rounds, we can assume without loss of generality that $R = 2T$ for the sake of reasoning about SP and SI. For rounds $r > T$, the allocations depend completely on the allocations at earlier rounds, and not on the agents' reports, so there is clearly no benefit to a agent for misreporting in these rounds. It remains to show that reporting $d'_{i,r} = d_{i,r}$ is optimal for rounds $r \leq T$.

We first show a monotonicity lemma, which says that, all else being equal, if agent i increases her report then her allocation will (weakly) increase, and all other agents' allocations will (weakly) decrease.

Lemma 4.4. *Let $a_{i,r}$ and $a'_{i,r}$ denote the allocations of agent i at round r when she reports $d_{i,r}$ and $d'_{i,r}$ respectively, holding fixed the reports of all agents $j \neq i$ and agent i's reports on all rounds other than r. If $d'_{i,r} < d_{i,r}$ then $a'_{i,r} \leq a_{i,r}$, and $a'_{j,r} \geq a_{j,r}$ for all $j \neq i$.*

Suppose that i reports $d'_{i,r} \neq d_{i,r}$ for some round r, but this misreport does not change i's allocation (that is, $a'_{i,r} = a_{i,r}$). Then, by Lemma 4.4, $a'_{j,r} = a_{j,r}$ for all $j \neq i$ also. Therefore, i's misreport has not changed the allocations at round r. Since all future rounds take into account allocations at previous rounds but not reports, i's misreport has had no effect on the allocations in any round. Thus, i did not benefit from this misreport. We therefore assume that $a'_{i,r} \neq a_{i,r}$ for any round r where i reports $d'_{i,r} \neq d_{i,r}$ in the remainder of this section.

The next lemma says that if i obtains fewer resources from misreporting at round r, then those resources are all high-valued resources.

Lemma 4.5. *Hold the reports of all agents $j \neq i$ fixed, and the reports of agent i on all rounds other than r fixed. If i reports $d'_{i,r} < d_{i,r}$ and receives $a'_{i,r} < a_{i,r}$, then $a_{i,r} \leq d_{i,r}$.*

Proof. Note that $D' \leq D$, since $d'_{i,r} < d_{i,r}$ and $d'_{j,r} = d_{j,r}$ for all agents $j \neq i$. If $E \leq D$, then by the definition of the T-period mechanism we have:

$$a_{i,r} \leq \bar{d}_{i,r} = \min(e_i + b_i, d_{i,r}) \leq d_{i,r}.$$

Next, assume that $D' \leq D < E$. Then $a'_{i,r} < a_{i,r}$ implies that there is at least an agent j with $a'_{j,r} > a_{j,r}$. In the proof of Lemma 4.4 we show that if $x' < x$, then $a'_{k,r} = a_{k,r}$ for all k. Therefore, it has to be the case that $x' \geq x$. By the definition of the T-period mechanism and Lemma 4.1, $a_{i,r} = \min(e_i + b_{i,r}, \max(d_{i,r}, xe_i))$ and $a'_{i,r} = \min(e_i + b_i, \max(d'_{i,r}, x'e_i))$. It is easy to see that if $d_{i,r} < xe_i$, then $a'_{i,r} \geq a_{i,r}$, which is a contradiction. Therefore, we have:

$$a_{i,r} = \min(e_i + b_i, \max(d_{i,r}, xe_i)) = \min(e_i + b_i, d_{i,r}) \leq d_{i,r}.$$

□

Corollary 4.6. *Hold the reports of all agents $j \neq i$ fixed, and the reports of agent i on all rounds other than r fixed. If i reports $d'_{i,r} < d_{i,r}$ and receives $a'_{i,r} < a_{i,r}$, then $u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r}) = H(a_{i,r} - a'_{i,r})$.*

As a corollary we obtain a formula for the difference between the utility that agent i receives at round r under truthful reporting and misreporting, when i gets fewer resources in the misreported instance.

Proof. Because $a'_{i,r} < a_{i,r} \leq d_{i,r}$, we can substitute the utility values from Equation 2,

$$u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r}) = a_{i,r}H - a'_{i,r}H = H(a_{i,r} - a'_{i,r}).$$

□

The next lemma and corollary complement Lemma 4.5 and Corollary 4.6 in the case where i receives more resources in the misreported instance than the truthful instance at round r. The proofs are deferred to the appendix.

Lemma 4.7. *Hold the reports of all agents $j \neq i$ fixed, and the reports of agent i on all rounds other than r fixed. If i reports $d'_{i,r} > d_{i,r}$ and receives $a'_{i,r} > a_{i,r}$, then $a_{i,r} \geq d_{i,r}$.*

Corollary 4.8. *Hold the reports of all agents $j \neq i$ fixed, and the reports of agent i on all rounds other than r fixed. If i reports $d'_{i,r} > d_{i,r}$ and receives $a'_{i,r} > a_{i,r}$, then $u_{i,r}(a'_{i,r}) - u_{i,r}(a_{i,r}) = L(a'_{i,r} - a_{i,r})$.*

We now show that misreporting in round T is never beneficial to an agent.

Lemma 4.9. *A agent never improves her utility by reporting $d'_{i,T} \neq d_{i,T}$.*

Proof. Suppose first that agent i reports $d'_{i,T} < d_{i,T}$. Then, by Lemma 4.4, $a'_{i,T} \leq a_{i,T}$. If $a'_{i,T} = a_{i,T}$, then the misreport has had no effect on the allocations, since the allocation at rounds $r \leq T$ is unchanged, and the allocations at rounds $r > T$ depend only on the allocations at rounds $r \leq T$, not the reports. So assume that $a'_{i,T} = a_{i,T} - k$ for some $k > 0$. By the definition of the T-Period mechanism, i's allocation increases by $\frac{k}{T}$ for each of rounds $T+1, \ldots, 2T$. The difference between her utility from truthfully reporting at round T and from misreporting at round T is given by

$$\begin{aligned}
u_{i,R}(\mathbf{a_i}) - u_{i,R}(\mathbf{a'_i}) &= \sum_{r=1}^{R} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})\right) \\
&= u_{i,T}(a_{i,T}) - u_{i,T}(a'_{i,T}) + \sum_{r=T+1}^{2T} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})\right) \\
&= kH + \sum_{r=T+1}^{2T} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a_{i,r} + \frac{k}{T})\right) \\
&\geq kH - kH = 0
\end{aligned}$$

where the second transition follows from the assumption that $d'_{i,r} = d_{i,r}$ for all rounds $r < T$, the third transition from Corollary 4.6, and the final transition because the extra resources received in the misreported case for rounds $r > T$ can each be worth at most H to i.

Next suppose that agent i reports $d'_{i,T} > d_{i,T}$. Then, by Lemma 4.4, $a'_{i,T} \geq a_{i,T}$. As before, assume that $a'_{i,T} \neq a_{i,T}$. That is, $a'_{i,T} = a_{i,T} + k$ for some $k > 0$. By the definition of the T-Period mechanism, i's allocation decreases by $\frac{k}{T}$ for each of rounds $T+1, \ldots, 2T$. The difference between her utility from truthfully reporting at round T

and from misreporting at round T is given by

$$\begin{aligned} u_{i,R}(\mathbf{a_i}) - u_{i,R}(\mathbf{a'_i}) &= \sum_{r=1}^{R} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})\right) \\ &= u_{i,T}(a_{i,T}) - u_{i,T}(a'_{i,T}) + \sum_{r=T+1}^{2T} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})\right) \\ &= -kL + \sum_{r=T+1}^{2T} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a_{i,r} - \frac{k}{T})\right) \\ &\geq -kL + kL = 0 \end{aligned}$$

where the second transition follows from the assumption that $d'_{i,r} = d_{i,r}$ for all rounds $r < T$, the third transition from Corollary 4.8, and the final transition because the extra resources received in the truthful case for rounds $r > T$ are each worth at least L to i. □

As a corollary, we immediately have that the 1-Period mechanism is strategy-proof, because misreporting at round $r = 1 = T$ is not beneficial, and misreporting at round $r = 2 > T$ is not beneficial by our earlier argument.

Corollary 4.10. *The 1-Period mechanism satsifies SP.*

Our next lemma is a monotonicity statement for the borrowing limits: if i's borrowing limit at round r increases, and all other agents' borrowing limits decrease, then i's allocation (weakly) increases and all other agents' allocations (weakly) decrease.

Lemma 4.11. *Suppose that $r \leq T$. If $b'_{i,r} \geq b_{i,r}$ and $b'_{j,r} \leq b_{j,r}$ for all $j \neq i$, and $d'_{k,r} = d_{k,r}$ for all agents k, then $a'_{i,r} \geq a_{i,r}$ and $a'_{j,r} \leq a_{j,r}$ for all agents $j \neq i$.*

We now show that the 2-Period mechanism is strategy-proof. Some details are omitted; the full proof appears in the appendix.

Theorem 4.12. *The 2-Period mechanism satisfies SP.*

Proof Sketch. By Lemma 4.9, no agent can benefit by reporting $d'_{i,2} \neq d_{i,2}$. Similarly, no agent can benefit by reporting $d'_{i,r} \neq d_{i,r}$ for $r \in \{3, 4\}$, because the 2-Period mechanism ignores reports for those rounds. We may therefore assume that $d'_{i,r} = d_{i,r}$ for all agents i and all rounds $r \geq 2$.

We will show that an agent cannot benefit from reporting $d'_{i,1} < d_{i,1}$. The proof that reporting $d'_{i,1} > d_{i,1}$ is not beneficial is very similar. If $a'_{i,1} = a_{i,1}$, then $a'_{j,1} = a_{j,1}$, by Lemma 4.4. Therefore, the allocations are unchanged for all rounds i, as the 2-Period mechanism takes into account allocations at earlier rounds, but not reports, and the allocations at round 1 are the same in the truthful and misreported instances. We therefore assume that $a_{i,1} = a'_{i,1} + k$, for some $k > 0$. This implies that $b_{i,2} = b'_{i,2} - k_i$, for some $k_i \leq k$. By Lemma 4.5, i receives kH more utility in round 1 under truthful reporting than under misreporting. For every $j \neq i$, $a_{j,1} \leq a'_{j,1}$, and $b_{j,2} = b'_{j,2} + k_j$, where $\sum_{j \neq i} k_j \leq k$. By Lemma 4.11, $a'_{i,2} \geq a_{i,2}$. In the complete proof, we show that $a'_{i,2} \leq a_{i,2} + k$. This implies that $a'_{i,1} + a'_{i,2} \leq a_{i,1} + a_{i,2}$, which means that $a'_{i,3} \geq a_{i,3}$ and $a'_{i,4} \geq a_{i,4}$.

Consider the difference in utility across all four rounds between the truthful and misreported instances.

$$\begin{aligned} u_{i,R}(\mathbf{a_i}) - u_{i,R}(\mathbf{a'_i}) &= \sum_{r=1}^{4} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})\right) \\ &= kH + \sum_{r=2}^{4} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})\right) \\ &\geq kH - kH = 0 \end{aligned}$$

The second transition is by Corollary 4.6, and the final transition because each $a'_{i,r} \geq a_{i,r}$ for all $r \in \{1, 2, 3\}$, $\sum_{r=2}^{4}(a'_{i,r} - a_{i,r}) = k$, and each resource can be worth at most H to agent i. □

Given that the 1-Period and 2-Period mechanisms satisfy SP, it is easy to see that they satisfy SI also. By strategy-proofness, the utility that an agent gets from truthfully reporting her demands is at least the utility she gets from reporting $d'_{i,r} = e_i$ for all rounds r. Sharing incentives therefore follows as a corollary of the following proposition.

Proposition 4.13. *Under the T-Period mechanism, any agent that reports $d'_{i,r} = e_i$ for all rounds r receives $a_{i,r} = e_i$ for all rounds r.*

Proof. Let $r \leq T$. First suppose that $D < E$. Then i's minimum allocation is $\bar{d}_{i,r} = \min(d'_{i,r}, e_i + b_{i,r}) = e_i$. So we know that $a_{i,r} \geq e_i$. Suppose for contradiction that $a_{i,r} > e_i$. Then there must be some agent $j \neq i$ with $a_{j,r} \leq e_j$. But now we could obtain a smaller value of x in the PSWC program by assigning slightly higher allocation to j, and slightly lower allocation to any agent with $a_{k,r}/e_k = x$ (we know that j is not one of these agents since $a_{j,r}/e_j < 1 < a_{i,r}/e_i \leq x$). This contradicts optimality of the PSWC program, therefore $a_{i,r} = e_i$.

Next, suppose that $D \geq E$. Then i's limit allocation is $\bar{d}_{i,r} = \min(d'_{i,r}, e_i + b_{i,r}) = e_i$. So we know that $a_{i,r} \leq e_i$. Suppose for contradiction that $a_{i,r} < e_i$. Then there must exist some agent j with $a_{j,r} > e_j$. But now the objective value x of the call to PSWC could be improved by transferring some small amount of allocation to i from all agents k with $a_{k,r}/e_k = x$ (we know that i is not one of these agents since $a_{i,r}/e_i < 1 < a_{j,r}/e_j \leq x$). This contradicts optimality of the PSWC program, therefore $a_{i,r} = e_i$. □

Corollary 4.14. *The T-Period mechanism satisfies SI for $T \leq 2$.*

One may hope to continue increasing flexibility, and therefore performance, by increasing the length of the 'borrowing' and 'payback' periods, potentially all the way to having a single borrowing period of length $R/2$ and a single payback period of length $R/2$. Unfortunately, even for periods of length 3, strategy-proofness is violated.

Example 4.15. Consdier the 3-Period mechanism. Suppose that $n = 5$ and $R = 6$. Each agent has endowment $e_i = 1$ (so each agent can borrow a total of three resources over the first three rounds, corresponding to the sum of their endowment across the final three rounds). Truthful demands are given by the following table.

	$d_{i,1}$	$d_{i,2}$	$d_{i,3}$	$d_{i,4}$	$d_{i,5}$	$d_{i,6}$
$i=1$	3	3	0	1	1	1
$i=2$	0	3	3	1	1	1
$i=3$	0	0	0	0	0	0
$i=4$	0	0	0	0	0	0
$i=5$	0	0	0	0	0	0

The corresponding allocations are given by:

	$a_{i,1}^{3-p}$	$a_{i,2}^{3-p}$	$a_{i,3}^{3-p}$	$a_{i,4}^{3-p}$	$a_{i,5}^{3-p}$	$a_{i,6}^{3-p}$
$i=1$	3	2	0.75	0.08	0.08	0.08
$i=2$	0.5	3	2	0.17	0.17	0.17
$i=3$	0.5	0	0.75	1.58	1.58	1.58
$i=4$	0.5	0	0.75	1.58	1.58	1.58
$i=5$	0.5	0	0.75	1.58	1.58	1.58

Agent 1's utility is $5.25H + 0.75L$. If agent 1 misreports $d'_{1,1} = 2$, it can be checked that her allocations become 2, 2.5, 0.625, 0.292, 0.292, 0.292. Her utility is then $5.375H + 0.625L$, which is higher than her utility from reporting truthfully.

5 RELAXING SHARING INCENTIVES TO ACHIEVE HIGHER EFFICIENCY

In this section, we seek an algorithm with improved performance over the 2-Period mechanism. To do so, we will sacrifice the sharing incentives property, while retaining a 0.5 approximation guarantee to it. The idea of our algorithm is to achieve greater flexibility by not having explicit borrowing and payback rounds, but by instead allocating resources locally according to the PSWC algorithm at each round. We do, however, preserve the property that each agent receives exactly Re_i resources across all rounds. We do this by simply cutting an agent off once she has received Re_i resources in total. We keep track of the resources each agent has received with a running token count t_i, effectively 'charging' each agent a token for every resource she receives. We denote by $t_{i,r}$ the number of tokens that agent i has remaining at the start of round r. The number of tokens that an agent has remaining thus puts a hard limit on the number of resources they can receive at any given round, in the same way that an agent's borrowing limit does in the T-Period mechanism. The token mechanism is presented as Algorithm 2; as with the T-period mechanism, it is split into two cases according to whether the total allocatable demand is greater than or less than the total supply.

Algorithm 2 Token Mechanism

```
t = Re                                          ▷ Initialize token count
for r ∈ {1, . . . , R} do
    d̄ ← min(d'·,r, t)                           ▷ d̄_i is i's allocatable demand
    D ← Σ_{i∈[n]} d̄_i
    if D ≥ E then
        a·,r ← PWSC(A = E, l = d̄, m = 0, w = e)
    else
        a·,r ← PWSC(A = E, l = t, m = d̄, w = e)
    end if
    t ← t − a·,r
end for
```

We illustrate the token mechanism with an example.

Example 5.1. Consider the instance from Example 4.2, where each agent has endowment 1 and demands are given by:

	$d_{i,1}$	$d_{i,2}$	$d_{i,3}$	$d_{i,4}$
$i=1$	3	1	1	0
$i=2$	0	2	1	2
$i=3$	0	0	0	4

The token mechanism (TM) allocations are given by the following table:

	$a_{i,1}^{TM}$	$a_{i,2}^{TM}$	$a_{i,3}^{TM}$	$a_{i,4}^{TM}$
$i=1$	3	1	0	0
$i=2$	0	2	1.5	1
$i=3$	0	0	1.5	3

While all agents have tokens remaining, the token mechanism efficiently allocates resources. However, in round 3, agent 1 has no tokens remaining and therefore the supply of resources exceeds the allocatable demand. In this case, resources are evenly divided between agents 2 and 3. In the final round, agent 2 can receive only 0.5 resources before running out of tokens, so the rest of the resources are allocated to agent 3.

We will make extensive use of the following lemma, which is analagous to Lemma 4.3.

Lemma 5.2. *Let x denote the objective value of a call to PSWC at round r. If $D \geq E$, then $a_{i,r} = \min(xe_i, d_{i,r}, t_{i,r})$. If $D < E$, then $a_{i,r} = \min(t_{i,r}, \max(d_{i,r}, xe_i))$.*

Proof. Suppose first that $\sum_{i\in[n]} \bar{d}_{i,r} \geq E$. Substituting the relevant terms into Lemma 4.1, we have that

$$a_{i,r} = \max(0, \min(\min(d_{i,r}, t_{i,r}), xe_i)) = \min(xe_i, d_{i,r}, t_{i,r}).$$

If instead $\sum_{i\in[n]} \bar{d}_{i,r} < E$, then again substuting into Lemma 4.1 gives

$$a_{i,r} = \max(\min(d_{i,r}, t_{i,r}), \min(t_{i,r}, xe_i)) = \min(t_{i,r}, \max(d_{i,r}, xe_i)).$$

□

We use Lemma 5.2 to prove a basic monotonicity result: that if we shift some tokens to a single agent from all other agents, then the agent with more tokens achieves a (weakly) higher allocation.

Lemma 5.3. *Consider some agent i, and suppose that $t'_{j,r} \leq t_{j,r}$ for all $j \neq i$, and $d_{k,r} = d'_{k,r}$ for all $k \in [n]$. Then $a'_{i,r} \geq a_{i,r}$.*

In the following subsection we show our main technical result: that the token mechanism is strategy-proof.

5.1 Proof of strategy-proofness

Suppose agent i reports demands that are not equal to her true demands. Let r' be the latest round for which i misreports. That is, $r' = \max\{r : d'_{i,r} \neq d_{i,r}\}$. Suppose that $d'_{i,r'} < d_{i,r'}$. We will show that, all else being equal, i could (weakly) improve her utility by instead reporting $d'_{i,r'} = d_{i,r'}$. The proof that reporting $d'_{i,r'} > d_{i,r'}$ is also (weakly) worse than reporting $d'_{i,r'} = d_{i,r'}$ is almost identical and can be found in Appendix B. It follows from this that the token mechanism is strategy-proof, since any non-truthful reports can be converted to truthful reports one round at a time, never decreasing i's utility.

As in the previous section, we will denote allocations and tokens in the instance where i reports $d'_{i,r'}$ at round r' (the 'misreported instance') using a' and t' respectively. We will also consider an alternative universe in which i truthfully reports $d_{i,r'}$ at round r', while holding all her other reports, as well as the reports of all other agents, constant (the 'truthful instance,' even though i's reports prior to r' may yet be nontruthful). Allocations and tokens in that universe are denoted by a and t. We will denote by D_r and D'_r the value of D at round r in the truthful and misreported instances respectively.

The following lemma holds because $d'_{k,r} = d_{k,r}$ for all $r < r'$.

Lemma 5.4. *For all rounds $r < r'$ and for all agents j, $a'_{j,r} = a_{j,r}$.*

We next state a monotonicity lemma analagous to Lemma 4.4.

Lemma 5.5. *For all agents $j \neq i$, we have that $a'_{j,r'} \geq a_{j,r'}$. Further, $a'_{i,r'} \leq a_{i,r'}$.*

If it is the case that $a'_{i,r'} = a_{i,r'}$, then it must also be that $a'_{j,r'} = a_{j,r'}$ for all $j \neq i$. That is, allocations at round r' are the same in the misreported instance as the truthful instance. Therefore, for all rounds $r \leq r'$, allocations in both universes would be the same. In all rounds $r > r'$, reports in both universes are the same. Together, these imply that allocations for all rounds $r > r'$ would be the same in both universes. In particular, i does not profit from her misreport and could weakly improve her utility by reporting $d'_{i,r'} = d_{i,r'}$. So, for the remainder of this section, we assume that $a'_{i,r'} < a_{i,r'}$.

Our next lemma says that the resources that i sacrifices in round r' are *high* valued resources for her. The proof is similar to the proof of Lemma 4.5, and is deferred to the appendix.

Lemma 5.6. *If $a'_{i,r'} < a_{i,r'}$, then $a_{i,r'} \leq d_{i,r'}$.*

As a corollary, we can write the difference in utility between the truthful and misreported instances that i derives from round r'.

Corollary 5.7. $u_{i,r'}(a_{i,r'}) - u_{i,r'}(a'_{i,r'}) = H(a_{i,r'} - a'_{i,r'})$.

For a fixed agent k, denote by r_k the round at which agent k runs out of tokens in the truthful universe. That is, r_k is the first (and only) round with $a_{r_k} = t_{k,r_k} > 0$. Note that $r_i \geq r'$, since $a_{i,r'} > 0$.

Our next lemma states that, under certain conditions, the effect of i's misreport $d'_{i,r} < di, r$ is to increase the objective value x in the call to the PSWC procedure.

Lemma 5.8. *Let $r < r_i$ (that is, $a_{i,r} < t_{i,r}$). Suppose $t'_{j,r} \leq t_{j,r}$ for all agents $j \neq i$. Suppose that either $\min(D_r, D'_r) \geq E$ or $\max(D_r, D'_r) < E$. Then $x' \geq x$, where x' denotes the objective value in the algorithm's call to PSWC in the misreported instance and x in the truthful instance.*

Using Lemmas 5.8 and 5.8, we show our main lemma. It will allow us to make an inductive argument that, after giving up some resources in round r', i's allocation is (weakly) larger for all other rounds in the misreported instance than the truthful instance.

Lemma 5.9. *Let $r' < r < r_i$ (that is, $a_{i,r} < t_{i,r}$). Suppose that $t'_{j,r} \leq t_{j,r}$ for all agents $j \neq i$. Then for all $j \neq i$, either*

(1) $a'_{j,r} = t'_{j,r}$, *OR*
(2) $a'_{j,r} \geq a_{j,r}$.

Proof. Note that $t'_{j,r} \leq t_{j,r}$ for all $j \neq i$ implies that $t'_{i,r} \geq t_{i,r}$, which we will use in the proof. Also, because $r' < r$, we know that $d'_{i,r} = d_{i,r}$, as r' is the last round for which $d'_{i,r} \neq d_{i,r}$.

Let $j \neq i$. We will assume that condition (1) from the lemma statement is false – that is, $a'_{j,r} < t'_{j,r}$ – and show that condition (2) must hold. Suppose first that $D_r < M$. Then, because $a_{i,r} < t_{i,r}$, we know that $d_{i,r} \leq t_{i,r} \leq t'_{i,r}$. So $\min(d_{i,r}, t_{i,r}) = \min(d_{i,r}, t'_{i,r}) = d_{i,r}$. Since all $k \neq i$ have $t'_{k,r} \leq t_{k,r}$, we have $\min(d_{k,r}, t'_{k,r}) \leq \min(d_{k,r}, t_{k,r})$. Therefore, it is the case that $D'_r \leq D_r < E$.

By Lemma 5.2 and the assumption that $a'_{j,r} < t'_{j,r}$, it must be the case that $a'_{j,r} = \max(d_{j,r}, x' e_j)$. Further, by Lemma 5.8, we know that $x' \geq x$. Therefore $\max(d_{j,r}, x e_j) \leq \max(d_{j,r}, x' e_j) < t'_{j,r} \leq t_{j,r}$, so $a_{j,r} = \max(d_{j,r}, x e_j) \leq \max(d_{j,r}, x' e_j) = a'_{j,r}$. That is, condition (2) from the lemma statement holds.

Now suppose that $D_r \geq M$. Then, from the definition of the mechanism, we have that $a_{j,r} \leq \min(d_{j,r}, t_{j,r}) \leq d_{j,r}$. If it is the case that $D'_r < M$ then we have that $a'_{j,r} \geq \min(d_{j,r}, t'_{j,r}) = d_{j,r}$. The equality $\min(d_{j,r}, t'_{j,r}) = d_{j,r}$ holds because otherwise we would have $a'_{j,r} \geq \min(d_{j,r}, t'_{j,r}) = t'_{j,r}$, violating the assumption that $a'_{j,r} < t'_{j,r}$. Using these inequalities, we have $a'_{j,r} \geq d_{j,r} \geq a_{j,r}$, so condition (2) from the statement of the lemma holds.

Finally, it may be the case that $D_r \geq M$ and $D'_r \geq M$. By Lemma 5.2 and the assumption that $a'_{j,r} < t'_{j,r}$, we have that $a'_{j,r} = \min(d_{j,r}, x' e_k) \geq \min(d_{j,r}, x e_k) = a_{j,r}$, where the inequality follows from Lemma 5.8. So condition (2) of the lemma statement holds. □

Finally, we show that the mechanism is strategy-proof.

Theorem 5.10. *The token mechanism satisfies SP.*

Proof. This proof establishes that misreporting $d'_{i,r}$ is never beneficial for an agent. We first observe that for every $r \leq r_i$, $t'_{j,r} \leq t_{j,r}$ for every $j \neq i$. This is true for every $r \leq r'$ because $a'_{j,r} = a_{j,r}$ for $r < r'$, by Lemma 5.4. For $r = r' + 1$, it follows from Lemma 5.5, which says that $a'_{j,r'} \geq a_{j,r'}$. For all subsequent rounds, up to and including $r = r_i$, it follows inductively from Lemma 5.9: $t'_{j,r} \leq t_{j,r}$ implies that either $a'_{j,r} = t'_{j,r}$ (in which case $t'_{j,r+1} = 0 \leq t_{j,r+1}$), or $a'_{j,r} \geq a_{j,r}$ (in which case $t'_{j,r+1} = t'_{j,r} - a'_{j,r} \leq t_{j,r} - a_{j,r} = t_{j,r+1}$).

Consider an arbitrary round $r \neq r'$, with $r \leq r_i$. By the above argument, we know that $t'_{j,r} \leq t_{j,r}$ for all $j \neq i$. Further, because reports in our two instances are identical on all rounds $r \neq r'$, we have that $d_{k,r} = d'_{k,r}$ for all $k \in [n]$. Therefore, by Lemma 5.3, $a'_{i,r} \geq a_{i,r}$. For rounds $r > r_i$, it is also true that $a'_{i,r} \geq a_{i,r}$, since $a_{i,r} = 0$ for these rounds by the definition of r_i. Finally,

$$
\begin{aligned}
u_{i,R}(\mathbf{a_i}) - u_{i,R}(\mathbf{a'_i}) &= \sum_{r=1}^{R} (u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})) \\
&= (u_{i,r'}(a_{i,r'}) - u_{i,r'}(a'_{i,r'})) + \sum_{r \neq r'} (u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})) \\
&= H(a_{i,r'} - a'_{i,r'}) - \sum_{r \neq r'} (u_{i,r}(a'_{i,r}) - u_{i,r}(a_{i,r})) \\
&\geq H(a_{i,r'} - a'_{i,r'}) - H(a_{i,r'} - a'_{i,r'}) = 0
\end{aligned}
$$

Here, the third transition follows from Lemma 5.7, and the final transition because $\sum_{r \neq r'}(a'_{i,r} - a_{i,r}) = a_{i,r'} - a'_{i,r'}$, and every term in the sum is positive.

The proof for the case where $d'_{i,r'} > d_{i,r'}$ is in the appendix. Together, they show that i achieves (weakly) higher utility by truthfully reporting her demand $d_{i,r'}$ at round r, rather than misreporting $d'_{i,r'} \neq d_{i,r'}$. By the argument at the start of this subsection, this is sufficient to prove strategy-proofness. □

5.2 Approximating Sharing Incentives

Unfortunately, the token mechanism fails to satisfy SI, and may give an agent as little as $\frac{1}{2}$ of her SI share.

Theorem 5.11. *The token mechanism does not satisfy α-SI for any $\alpha > \frac{1}{2}$.*

Proof. Consider an instance with T rounds, and $T + 1$ agents, each with endowment $e_i = 1$. Agent 1 has $d_{1,1} = d_{1,T} = 1$ and $d_{1,2} = \ldots = d_{1,T-1} = 0$, agent 2 has $d_{i,t} = T$ for all rounds r, and all other agents have $d_{i,r} = 0$ for all rounds r. In round 1, agent 1 receives allocation $a_{1,1} = 1$ and agent 2 receives $a_{2,1} = T$. For rounds $r = 2, \ldots, T - 1$, each agent $j \neq 2$ receives allocation $a_{j,r} = 1 + \frac{1}{T}$. Therefore, in round T, agent 1 receives $a_{1,T} = T - (T - 1 + \frac{T-2}{T}) = \frac{2}{T}$. Her total utility is therefore $\frac{T+2}{T}H + (T - 1 - \frac{2}{T})L$, compared to total utility $2H + (T - 2)L$ that she would receive by not participating in the mechanism. For $L = 0$, the ratio of these utilities approaches $\frac{1}{2}$ as $T \to \infty$. □

However, the token mechanism does provide a $\frac{1}{2}$ approximation guarantee, in the sense that each agent is guaranteed at least half the utility she would receive from not participating in the mechanism.

Theorem 5.12. *Suppose that agent i receives x high-valued resources by not participating in any sharing mechanism. Then i receives at least $\frac{x}{2}$ high-valued resources by (truthfully) participating in the token mechanism.*

Note that Theorem 5.12 implies the desired approximiation. Suppose that i obtains utility $xH + (Te_i - x)L$ by not participating in the mechanism. Theorem 5.12 in combination with the fact that she will receive the same number of resources overall whether she participates or not, implies that, by participating, she gets at least $\frac{x}{2}H + (Te_i - \frac{x}{2})L \geq \frac{x}{2}H + (\frac{T}{2}e_i - \frac{x}{2})L = \frac{xH+(Te_i-x)L}{2}$.

5.3 Limit Efficiency for Symmetric Agents

In this section we prove that, under certain assumptions, the token mechanism is efficient in the limit as the number of rounds grows large. Suppose that each agent has the same endowment (say, without loss of generality, that each agent has $e_i = 1$). Further, suppose that demands are drawn i.i.d. across rounds and that the distribution within rounds treats agents symmetrically (either demands are drawn i.i.d. across agents, or there is correlation that treats all agents symmetrically).

Theorem 5.13. *In the limit as the number of rounds grows large, the token mechanism approaches full efficiency for symmetric agents.*

6 SIMULATIONS

In this section we evaluate different mechanisms using real and synthetic benchmarks.

Benchmarks. For real benchmarks, we use the Google cluster traces [4, 28]. Google traces provides data collected from a 12.5k-machine cluster over about a month-long period in May 2011. All the machines in the cluster share a common cluster manager that allocates agent tasks to machines.

Agents submit a set of resource demands for each task (*e.g.* required CPU, memory, or disk space). These demands are normalized relative to the largest capacity of the resource on any machine in the traces. Cluster manger records any changes in the status of tasks (*e.g.* being evicted, failed, or killed) during their lifecycle in a task event table. We use the task event table to create agent demands for our evaluation. Since in this paper we only focus on a single resource type, our agent demands only capture agents' CPU requests. Note that since all demands are scaled by the same factor, we safely use normalized demands as actual demands.

We divide time into 15 min intervals.[2] We define agents' demands for each interval to be the sum of their demands for all tasks they run in that interval. After processing the traces, we remove agents with constant demands or with average demand less than some marginal threshold.

In many real-world instances, sharing schemes are unreliable so participants buy resources to align with their maximum possible demand more than their average demand. However, with a well-functioning resource sharing scheme in place, we would expect agents to contribute something closer to their average demand and rely on sharing for periods of high demand. Hence we assume agents' endowments are equal to their average demands.

We observe that for each agent, demands computed from Google traces have high correlations over time. An agent with high demand at 12am has typically high demand at 12:15am as well. In some deployment scenarios, demands may not be highly correlated. For example, when university cluster machines are allocated to professors and researchers on a daily basis, a researcher may have some jobs today, but may not want to use the cluster tomorrow. To evaluate mechanisms in such scenarios, we use synthetic benchmarks.

For synthetic benchmarks, we create random agent populations and random number of rounds. For each agent, we uniformly and randomly assign an endowment from 1 to 20. Once agents' endowments are set, we uniformly and randomly generate agent demands such that their average is equal to agents' endowments (*i.e.* $d_{i,r} \sim u[0, 2e_i]$)

Metrics. For our evaluations, we assume $L = 0$, which implies

$$u_{i,R} = \sum_r u_{i,r} = \sum_r \min(d_{i,r}, a_{i,r}).$$

The $L = 0$ case is the worst case for our performance analysis since it results in the largest relative efficiency loss for allocating an L resource instead of an H. We define system performance as the weighted average of agent utilities. For weights we use agents'

[2] We have created demands for varying time intervals. Since the results do not change significantly for different interval lengths, we only include results on 15-min-long intervals.

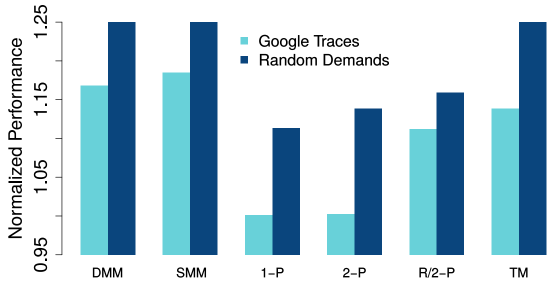

Figure 2: Normalized System Performance. Weighted system performance of different dynamic allocation mechanism normalized to that of static allocations for Gooole cluster traces and 100 instances of random demands.

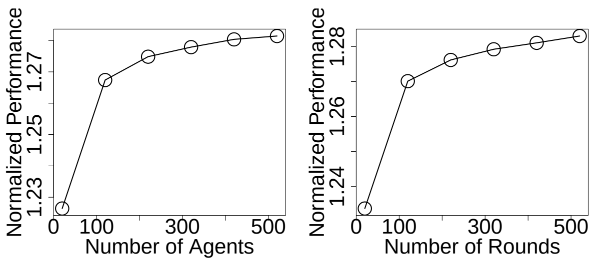

Figure 3: Sensitivity of Token Mechanism. Normalized performance of token mechanism for varying agent population sizes and number of rounds. We fix number of rounds to 50 when we vary number of agents, and fix number of agents to 50 when we vary number of rounds.

endowments, as they reflect system priorities.

$$\text{System Performance} = (1/E) \sum_i e_i \, u_i \quad (1)$$

6.1 Performance Evaluation

Figure 2 presents performance for varied allocation mechanisms for both Google and random traces normalized to the performance of static allocations. DMM and SMM achieve the highest performances as they always allocate resources to those agents with high valuations. DMM has a slightly lower performance as it may choose agents with low endowments over agents with high endowments because low-endowment agents have lower cumulative allocations. Note that SMM and DMM both fail to guarantee strategy-proofness so in the presence of strategic behavior. Therefore, when agents report strategically, for all we now they could be as inefficient as static allocation.

The 1-Period mechanism achieves the lowest performance as it has the least flexibility. Increasing the period length to 2 slightly improves the performance of the T-Period mechanism. The $R/2$-Period mechanism achieves 87% of SMM performance, but fails to provide strategy proofness.

As can be seen, the performance of the token mechanism is competitive with the state-of-the-art dynamic allocation mechanisms. The token mechanism achieves 96% of the performance of SMM for Google traces and 98% for random demands. This shows that in practice, strong game-theoretic desiderata do not come with high performance prices.

Figure 3 depicts how the performance of the token mechanism changes for varying population sizes and number of rounds. As the population size increases, the diversity between agents' demands at each round increases. Complementary demands improve the performance of the token mechanism as fewer will be forced to spend tokens on low-valued resources. Moreover, as the number of rounds increases, agents' flexibility in spending their tokens on high-valued resources increases. We prove in §5.3 that, at least when endowments are equal, the token mechanism approximates efficiency.

6.2 Sharing Incentives

Let us define the *sharing index* of agent i to be the ratio between agent i's utility under the token mechanism and her utility under static allocation. In §5.2 we show that the token mechanism guarantees that the sharing index of agents is always at least 1/2. In other words, under the token mechanism, agents receive at least half the utility they would receive under static allocation. In practice, however, our results show that the sharing index is much higher.

Figure 4 shows the sharing index for all agents in the Google cluster traces, sorted in an increasing order. The minimum sharing index across all agents is 0.98, and on average agents receive 15x more utility under the token mechanism compared to static allocations. As can be seen, there is high variance in sharing index across agents. Agent with high index are those who have zero demand, at most of the rounds and very high demand at a few rounds. These agents benefit the most from sharing. When they have zero demand, they do not spend any tokens. Once they have a high demand they spend their tokens to receive resources they need.

Figure 5 shows agents' sharing index for an instance with random demands. Since agents do not have correlated demands, the variance in sharing index is significantly lower compared to the Google cluster traces. Moreover, we observe a minimum sharing index of 1 across all agents over 100 random instances.

7 RELATED WORK

There is a body of work in the *mechanism design without money* literature that is related to our work. Gorokh et al. [19] consider a setting where a single item is to be allocated repeatedly, and extend to more general settings in a follow-up paper [20]. They do so by endowing each user with a fixed amount of artificial currency and then treating it similarly to if it were real money. They show that, for a large enough number of rounds, incentives to misreport and welfare loss both vanish. However, their notion of strategy-proofness is ex-ante Bayesian, requiring users (and the mechanism) to know the distribution from which other users' demands are drawn and truthful reporting is optimal only in expectation. Our notion of SP is ex-post, meaning that an agent never regrets truthful reporting.

Various other work does not explicitly use artificial currency, but by keeping track of how much utility an agent should receive in the future, achieve guarantees in a way that resembles the use of

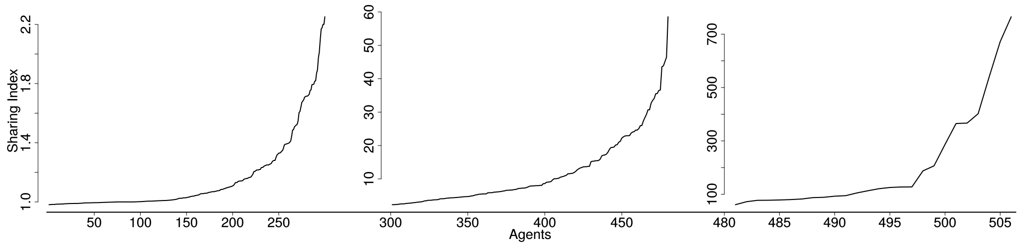

Figure 4: Sorted Sharing Incentive Index for Google Cluster Traces. The ratio between agents' performance under token mechanism and static allocations sorted across all agents.

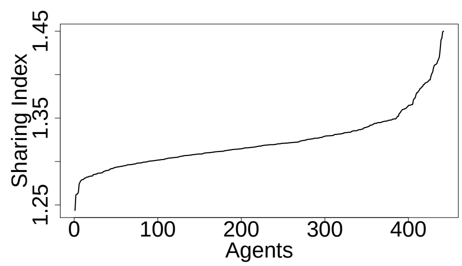

Figure 5: Sensitivity of the Token Mechanism for Random Demands.

artificial currency [5, 10, 22]. Again, these results are for a weaker notion of SP.

In a similar setting, Aleksandrov et al. [6, 7] consider a stream of resources arriving one at a time that must be allocated among competing strategic agents. They consider two mechanisms, one of which is similar to max-min and the other randomly allocates among all agents that request the resource. They obtain both positive and negative results for these mechanisms, however they primarily consider the case where agent utilities are 0 or 1, corresponding to our $L = 0$ case. They also consider only the symmetric agent setting, rather than our setting that allows unequal endowments.

There also exists literature on *dynamic fair division* [15, 24, 34], but this work predominantly focuses on agents arriving and departing over time, rather than the preferences themselves being dynamic, as in our work.

In the systems literature, in recent years, there has been a growing body of work on using economic game-theory to allocate resources [17, 35, 36]. These works only consider one-shot allocations and do not study allocations over time. Ghodsi et al. [16] consider dynamic allocations over time but in a completely different allocation setting than ours. Their proposed mechanism allocates resources to packets in a queue. In such a setting, time cannot be divided into fixed intervals, because processing packets take different times, which means a packet could stall all other packets until its processed. As a result, proportional allocations have to be approximated through discrete packet scheduling decisions [14, 27].

In a work that is close to our setting, Tang *et al.*[31] propose a dynamic allocation policy that resembles DMM. We study the characteristics of DMM in §3 and evaluate its performance in §6.

Another related work in this area is [29]. The authors propose a scheduler that allocates resources between users with dynamically changing demands. This work deploys heuristics and does not provide any theoretical guarantees that we study in this paper.

8 CONCLUSION

We have considered the problem of designing mechanisms for dynamic proportional sharing in a high-low utility model that both incentivize users to participate and share their resources (sharing incentives), as well as truthfully report their resource requirements to the system (strategy-proofness). We show that while each of these properties is incompatible with full efficiency, it is possible to satisfy both of them and still obtain some efficiency gains from sharing.

The main mechanism that we present, the *token mechanism*, is strategy-proof and provides each user a theoretical guarantee of at least half her sharing incentives share. While we do not guarantee full sharing incentives, we show via simulations on both real and synthetic data that in practical situations, no users are significantly worse off by participating in the sharing scheme (and the majority are vastly better off). We show that under certain assumptions, the token mechanism provides full efficiency in the large round limit, which is supported by our simulation results. By incentivizing truthful reporting, we posit that the token mechanism will in fact produce significant efficiency gains in settings where agents are strategic.

Many directions for future work remain. The 2-Period mechanism fully satisfies both SP and SI, but remains very inflexible in its allocations. A key challenge is the design of a more flexible mechanism that satisfies both properties (or some upper bound on the efficiency that such mechanisms can achieve). Another direction is to extend the utility model. The high/low model is crucial to the positive strategic results that we obtain because tradeoffs are well-defined: swapping an L resource for an H resource is always bad. Even introducing a medium (M) value complicates the situation considerably, and extending to such a setting would represent an exicting step forward.

REFERENCES

[1] Capacity Scheduler. https://hadoop.apache.org/docs/current/hadoop-yarn/hadoop-yarn-site/CapacityScheduler.html. (????).
[2] Fair Scheduler. https://hadoop.apache.org/docs/current/hadoop-yarn/hadoop-yarn-site/FairScheduler.html. (????).
[3] Spark Dynamic Allocation. https://spark.apache.org/docs/latest/job-scheduling.html. (????).

[4] 2011. More Google cluster data. Google research blog. (Nov. 2011). Posted at http://googleresearch.blogspot.com/2011/11/more-google-cluster-data.html.
[5] Atila Abdulkadiroğlu and Kyle Bagwell. 2013. Trust, Reciprocity, and Favors in Cooperative Relationships. *American Economic Journal: Microeconomics* 5, 2 (2013), 213–259.
[6] Martin Aleksandrov, Haris Aziz, Serge Gaspers, and Toby Walsh. 2015. Online Fair Division: Analysing a Food Bank Problem.. In *IJCAI*. 2540–2546.
[7] Martin Aleksandrov and Toby Walsh. 2017. Pure Nash Equilibria in Online Fair Division. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI)*. 42–48.
[8] Anonymous. 2017. SHARCNET. https://www.sharcnet.ca. (2017).
[9] Anonymous for review. 2017. The *** Computer Cluster. http://***.edu. (2017).
[10] Susan Athey and Kyle Bagwell. 2001. Optimal Collusion with Private Information. *RAND Journal of Economics* 32, 3 (2001), 428–465.
[11] L. Barroso and U. Hoelzle. 2007. The case for energy-proportional computing. *IEEE Computer* (2007).
[12] Garrett Birkhoff. 1946. Tres observaciones sobre el algebra lineal. *Univ. Nac. Tucumán Rev. Ser. A* 5 (1946), 147–151.
[13] Eric Boutin, Jaliya Ekanayake, Wei Lin, Bing Shi, Jingren Zhou, Zhengping Qian, Ming Wu, and Lidong Zhou. 2014. Apollo: Scalable and coordinated scheduling for cloud-scale computing. In *Proceedings of the 11th USENIX conference on Operating Systems Design and Implementation (OSDI)*. USENIX Association, 285–300.
[14] Alan Demers, Srinivasan Keshav, and Scott Shenker. 1989. Analysis and simulation of a fair queueing algorithm. In *ACM SIGCOMM Computer Communication Review*, Vol. 19. ACM, 1–12.
[15] Eric Friedman, Christos-Alexandros Psomas, and Shai Vardi. 2017. Controlled Dynamic Fair Division. In *Proceedings of the 2017 ACM Conference on Economics and Computation*. ACM, 461–478.
[16] Ali Ghodsi, Vyas Sekar, Matei Zaharia, and Ion Stoica. 2012. Multi-resource fair queueing for packet processing. *ACM SIGCOMM Computer Communication Review* 42, 4 (2012), 1–12.
[17] Ali Ghodsi, Matei Zaharia, Benjamin Hindman, Andy Konwinski, Scott Shenker, and Ion Stoica. 2011. Dominant Resource Fairness: Fair Allocation of Multiple Resource Types.. In *NSDI*, Vol. 11. 24–24.
[18] Ali Ghodsi, Matei Zaharia, Scott Shenker, and Ion Stoica. 2013. Choosy: Max-min fair sharing for datacenter jobs with constraints. In *Proceedings of the 8th ACM European Conference on Computer Systems*. ACM, 365–378.
[19] Artur Gorokh, Siddhartha Banerjee, and Krishnamurthy Iyer. 2016. Near-Efficient Allocation Using Artificial Currency in Repeated Settings. (2016).
[20] Artur Gorokh, Siddhartha Banerjee, and Krishnamurthy Iyer. 2017. From Monetary to Non-Monetary Mechanism Design Via Artificial Currencies. (2017).
[21] Ajay Gulati, Ganesha Shanmuganathan, Xuechen Zhang, and Peter J Varman. 2012. Demand Based Hierarchical QoS Using Storage Resource Pools.. In *USENIX Annual Technical Conference*. 1–13.
[22] Mingyu Guo, Vincent Conitzer, and Daniel M. Reeves. 2009. Competitive Repeated Allocation Without Payments. In *Proceedings of the Fifth Workshop on Internet and Network Economics (WINE)*. Rome, Italy, 244–255.
[23] Gary J Henry. 1984. The UNIX system: The fair share scheduler. *AT&T Bell Laboratories Technical Journal* 63, 8 (1984), 1845–1857.
[24] Ian Kash, Ariel D Procaccia, and Nisarg Shah. 2014. No agent left behind: Dynamic fair division of multiple resources. *Journal of Artificial Intelligence Research* 51 (2014), 579–603.
[25] J. Kay and P. Lauder. 1988. A Fair Share Scheduler. *Commun. ACM* 31, 1 (1988), 44–55.
[26] Lawrence Berkeley National Laboratory. 2017. National Energy Research Scientific Computing Center. https://www.nersc.gov. (2017).
[27] Abhay K Parekh and Robert G Gallager. 1993. A generalized processor sharing approach to flow control in integrated services networks: The single-node case. *IEEE/ACM Transactions on Networking* 1, 3 (1993), 344–357.
[28] Charles Reiss, John Wilkes, and Joseph L. Hellerstein. 2011. *Google cluster-usage traces: format + schema*. Technical Report. Google Inc., Mountain View, CA, USA. Revised 2014-11-17 for version 2.1. Posted at https://github.com/google/cluster-data.
[29] Thomas Sandholm and Kevin Lai. 2010. Dynamic proportional share scheduling in Hadoop. In *Job scheduling strategies for parallel processing*. Springer, 110–131.
[30] Alan Shieh, Srikanth Kandula, Albert G Greenberg, Changhoon Kim, and Bikas Saha. 2011. Sharing the Data Center Network.. In *NSDI*, Vol. 11. 23–23.
[31] Shanjiang Tang, Bu-Sung Lee, Bingsheng He, and Haikun Liu. 2014. Long-term resource fairness: Towards economic fairness on pay-as-you-use computing systems. In *Proceedings of the 28th ACM international conference on Supercomputing*. ACM, 251–260.
[32] Abhishek Verma, Luis Pedrosa, Madhukar Korupolu, David Oppenheimer, Eric Tune, and John Wilkes. 2015. Large-scale Cluster Management at Google with Borg. In *Proceedings of the Tenth European Conference on Computer Systems (EuroSys)*. ACM, 18:1–18:17.
[33] Carl A. Waldspurger and William E. Weihl. 1994. Lottery Scheduling: Flexible Proportional-Share Resource Management. In *Proceedings of the First Symposium on Operating Systems Design and Implementation (OSDI)*. 1–11.
[34] Toby Walsh. 2011. Online cake cutting. *Algorithmic Decision Theory* (2011), 292–305.
[35] Xiaodong Wang and José F Martínez. 2015. XChange: Scalable Dynamic Multi-Resource Allocation in Multicore Architectures. In *Proceedings of the 21st IEEE International Symposium on High Performance Computer Architecture (HPCA)*.
[36] Seyed Majid Zahedi and Benjamin C. Lee. 2014. REF: Resource Elasticity Fairness with Sharing Incentives for Multiprocessors. In *Proceedings of the 19th International Conference on Architectural Support for Programming Languages and Operating Systems (ASPLOS)*. ACM, 145–160.

A OMITTED PROOFS

A.1 Proof of Lemma 4.4

If $r > T$, then the allocation of agent i is independent of her reported demand, thus $a_{i,r} = a'_{i,r}$. Now suppose that $r \leq T$. Let $\bar{d}_{i,r} = \min(d_{i,r}, e_i + b_{i,r})$ and $\bar{d}'_{i,r} = \min(d'_{i,r}, e_i + b_{i,r})$. Also, let x and x' denote the objective value in the T-period mechanism's call to PSWC when i reports $d_{i,r}$ and $d'_{i,r}$, respectively. Observe first that $D' = \bar{d}'_{i,r} + \sum_{j \neq i} \bar{d}_{j,r} \leq \bar{d}_{i,r} + \sum_{j \neq i} \bar{d}_{j,r} = D$.

Suppose first that $E \leq D' \leq D$. Let $a_{j,r}$ and $a'_{j,r}$ denote the allocations of j when i reports $d_{i,r}$ and $d'_{i,r}$, respectively. If $x' \geq x$, then for all $j \neq i$, by Lemma 4.3 we have:

$$a'_{j,r} = \min(e_j + b_{j,r}, d_{j,r}, x'e_j) \geq \min(e_j + b_{j,r}, d_{j,r}, xe_j) = a_{j,r}.$$

This immediately implies that $a'_{i,r} \leq a_{i,r}$, because $\sum_{k \in [n]} a_{k,r} = \sum_{k \in [n]} a'_{k,r} = E$. If $x' < x$, then again by Lemma 4.3 we have the following:

$$a'_{i,r} = \min(e_i + b_{i,r}, d'_{i,r}, x'e_i) \leq \min(e_i + b_{i,r}, d_{i,r}, xe_i) = a_{i,r}.$$

By the same lemma, for all $j \neq i$, we also have:

$$a'_{j,r} = \min(e_j + b_{j,r}, d_{j,r}, x'e_j) \leq \min(e_j + b_{j,r}, d_{j,r}, xe_j) = a_{i,r}.$$

Therefore, for all $k \in [n]$, $a_{k,r} \geq a'_{k,r}$. However, since $\sum_{k \in [n]} a_{k,r} = \sum_{k \in [n]} a'_{k,r} = E$, it has to be the case that $a_{k,r} = a'_{k,r}$ for all k.

Next, suppose that $D' < E \leq D$. By the definition of the T-period mechanism, for all $j \neq i$, $a'_{j,r} \geq \bar{d}_{j,r}$, and $a_{j,r} \leq \bar{d}_{j,r}$. Therefore, $a'_{j,r} \geq a_{j,r}$ which implies that $a'_{i,r} \leq a_{i,r}$.

Finally, suppose that $D' \leq D < E$. If $x' \geq x$, then by Lemma 4.3, for all $j \neq i$, we have:

$$\begin{aligned} a'_{j,r} &= \min(e_j + b_{j,r}, \max(d_{j,r}, x'e_j)) \\ &\geq \min(e_j + b_j, \max(d_{j,r}, xe_j)) = a_{j,r}. \end{aligned}$$

This implies $a'_{i,r} \leq a_{i,r}$. If $x' < x$, then, by Lemma 4.3 we have:

$$\begin{aligned} a'_{i,r} &= \min(e_i + b_{i,r}, \max(d'_{i,r}, x'e_i)) \\ &\leq \min(e_i + b_{i,r}, \max(d_{i,r}, xe_i)) = a_{i,r} \end{aligned}$$

By the same lemma, for all $j \neq i$, we also have:

$$\begin{aligned} a'_{j,r} &= \min(e_j + b_{j,r}, \max(d_{j,r}, x'e_j)) \\ &\leq \min(e_j + b_{j,r}, \max(d_{j,r}, xe_j)) = a_{j,r}. \end{aligned}$$

Therefore, for all $k \in [n]$, $a'_{k,r} \leq a_{k,r}$. However, since $\sum_{k \in [n]} a_{k,r} = \sum_{k \in [n]} a'_{k,r} = E$, it has to be the case that $a_{k,r} = a'_{k,r}$ for all k.

A.2 Proof of Lemma 4.7

Note that $D' \geq D$, since $d'_{i,r} > d_{i,r}$ and $d'_{j,r} = d_{j,r}$ for all agents $j \neq i$. If $D < E$, then $a_{i,r} \geq \bar{d}_{i,r} = \min(e_i + b_{i,r}, d_{i,r})$. We show that $e_i + b_{i,r} \geq d_{i,r}$, and therefore, $a_{i,r} \geq d_{i,r}$. Suppose for contradiction that $e_i + b_{i,r} < d_{i,r} < d'_{i,r}$, which means $\bar{d}_{i,r} = \bar{d}'_{i,r} = e_i + b_{i,r}$. By definition of the T-period mechanism, $\bar{d}_{i,r} \leq a_{i,r} \leq e_i + b_{i,r}$, which implies $a_{i,r} = e_i + b_{i,r}$. Also, by the definition of the mechanism, $a'_{i,r} \leq \bar{d}'_{i,r} = e_i + b_{i,r}$ if $D' \geq E$, and $a'_{i,r} \leq e_i + b'_{i,r} = e_i + b_{i,r}$ if $D' < E$. In both cases, $a'_{i,r} \leq e_i + b_{i,r} = a_{i,r}$, a contradiction.

If $D' \geq D \geq E$, then $a'_{i,r} > a_{i,r}$ implies that there is at least an agent j with $a'_{j,r} < a_{j,r}$. In the proof of Lemma 4.4 we show that if $x < x'$, then $a'_{k,r} = a_{k,r}$ for all k. Therefore, it has to be the case that $x \geq x'$. By the definition of the T-period mechanism and Lemma 4.1, $a_{i,r} = \min(e_i + b_{i,r}, d_{i,r}, xe_i)$ and $a'_{i,r} = \min(e_i + b_{i,r}, d'_{i,r}, x'e_j)$. It is easy to see that if $a_{i,r}$ is xe_i or $e_i + b_{i,r}$, then $a'_{i,r} \leq a_{i,r}$. Therefore, $a_{i,r} = d_{i,r}$, which means the lemma holds.

A.3 Proof of Corollary 4.8

Because $d_{i,r} \leq a_{i,r} < a'_{i,r}$, we can substitute the utility values from Equation 2,

$$\begin{aligned} u_{i,r}(a'_{i,r}) - u_{i,r}(a_{i,r}) &= d_{i,r}H + (a'_{i,r} - d_{i,r})L - (d_{i,r}H + (a_{i,r} - d_{i,r})L) \\ &= L(a'_{i,r} - a_{i,r}). \end{aligned}$$

A.4 Proof of Lemma 4.11

We will treat four cases, corresponding to whether or not supply exceeds demand in the truthful and misreported instances. Let x' denote the objective value in the T-Period mechanism's call to PSWC in the misreported instance, and x in the truthful instance. All cases rely heavily on the characterization of the allocation from Lemma 4.3.

Suppose first that $D_r \geq E$ and $D'_r \geq E$. Suppose that $x' \leq x$. Then, for all $j \neq i$, $a'_{j,r} = \min(x'e_j, d_{j,r}, e_j + b'_{j,r}) \leq \min(xe_j d_{j,r}, e_j + b_{j,r}) = a_{j,r}$, which implies that $a'_{i,r} \geq a_{i,r}$, since $\sum_{k\in[n]} a_{k,r'} = \sum_{k\in[n]} a'_{k,r'}$. On the other hand, if $x' > x$, then $a'_{i,r} = \min(x'e_i, d_{i,r}, e_i + b'_{i,r}) \geq \min(xe_i, d_{i,r}, e_i + b_{i,r}) = a_{i,r}$.

Second, suppose that $D_r \geq E$ and $D'_r < E$. Then $a'_{i,r} \geq \min(d_{i,r}, e_i + b'_{i,r}) \geq \min(d_{i,r}, e_i + b_{i,r}) \geq a_{i,r})$.

Third, suppose that $D_r < E$ and $D'_r \geq E$. Then $a'_{j,r} \leq \min(d_{j,r}, e_j + b'_{j,r}) \leq \min(d_{j,r}, e_j + b_{j,r}) \leq a_{j,r}$ for all $j \neq i$, which implies that $a'_{i,r} \geq a_{i,r}$.

Finally, suppose that $D_r < E$ and $D'_r < E$. If $x' \leq x$, then for all $j \neq i$, we have that $a'_{j,r} = \min(e_j + b'_{j,r}, \max(d_{j,r}, x'e_j)) \leq \min(e_j + b_{j,r}, \max(d_{j,r}, xe_j)) = a_{j,r}$, which implies that $a'_{i,r} \geq a_{i,r}$. If $x' > x$, then $a'_{i,r} = \min(e_i + b'_{i,r}, \max(d_{i,r}, x'e_i)) \geq \min(e_i + b_{i,r}, \max(d_{i,r}, xe_i)) = a_{i,r}$. Thus, the lemma holds in all cases.

A.5 Proof of Theorem 4.12

By Lemma 4.9, no agent can benefit by reporting $d'_{i,2} \neq d_{i,2}$. Similarly, no agent can benefit by reporting $d'_{i,r} \neq d_{i,r}$ for $r \in \{3, 4\}$, because the 2-Period mechanism ignores reports for those rounds. We may therefore assume that $d'_{i,r} = d_{i,r}$ for all agents i and all rounds $r \geq 2$.

We will show that an agent cannot benefit from reporting $d'_{i,1} < d_{i,1}$. The proof that reporting $d'_{i,1} > d_{i,1}$ is not beneficial is very similar. If $a'_{i,1} = a_{i,1}$, then $a'_{j,1} = a_{j,1}$, by Lemma 4.4. Therefore, the allocations are unchanged for all rounds i, as the 2-Period mechanism takes into account allocations at earlier rounds, but not reports, and the allocations at round 1 are the same in the truthful and misreported instances. We therefore assume that $a_{i,1} = a'_{i,1} + k$, for some $k > 0$. This implies that $b_{i,2} = b'_{i,2} - k_i$, for some $k_i \leq k$. By Lemma 4.5, i receives kH more utility in round 1 under truthful reporting than under misreporting. For every $j \neq i$, $a_{j,1} \leq a'_{j,1}$, and $b_{j,2} = b'_{j,2} + k_j$, where $\sum_{j\neq i} k_j \leq k$. By Lemma 4.11, $a'_{i,2} \geq a_{i,2}$. In the following, we show that $a'_{i,2} \leq a_{i,2} + k$.

Let x and x' denote the objective value in the T-period mechanism's call to PSWC when i reports $d_{i,r}$ and $d'_{i,r}$, respectively. We consider four cases, corresponding to whether resources in the truthful and misreported instances are over or under demanded at round 2. Suppose first that $D_2 \geq E$ and $D'_2 \geq E$. First, suppose that $x' < x$. Then, by Lemma 4.3,

$$\begin{aligned} a'_{i,2} &= \min(e_i + b'_i, d_{i,r}, x'e_i) \\ &= \min(e_i + b_i + k_i, d_{i,r}, x'e_i) \\ &\leq \min(e_i + b_i, d_{i,r}, x'e_i) + k_i \\ &\leq \min(e_i + b_i, d_{i,r}, xe_i) + k_i \leq a_{i,2} + k \end{aligned}$$

Next, suppose that $x' \geq x$. Then for all $j \neq i$,

$$\begin{aligned} a'_{j,2} &= \min(e_j + b'_j, d_{j,r}, x'e_j) \\ &= \min(e_j + b_j - k_j, d_{j,r}, x'e_j) \\ &\geq \min(e_j + b_j, d_{j,r}, x'e_j) - k_j \\ &\geq \min(e_j + b_j, d_{j,r}, xe_j) - k_j = a_{j,2} - k_j \end{aligned}$$

Taking the sum over all $j \neq i$ and noting that $\sum_{j\neq i} k_j \leq k$, we have that $\sum_{j\neq i} a'_{j,2} \geq \sum_{j\neq i} a_{j,2} - k$. Therefore, $a'_{i,2} \leq a_{i,2} + k$.

Second, suppose that $D_2 \geq E$ and $D'_2 < E$. Then, by the definition of the T-Period mechanism, $a_{j,2} \leq \min(e_j + b_j, d_{j,r})$ for all $j \neq i$. Further

$$\begin{aligned} a'_{j,2} &\geq \min(e_j + b'_j, d_{j,r}) \\ &= \min(e_j + b_j - k_j, d_{j,r}) \\ &\geq \min(e_j + b_j, d_{j,r}) - k_j \\ &\geq a_{j,2} - k_j \end{aligned}$$

By the same argument as in the previous case, this implies that $a'_{i,2} \leq a_{i,2} + k$.

Third, suppose that $D_2 < E$ and $D'_2 \geq E$. Then

$$\begin{aligned} a'_{i,2} &\leq \min(e_i + b'_i, d_{i,r}) \\ &= \min(e_i + b_i + k_i, d_{i,r}) \\ &\leq \min(e_i + b_i, d_{i,r}) + k_i \\ &\leq a_{i,r} + k \end{aligned}$$

Finally, suppose that $D_2 < E$ and $D_2' < E$. First, suppose that $x' < x$. Then

$$\begin{aligned} a'_{i,2} &= \min(e_i + b'_i, \max(d_{i,r}, x'e_i)) \\ &= \min(e_i + b_i + k_i, \max(d_{i,r}, x'e_i)) \\ &\le \min(e_i + b_i, \max(d_{i,r}, x'e_i)) + k_i \\ &\le \min(e_i + b_i, \max(d_{i,r}, xe_i)) + k_i \le a_{i,2} + k \end{aligned}$$

Next, suppose that $x' \ge x$. Then for all $j \ne i$,

$$\begin{aligned} a'_{j,2} &= \min(e_j + b'_j, \max(d_{j,r}, x'e_j)) \\ &= \min(e_j + b_j - k_j, \max(d_{j,r}, x'e_j)) \\ &\ge \min(e_j + b_j, \max(d_{j,r}, x'e_j)) - k_j \\ &\ge \min(e_j + b_j, \max(d_{j,r}, xe_j)) - k_j = a_{j,2} - k_j \end{aligned}$$

Again, this implies that $a'_{i,2} \le a_{i,2} + k$.

In all cases, we have that $a'_{i,2} \le a_{i,2} + k$. Therefore, $a'_{i,1} + a'_{i,2} \le a_{i,1} + a_{i,2}$, which means that $a'_{i,3} \ge a_{i,3}$ and $a'_{i,4} \ge a_{i,4}$. Consider the difference in utility across all four rounds between the truthful and misreported instances.

$$\begin{aligned} u_{i,R}(\mathbf{a_i}) - u_{i,R}(\mathbf{a'_i}) &= \sum_{r=1}^{4} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})\right) \\ &= kH + \sum_{r=2}^{4} \left(u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})\right) \\ &\ge kH - kH = 0 \end{aligned}$$

The second transition is by Corollary 4.6, and the final transition because each $a'_{i,r} \ge a_{i,r}$ for all $r \in \{1, 2, 3\}$, $\sum_{r=2}^{4}(a'_{i,r} - a_{i,r}) = k$, and each resource can be worth at most H to agent i.

A.6 Proof of Lemma 5.3

Note that the condition that $t'_{j,r} \le t_{j,r}$ for all $j \ne i$ implies that $t'_{i,r} \ge t_{i,r}$. We will use these assumptions, along with the characterization of the token mechanism allocations from Lemma 5.2, to prove the lemma.

We will treat four cases, corresponding to whether or not supply exceeds demand in the truthful and misreported instances. Let x' denote the objective value in the token mechanism's call to PSWC in the misreported instance, and x in the truthful instance.

Suppose first that $D_r \ge E$ and $D'_r \ge E$. Suppose that $x' \le x$. Then, for all $j \ne i$,

$$a'_{j,r} = \min(x'e_j, d_{j,r}, t'_{j,r}) \le \min(xe_j d_{j,r}, t_{j,r}) = a_{j,r},$$

which implies that $a'_{i,r} \ge a_{i,r}$, since $\sum_{k\in[n]} a_{k,r'} = \sum_{k\in[n]} a'_{k,r'}$. On the other hand, if $x' > x$, then

$$a'_{i,r} = \min(x'e_i, d_{i,r}, t'_{i,r}) \ge \min(xe_i, d_{i,r}, t_{i,r}) = a_{i,r}.$$

Second, suppose that $D_r \ge E$ and $D'_r < E$. Then

$$a'_{i,r} \ge \min(d_{i,r}, t'_{i,r}) \ge \min(d_{i,r}, t_{i,r}) \ge a_{i,r}.$$

Third, suppose that $D_r < E$ and $D'_r \ge E$. Then

$$a'_{j,r} \le \min(d_{j,r}, t'_{j,r}) \le \min(d_{j,r}, t_{j,r}) \le a_{j,r}$$

for all $j \ne i$, which implies that $a'_{i,r} \ge a_{i,r}$.

Finally, suppose that $D_r < E$ and $D'_r < E$. If $x' \le x$, then for all $j \ne i$, we have that

$$a'_{j,r} = \min(t'_{j,r}, \max(d_{j,r}, x'e_j)) \le \min(t_{j,r}, \max(d_{j,r}, xe_j)) = a_{j,r},$$

which implies that $a'_{i,r} \ge a_{i,r}$. If $x' > x$, then

$$a'_{i,r} = \min(t'_{i,r}, \max(d_{i,r}, x'e_i)) \ge \min(t_{i,r}, \max(d_{i,r}, xe_i)) = a_{i,r}.$$

Thus, the lemma holds in all cases.

A.7 Proof of Lemma 5.5

We prove the statment for all $j \ne i$. The statement for i follows immediately because the total number of resources to allocate is fixed.

Observe first that

$$D'_{r'} = \sum_{k\in[n]} \min(d'_{k,r'}, t_{k,r'}) \le \sum_{k\in[n]} \min(d_{k,r'}, t_{k,r'}) = D_{r'},$$

since i's demand decreases in the misreported instances but all other demands and token counts stay the same. Let x' denote the objective value in the token mechanism's call to PWSL in the misreported instance, and x in the truthful instance.

Suppose that $E \le D'_{r'} \le D_{r'}$. Suppose first that $x' > x$. Then, by Lemma 5.2,

$$a'_{j,r'} = \min(x'e_j, d_{j,r'}, t_{j,r'}) \ge \min(xe_j, d_{j,r'}, t_{j,r'}) = a_{j,r'}$$

for all $j \ne i$. Next, suppose that $x' \le x$. Then, again by Lemma 5.2 and the fact that $d'_{i,r'} < d_{i,r'}$,

$$a'_{i,r'} = \min(x'e_i, d'_{i,r'}, t_{i,r'}) \le \min(xe_i, d_{i,r'}, t_{i,r'}) = a_{i,r'}.$$

And, for all $j \ne i$,

$$a'_{j,r'} = \min(x'e_j, d_{j,r'}, t_{j,r'}) \le \min(xe_j, d_{j,r'}, t_{j,r'}) = a_{j,r'}.$$

Because $a'_{k,r'} \le a_{k,r'}$ for all agents k, and $\sum_{k\in[n]} a_{k,r'} = \sum_{k\in[n]} a'_{k,r'}$, it must be the case that $a'_{k,r'} = a_{k,r'}$ for all k, which satisfies the statement of the lemma.

Next, suppose that $D'_{r'} < E \le D_{r'}$. By the definition of the token mechanism, $a'_{k,r'} \ge \min(d'_{k,r'}, t_{k,r'})$ for all k, and $a_{k,r'} \le \min(d_{k,r'}, t_{k,r'})$ for all k. Since $\min(d'_{j,r'}, t_{j,r'}) = \min(d_{j,r'}, t_{j,r'})$ for all $j \ne i$, we have that $a'_{j,r'} \ge a_{j,r'}$, implying also that $a'_{i,r'} \le a_{i,r'}$.

Finally, suppose that $D'_{r'} \le D_{r'} < E$. Suppose first that $x' \le x$. Then, by Lemma 5.2 and the assumptions that $d'_{i,r'} < d_{i,r'}$ and $x' \le x$,

$$a'_{i,r'} = \min(t_{i,r'}, \max(x'e_i, d'_{i,r'})) \le \min(t_{i,r'}, \max(xe_i, d_{i,r'})) = a_{i,r'}$$

and

$$a'_{j,r'} = \min(t_{j,r'}, \max(x'e_j, d_{j,r'})) \le \min(t_{j,r'}, \max(xe_j, d_{j,r'})) = a_{j,r'}$$

for all $j \ne i$. Because $a'_{k,r'} \le a_{k,r'}$ for all agents k, and $\sum_{k\in[n]} a_{k,r'} = \sum_{k\in[n]} a'_{k,r'}$, it must be the case that $a'_{k,r'} = a_{k,r'}$ for all k, which satisfies the lemma statement. Next, suppose that $x' > x$. Then, again by Lemma 5.2, we have

$$a'_{j,r'} = \min(t_{j,r'}, \max(x'e_j, d_{j,r'})) \ge \min(t_{j,r'}, \max(xe_j, d_{j,r'})) = a_{j,r'}$$

for all $j \ne i$.

A.8 Proof of Lemma 5.6

Suppose for contradiction that the lemma statement is false; that $a_{i,r'} > d_{i,r'}$. It must therefore be the case that $D'_{r'} \le D_{r'} < E$, where the first inequality holds because $d'_{j,r'} = d_{j,r'}$ for all $j \neq i$ and $d'_{i,r'} < d_{i,r'}$. Let x denote the objective value of the program in the call to the PSWC algorithm in the truthful instance, and x' in the misreported instance. Suppose that $x' \le x$. Then, by Lemma 5.2 and the assumptions that $d'_{i,r'} < d_{i,r'}$ and $x' \le x$,

$$a'_{i,r'} = \min(t_{i,r'}, \max(x'e_i, d'_{i,r'})) \le \min(t_{i,r'}, \max(xe_i, d_{i,r'})) = a_{i,r'}$$

and

$$a'_{j,r'} = \min(t_{j,r'}, \max(x'e_j, d_{j,r'})) \le \min(t_{j,r'}, \max(xe_j, d_{j,r'})) = a_{j,r'}$$

for all $j \neq i$. Because $a'_{k,r'} \le a_{k,r'}$ for all agents k, and $\sum_{k\in[n]} a_{k,r'} = \sum_{k\in[n]} a'_{k,r'}$, it must be the case that $a'_{k,r'} = a_{k,r'}$ for all k. This contradicts the assumption that $a'_{i,r'} < a_{i,r'}$.

So suppose that $x' > x$. Note also that $xe_i > d_{i,r'} > d'_{i,r'}$, where the first inequality holds because $a_{i,r'} > d_{i,r'}$. Then, again by Lemma 5.2 and the previous observation, we have

$$\begin{aligned} a'_{i,r'} &= \min(t_{i,r'}, \max(x'e_i, d'_{i,r'})) = \min(t_{i,r'}, x'e_i) \\ &\ge \min(t_{i,r'}, xe_i) = \min(t_{i,r'}, \max(xe_i, d_{i,r'})) = a_{i,r'}, \end{aligned}$$

which contradicts that $a' - i, r < a_{i,r}$.

Since we arrive at a contradiction in all cases, the lemma statement must be true.

A.9 Proof of Lemma 5.8

Suppose for contradiction that $x' < x$. By Lemma 5.2,

$$a'_{j,r} = \min(x'e_j, d_{j,r}, t'_{j,r}) \le \min(xe_j, d_{j,r}, t_{j,r}) = a_{j,r}$$

for all $j \neq i$, where the inequality follows from the assumption that $x' < x$ and that $t'_{j,r} \le t_{j,r}$. Further,

$$\begin{aligned} a'_{i,r} &= \min(x'e_i, d_{i,r}, t'_{i,r}) \le \min(x'e_i, d_{i,r}) \\ &\le \min(xe_i, d_{i,r}) = \min(xe_i, d_{i,r}, t_{i,r}) = a_{i,r}, \end{aligned}$$

where the second inequality follows from the assumption that $x' < x$ and the second to last equality from the assumption $a_{i,r} < t_{i,r}$.

Therefore, $a'_{k,r} \le a_{k,r}$ for all agents k. Since $\sum a'_{k,r} = \sum a_{k,r}$, it must be the case that $a'_{k,r} = a_{k,r}$ for all agents k.

Now, suppose that $\min(D_r, D'_r) \ge E$. By the definition of the token mechanism in this case, $a_{k,r}/e_k \le x' < x$ for all agents k with $a_{k,r} > m_k = 0$. Therefore x is not the optimal objective value of the PSWC program in the truthful instance, a contradiction. Thus, $x' \ge x$.

Next, suppose that $\max(D_r, D'_r) < E$. Consider all agents with $\min(d_{k,r}, t_{k,r}) < a_{k,r}$ (that is, those agents for which the first constraint in the PSWC program binds in the truthful instance). For all such agents, we have

$$\begin{aligned} & \min(d_{k,r}, t_{k,r}) < a_{k,r} \\ \Longrightarrow\quad & d_{k,r} < a_{k,r} \le t_{k,r} \\ \Longrightarrow\quad & d_{k,r} < a'_{k,r} \le t'_{k,r} \\ \Longrightarrow\quad & \min(d_{k,r}, t'_{k,r}) < a'_{k,r}, \end{aligned}$$

so the constraints bind in the misreported instance as well. Therefore, $a'_{k,r}/e_k \le x' < x$ for all agents k for which the first constraint binds in the truthful instance. Therefore x is not the optimal objective value of the PSWC program in the truthful instance, a contradiction. Thus, $x' \ge x$.

A.10 Proof of Theorem 5.12

We will suppose that agent i truthfully reports her demand $d_{i,r}$ for all rounds (since the token mechanism is SP).

Recall that for every agent i, we denote by r_i the round at which $a_{i,r_i} = t_{i,r_i}$. For every agent i, define sets B_i and A_i to be the agents that run out of tokens before and after i, respectively. Formally,

$$\begin{aligned} B_i &= \{j : r_j \le r_i \text{ and } \frac{a_{j,r_i}}{e_j} < \frac{a_{i,r_i}}{e_i}\} \\ A_i &= \{j : r_j \ge r_i \text{ and } r_j = r_i \implies \frac{a_{j,r_i}}{e_j} \ge \frac{a_{i,r_i}}{e_i}\}. \end{aligned}$$

For a round r, define

$$s_{i,r} = a_{i,r} - e_i \frac{\sum_{j\in A_i} a_{j,r}}{\sum_{j\in A_i} e_j}.$$

That is, $s_{i,r}$ is the number of resources i gets more than the (endowment weighted) average number of resources for agents in A_i.

Lemma A.1. *For every agent i and every round r,*

$$s_{i,r} \le \min(d_{i,r}, a_{i,r}).$$

Proof. If $a_{i,r} \le d_{i,r}$, then the lemma statement says that $s_{i,r} \le a_{i,r}$, which is obviously true from the definition of $s_{i,r}$.

If $a_{i,r} > d_{i,r}$, then we know from the definition of the token mechanism that $\sum_{j\in[n]} \min(d_{j,r}, t_{j,r}) < M$, and $a_{i,r} = \min(xe_i, t_{i,r})$, where x is the objective value of the algorithm's call to the PSWC program. Further, all agents with $\frac{a_{j,r}}{e_j} < \frac{a_{i,r}}{e_i} \le x$ are those with $a_{j,r} = t_{j,r}$, so by definition $r_j \le r_i$ and $\frac{a_{j,r}}{e_j} < \frac{a_{i,r}}{e_i}$. Therefore, by definition, $j \in B_i$. So $\frac{a_{j,r}}{e_j} \ge \frac{a_{i,r}}{e_i}$ for all $j \in A_i$. To complete the proof, note that

$$\begin{aligned} s_{i,r} = a_{i,r} - e_i \frac{\sum_{j\in A_i} a_{j,r}}{\sum_{j\in A_i} e_j} &\le a_{i,r} - e_i \frac{a_{i,r}}{e_i} \\ &= 0 \le d_{i,r} = \min(d_{i,r}, a_{i,r}). \end{aligned}$$

□

While i has tokens remaining, she is guaranteed to get as many resources as she demands, up to her endowment e_i. Thus, for these rounds, she would obtain no additional high-valued resources from not participating in the mechanism. However, there is the possibility that by participating in the mechanism, she runs out of tokens prematurely, thus missing out on resources in later rounds that she wants, and would have received by not participating in the mechanism (as in the proof of Theorem 5.11). The proof proceeds by showing that for every resource that i does not receive due to a lack of tokens, she must have received at least one H valued resource in an earlier round.

Suppose first that $a_{i,r_i} \geq e_i$. We have the following inequality:

$$\sum_{r \leq r_i} \min(d_{i,r}, a_{i,r}) \geq \sum_{r \leq r_i} s_{i,r} = -\sum_{r > r_i} s_{i,r} = \sum_{r > r_i} \left(e_i \frac{\sum_{j \in A_i} a_{j,r}}{\sum_{j \in A_i} e_j} \right)$$
$$= \sum_{r > r_i} \left(\frac{E}{\sum_{j \in A_i} e_j} \right) e_i \geq (T - r_i) e_i. \quad (2)$$

The first inequality follows from Lemma A.1, and the second inequlity because $\sum_{j \in A_i} e_j \leq E$. The first equality holds because $\sum_{r=1}^{R} s_{i,r} = 0$, and the second equality holds because $a_{i,r} = 0$ for all $r > r_i$. The third equality holds because for rounds $r > r_i$, only agents in A_i remain active, so all resources are allocated to them.

Note that $\sum_{r \leq r_i} \min(d_{i,r}, a_{i,r})$ is the total number of H valued resources that i receives. Further, were i not to have participated in the mechanism, i would have received at most this many H valued resources over the first r_i rounds, because the token mechanism guarantees each agent at least $\min(d_{i,r}, e_i)$ resources, as long as they have sufficient tokens remaining. The total number of H valued resources that i would have received in rounds $r > r_i$ were she not to have shared is $e_i(T - r_i)$. Therefore, the total number of H valued resources i would receive if she did not participate in the mechanism is at most $e_i(T - r_i) + \sum_{r \leq r_i} \min(d_{i,r}, a_{i,r}) \leq 2 \sum_{r \leq r_i} \min(d_{i,r}, a_{i,r})$. From participating in the token mechanism, i receives exactly $\sum_{r \leq r_i} \min(d_{i,r}, a_{i,r})$ H valued resources, which proves the lemma in this case.

Second, suppose that $a_{i,r_i} < e_i$. We have the following inequality:

$$\sum_{r \leq r_i} \min(d_{i,r}, a_{i,r}) \geq \sum_{r < r_i} s_{i,r} = -\sum_{r > r_i} s_{i,r} - s_{i,r_i}$$
$$\geq e_i(T - r_i) + e_i \frac{\sum_{j \in A_i} a_{j,r_i}}{\sum_{j \in A_j} e_j} - a_{i,r_i} \geq e_i(T - r_i) + e_i - a_{i,r_i}$$
$$= e_i(T - r_i + 1) - a_{i,r_i}.$$

The first inequality again follows from Lemma A.1, and the second inequality holds from Equation 2. The third inequality holds because at round r_i, agent i is receiving allocation $a_{i,r} < e_i$, therefore every agent $j \in B_i$ is receiving allocation $a_{j,r} < e_j$, therefore $\sum_{j \in A_i} a_{j,r_i} \geq \sum_{j \in A_i} e_j$.

As with the previous case, $e_i(T - r_i + 1) - a_{i,r_i}$ is exactly the number of H valued resources that i may have been able to receive in rounds $r \geq r_i$ had she not participated in the mechanism, over and above those she receives by participating. $\sum_{r \leq r_i} \min(d_{i,r}, a_{i,r})$ is the number of H valued resources she receives by participating in the mechanism. Therefore $\sum_{r \leq r_i} \min(d_{i,r}, a_{i,r}) + e_i(T - r_i + 1) - a_{i,r_i} \leq 2 \sum_{r \leq r_i} \min(d_{i,r}, a_{i,r})$ is an upper bound on the number of H valued resources i would receive by not participating in the mechanism. Therefore, i receives at least half as many H valued resources from participating as she would have by not participating.

A.11 Proof of Theorem 5.13

Let Q be a random variable denoting how many tokens a single agent i would spend (that is, how many resources i would be allocated) in a single round, supposing that no agent runs out of tokens at or before that round. Note that by the symmetry of the agents, Q is independent of the identity of any single agent, and independent of the particular round since the token mechanism allocates independently of the round, provided that all agents have sufficient numbers of tokens remaining.

By symmetry, $\mathbb{E}(Q) = 1$. Let stddev$Q = \sigma$. Let $r = R - R^{\frac{2}{3}}$ and let Q_r be a random variable denoting the number of tokens i would spend before the start of round $r + 1$. Because demands are drawn independently across rounds, $\mathbb{E}(Q_r) = r$ and $\text{sd}(Q_r) = \sqrt{r}\sigma$.

Consider the probability that agent i, in a world with unlimited tokens, spends at least R tokens in the first r rounds, which is:

$$\begin{aligned}
P(Q_r \geq R) &= P(Q_r - \mathbb{E}(Q_r) \geq R - r) \\
&= P(Q_r - \mathbb{E}(Q_r) \geq R^{\frac{2}{3}} \\
&= P(Q_r - \mathbb{E}(Q_r) \geq \frac{R^{\frac{1}{6}}}{\sigma} \sqrt{R}\sigma) \\
&\leq P(Q_r - \mathbb{E}(Q_r) \geq \frac{R^{\frac{1}{6}}}{\sigma} \sqrt{r}\sigma) \\
&\leq \frac{\sigma^2}{R^{\frac{1}{3}}}
\end{aligned}$$

Here the final inequality follows from Chebyshev's concentration inequality, because $\sqrt{r}\sigma$ is the standard deviation of Q_r.

Taking a union bound over all n agents, the probability that any agent spends at least R tokens in the first r rounds (if this is not the case then it is also not the case when tokens are limited) is at most $\frac{n\sigma^2}{R^{\frac{1}{3}}}$. This approaches 0 as $R \to \infty$.

So, with probability going to 1, no agent runs out of tokens before round r. By the definition of the token mechanism, full efficiency is achieved on all rounds for which no agents have their allocation limited by lack of tokens. Therefore, with probability going to 1, the token mechanism allocates efficiently for the first r rounds. Therefore (because demands are i.i.d. across rounds),the expected efficiency of the mechanism approaches at least an $\frac{r}{R} = \frac{R - R^{\frac{2}{3}}}{R}$ fraction of the optimal efficiency. This fraction approaches 1 as $R \to \infty$.

B OVER-REPORTING DEMAND IS NOT ADVANTAGEOUS

In this section we assume that $d'_{i,r'} > d_{i,r'}$. The setup otherwise mirrors that of Section 5.1.

Lemma B.1. *For all agents $j \neq i$, we have that $a'_{j,r'} \leq a_{j,r'}$. Further, $a'_{i,r'} \geq a_{i,r'}$.*

Proof. We prove the statment for all $j \neq i$. The statement for i follows immediately because the total number of resources to allocate is fixed.

Observe first that $D_{r'} = \sum_{k \in [n]} \min(d_{k,r'}, t_{k,r'}) \leq \sum_{k \in [n]} \min(d'_{k,r'}, t_{k,r'}) = D'_{r'}$, since i's demand increases in the misreported instances but all other demands and token counts stay the same. Let x' denote the objective value in the token mechanism's call to PWSL in the misreported instance, and x in the truthful instance.

Suppose that $E \leq D_{r'} \leq D'_{r'}$. Suppose first that $x' < x$. Then, by Lemma 5.2,

$$a_{j,r'} = \min(xe_j, d_{j,r'}, t_{j,r'}) \geq \min(x'e_j, d_{j,r'}, t_{j,r'}) = a'_{j,r'}$$

for all $j \neq i$. Next, suppose that $x' \geq x$. Then, again by Lemma 5.2 and the fact that $d'_{i,r'} > d_{i,r'}$,

$$a'_{i,r'} = \min(x'e_i, d'_{i,r'}, t_{i,r'}) \geq \min(xe_i, d_{i,r'}, t_{i,r'}) = a_{i,r'}.$$

And, for all $j \neq i$,

$$a'_{j,r'} = \min(x'e_j, d_{j,r'}, t_{j,r'}) \geq \min(xe_j, d_{j,r'}, t_{j,r'}) = a_{j,r'}.$$

Because $a'_{k,r'} \geq a_{k,r'}$ for all users k, and $\sum_{k\in[n]} a_{k,r'} = \sum_{k\in[n]} a'_{k,r'}$, it must be the case that $a'_{k,r'} = a_{k,r'}$ for all k, which satisfies the statement of the lemma.

Next, suppose that $D_{r'} < E \leq D'_{r'}$. By the definition of the token mechanism, $a_{k,r'} \geq \min(d_{k,r'}, t_{k,r'})$ for all k, and $a'_{k,r'} \leq \min(d'_{k,r'}, t_{k,r'})$ for all k. Since $\min(d'_{j,r'}, t_{j,r'}) = \min(d_{j,r'}, t_{j,r'})$ for all $j \neq i$, we have that $a_{j,r'} \geq a'_{j,r'}$, implying also that $a_{i,r'} \leq a'_{i,r'}$.

Finally, suppose that $D_{r'} \leq D'_{r'} < E$. Suppose first that $x \leq x'$. Then, by Lemma 5.2 and the assumptions that $d_{i,r'} < d'_{i,r'}$ and $x \leq x'$,

$$a_{i,r'} = \min(t_{i,r'}, \max(xe_i, d_{i,r'})) \leq \min(t_{i,r'}, \max(x'e_i, d'_{i,r'})) = a'_{i,r'}$$

and

$$a_{j,r'} = \min(t_{j,r'}, \max(xe_j, d_{j,r'})) \leq \min(t_{j,r'}, \max(x'e_j, d_{j,r'})) = a'_{j,r'}$$

for all $j \neq i$. Because $a_{k,r'} \leq a'_{k,r'}$ for all users k, and $\sum_{k\in[n]} a'_{k,r'} = \sum_{k\in[n]} a_{k,r'}$, it must be the case that $a_{k,r'} = a'_{k,r'}$ for all k, which satisfies the lemma statement. Next, suppose that $x > x'$. Then, again by Lemma 5.2, we have

$$a_{j,r'} = \min(t_{j,r'}, \max(xe_j, d_{j,r'})) \geq \min(t_{j,r'}, \max(x'e_j, d_{j,r'})) = a'_{j,r'}$$

for all $j \neq i$. □

If it is the case that $a'_{i,r'} = a_{i,r'}$, then it must also be that $a'_{j,r'} = a_{j,r'}$ for all $j \neq i$. So allocations at round r' are the same in the misreported instance as the truthful instance. Therefore, for all rounds $r \leq r'$, allocations in both universes are the same. In all rounds $r > r'$, reports in both universes are the same. Together, these imply that allocations for all rounds $r > r'$ are the same in both universes. In particular, i does not profit from her misreport and could weakly improve her utility by reporting $d'_{i,r'} = d_{i,r'}$. So, for the remainder of this section, we assume that $a'_{i,r'} > a_{i,r'}$.

Our next lemma says that the additional resources that i receives in round r' are *low* valued resources for her.

Lemma B.2. *If $a'_{i,r'} > a_{i,r'}$, then $a_{i,r'} \geq d_{i,r'}$.*

Proof. Suppose for contradiction that the lemma statement is false; $a_{i,r'} < d_{i,r'}$. We also know that $a_{i,r'} < a'_{i,r'} \leq t'_{i,r'} = t_{i,r'}$, where the last inequality holds because allocations before round r' are identical in the truthful and misreported instances. It must therefore be the case that $D'_{r'} \geq D_{r'} > E$, where the first inequality holds because $d'_{j,r'} = d_{j,r'}$ for all $j \neq i$ and $d'_{i,r'} > d_{i,r'}$, and the second because $a_{i,r'} < \min(t_{i,r'}, d_{i,r'})$. Let x denote the objective value of the program in the call to the PSWC algorithm in the truthful instance, and x' in the misreported instance. Suppose that $x \leq x'$. Then, by Lemma 5.2 and the assumptions that $d_{i,r'} < d'_{i,r'}$ and $x \leq x'$,

$$a_{i,r'} = \min(t_{i,r'}, xe_i, d_{i,r'}) \leq \min(t_{i,r'}, x'e_i, d'_{i,r'}) = a'_{i,r'},$$

and

$$a_{j,r'} = \min(t_{j,r'}, xe_j, d_{j,r'}) \leq \min(t_{j,r'}, x'e_j, d_{j,r'}) = a'_{j,r'}$$

for all $j \neq i$. Because $a_{k,r'} \leq a'_{k,r'}$ for all users k, and $\sum_{k\in[n]} a_{k,r'} = \sum_{k\in[n]} a'_{k,r'}$, it must be the case that $a'_{k,r'} = a_{k,r'}$ for all k. This contradicts the assumption that $a_{i,r'} < a'_{i,r'}$.

So suppose that $x > x'$. Then, again by Lemma 5.2 and the assumptions that $x > x'$ and $d'_{i,r'} > d_{i,r'}$, we have

$$a'_{i,r'} = \min(t_{i,r'}, x'e_i, d'_{i,r'}) \leq \min(t_{i,r'}, xe_i, d_{i,r'}) = a_{i,r'},$$

which contradicts that $a_{i,r'} < a'_{i,r'}$.

Since we arrive at a contradiction in all cases, the lemma statement must be true. □

As a corollary, we can write the difference in utility between the truthful and misreported instances that i derives from round r'.

Corollary B.3. $u_{i,r'}(a'_{i,r'}) - u_{i,r'}(a_{i,r'}) = L(a'_{i,r'} - a_{i,r'})$.

For a fixed user k, denote by r'_k the round at which agent k runs out of tokens in the misreported universe. That is, r'_k is the first (and only) round with $a'_{r_k} = t'_{k,r_k} > 0$. Note that $r'_i \geq r'$, since $a'_{i,r'} > 0$.

Lemma B.4. *Let $r < r'_i$ (that is, $a'_{i,r} < t'_{i,r}$). Suppose $t_{j,r} \leq t'_{j,r}$ for all agents $j \neq i$. Suppose that $D'_r \geq E$ and $D_r \geq E$. Then $x \geq x'$, where x' denotes the objective value in the algorithm's call to PSWC in the misreported instance and x in the truthful instance.*

Proof. Suppose for contradiction that $x < x'$. By Lemma 5.2,

$$a_{j,r} = \min(xe_j, d_{j,r}, t_{j,r}) \leq \min(x'e_j, d_{j,r}, t'_{j,r}) = a'_{j,r}$$

for all $j \neq i$, where the inequality follows from the assumption that $x < x'$ and that $t_{j,r} \leq t'_{j,r}$. Further,

$$\begin{aligned} a_{i,r} &= \min(xe_i, d_{i,r}, t_{i,r}) \leq \min(xe_i, d_{i,r}) \leq \min(x'e_i, d_{i,r}) \\ &= \min(x'e_i, d_{i,r}, t'_{i,r}) = a'_{i,r}, \end{aligned}$$

where the second inequality follows from the assumption that $x < x'$ and the second to last equality from the assumption $a'_{i,r} < t'_{i,r}$.

Therefore, $a_{k,r} \leq a'_{k,r}$ for all users k. Since $\sum a'_{k,r} = \sum a_{k,r}$, it must be the case that $a'_{k,r} = a_{k,r}$ for all users k. Therefore, by the definition of the local token mechanism, $a'_{k,r}/e_k \leq x < x'$ for all users k with $a'_{k,r} > m_k = 0$. Therefore x' is not the optimal objective value of the PSWC program in the misreported instance, a contradiction. Thus, $x \geq x'$. □

Lemma B.5. *Let $r < r'_i$ (that is, $a'_{i,r} < t'_{i,r}$). Suppose $t_{j,r} \leq t'_{j,r}$ for all agents $j \neq i$. Suppose that $D'_r < E$ and $D_r < E$. Then $x \geq x'$, where x' denotes the algorithm's objective value in the algorithm's call to PSWC in the misreported instance and x in the truthful instance.*

Proof. Suppose for contradiction that $x < x'$. By Lemma 5.2,

$$a_{j,r} = \min(t_{j,r}, \max(xe_j, d_{j,r})) \leq \min(t'_{j,r}, \max(x'e_j, d_{j,r})) = a'_{j,r}$$

for all $j \neq i$, where the inequality follows from the assumption that $x < x'$ and that $t_{j,r} \leq t'_{j,r}$. Further,

$$\begin{aligned} a_{i,r} &= \min(t_{i,r}, \max(xe_i, d_{i,r})) \leq \max(xe_i, d_{i,r}) \leq \max(x'e_i, d_{i,r}) \\ &= \min(t'_{i,r}, \max(x'e_i, d_{i,r})) = a'_{i,r}, \end{aligned}$$

where the second inequality follows from the assumption that $x < x'$ and the second to last equality from the assumption $a'_{i,r} < t'_{i,r}$.

Therefore, $a_{k,r} \leq a'_{k,r}$ for all users k. Since $\sum a'_{k,r} = \sum a_{k,r}$, it must be the case that $a'_{k,r} = a_{k,r}$ for all users k. Consider all users with $\min(d_{k,r}, t'_{k,r}) < a'_{k,r}$ (that is, those users for which the first constraint in the PSWC program binds in the misreported instance). For all such users, we have

$$\begin{aligned} \min(d_{k,r}, t'_{k,r}) &< a'_{k,r} \\ &\implies d_{k,r} < a'_{k,r} \leq t'_{k,r} \\ &\implies d_{k,r} < a_{k,r} \leq t_{k,r} \\ \implies \min(d_{k,r}, t_{k,r}) &< a_{k,r}, \end{aligned}$$

so the constraints bind in the truthful instance as well. Therefore, $a_{k,r}/e_k \leq x < x'$ for all users k for which the first constraint binds in the misreported instance. Therefore x' is not the optimal objective value of the PSWC program in the misreported instance, a contradiction. Thus, $x \geq x'$. □

Using Lemmas B.4 and B.5, we show our main lemma. It will allow us to make an inductive argument that, after gaining some extra resources in round r', i's allocation is (weakly) smaller for all other rounds in the mireported instance than the truthful instance.

Lemma B.6. *Let $r' < r < r'_i$ (that is, $a'_{i,r} < t'_{i,r}$). Suppose that $t_{j,r} \leq t'_{j,r}$ for all agents $j \neq i$. Then for all $j \neq i$, either*

(1) $a_{j,r} = t_{j,r}$, *OR*
(2) $a_{j,r} \geq a'_{j,r}$

Proof. Note that $t_{j,r} \leq t'_{j,r}$ for all $j \neq i$ implies that $t_{i,r} \geq t'_{i,r}$, which we will use in the proof. Also, because $r' < r$, we know that $d'_{i,r} = d_{i,r}$, as r' is the last round for which $d'_{i,r} \neq d_{i,r}$.

Let $j \neq i$. We will assume that condition 1) from the lemma statement is false – that is, $a_{j,r} < t_{j,r}$ – and show that condition 2) must hold. Suppose first that $D'_r < M$. Then, because $a'_{i,r} < t'_{i,r}$, we know that $d'_{i,r} \leq t'_{i,r} \leq t_{i,r}$. So $\min(d_{i,r}, t'_{i,r}) = \min(d_{i,r}, t_{i,r}) = d_{i,r}$. Since all $k \neq i$ have $t_{k,r} \leq t'_{k,r}$, we have $\min(d_{k,r}, t_{k,r}) \leq \min(d_{k,r}, t'_{k,r})$. Therefore, it is the case that $D_r \leq D'_r < E$.

By Lemma 5.2 and the assumption that $a_{j,r} < t_{j,r}$, it must be the case that $a_{j,r} = \max(d_{j,r}, xe_j)$. Further, by Lemma B.5, we know that $x \geq x'$. Therefore $\max(d_{j,r}, x'e_j) \leq \max(d_{j,r}, xe_j) < t_{j,r} \leq t'_{j,r}$, so $a'_{j,r} = \max(d_{j,r}, x'e_j) \leq \max(d_{j,r}, xe_j) = a_{j,r}$. That is, condition 2) from the lemma statement holds.

Now suppose that $D'_r \geq M$. Then, from the definition of the mechanism, we have that $a'_{j,r} \leq \min(d_{j,r}, t'_{j,r}) \leq d_{j,r}$. If it is the case that $D_r < M$ then we have that $a_{j,r} \geq \min(d_{j,r}, t_{j,r}) = d_{j,r}$. The equality $\min(d_{j,r}, t_{j,r}) = d_{j,r}$ holds because otherwise we would have $a_{j,r} \geq \min(d_{j,r}, t_{j,r}) = t_{j,r}$, violating the assumption that $a_{j,r} < t_{j,r}$. Using these inequalities, we have $a_{j,r} \geq d_{j,r} \geq a'_{j,r}$, so condition 2) from the statement of the lemma holds.

Finally, it may be the case that $D'_r \geq M$ and $D_r \geq M$. By Lemma 5.2 and the assumption that $a_{j,r} < t_{j,r}$, we have that $a_{j,r} = \min(d_{j,r}, xe_k) \geq \min(d_{j,r}, x'e_k) = a'_{j,r}$, where the inequality follows from Lemma B.4. So condition 2) of the lemma statement holds. □

Lemma B.7. *Suppose that $t_{j,r} \leq t'_{j,r}$ for all $j \neq i$, and $d_{k,r} = d'_{k,r}$ for all $k \in [n]$. Then $a_{i,r} \geq a'_{i,r}$.*

Proof. Note that the condition that $t_{j,r} \leq t'_{j,r}$ for all $j \neq i$ implies that $t_{i,r} \geq t'_{i,r}$. We will use these assumptions, along with the characterization of the token mechanism allocations from Lemma 5.2, to prove the lemma.

We will treat four cases, corresponding to whether or not supply exceeds demand in the truthful and misreported instances. Let x' denote the objective value in the token mechanism's call to PSWC in the misreported instance, and x in the truthful instance.

Suppose first that $D'_r \geq E$ and $D_r \geq E$. Suppose that $x \leq x'$. Then, for all $j \neq i$, $a_{j,r} = \min(xe_j, d_{j,r}, t_{j,r}) \leq \min(x'e_j d_{j,r}, t'_{j,r}) = a'_{j,r}$, which implies that $a_{i,r} \geq a'_{i,r}$, since $\sum_{k \in [n]} a'_{k,r'} = \sum_{k \in [n]} a_{k,r'}$. On the other hand, if $x > x'$, then $a_{i,r} = \min(xe_i, d_{i,r}, t_{i,r}) \geq \min(x'e_i, d_{i,r}, t'_{i,r}) = a'_{i,r}$.

Second, suppose that $D'_r \geq E$ and $D_r < E$. Then $a_{i,r} \geq \min(d_{i,r}, t_{i,r}) \geq \min(d_{i,r}, t'_{i,r}) \geq a'_{i,r})$.

Third, suppose that $D'_r < E$ and $D_r \geq E$. Then $a_{j,r} \leq \min(d_{j,r}, t_{j,r}) \leq \min(d_{j,r}, t'_{j,r}) \leq a'_{j,r}$ for all $j \neq i$, which implies that $a_{i,r} \geq a'_{i,r}$.

Finally, suppose that $D'_r < E$ and $D_r < E$. If $x \leq x'$, then for all $j \neq i$, we have that $a_{j,r} = \min(t_{j,r}, \max(d_{j,r}, xe_j)) \leq \min(t'_{j,r}, \max(d_{j,r}, x'e_j)) = a'_{j,r}$, which implies that $a_{i,r} \geq a'_{i,r}$. If $x > x'$, then $a_{i,r} = \min(t_{i,r}, \max(d_{i,r}, xe_i)) \geq \min(t'_{i,r}, \max(d_{i,r}, x'e_i)) = a'_{i,r}$. Thus, the lemma holds in all cases. □

Finally, we show that the mechanism is strategy-proof.

Theorem B.8. *Agent i never benefits from reporting $d_{i,r'} > d_{i,r'}$.*

Proof. We first observe that for every $r \leq r'_i$, $t_{j,r} \leq t'_{j,r}$ for every $j \neq i$. This is true for every $r \leq r'$ because $a'_{j,r} = a_{j,r}$ for $r < r'$, by Lemma 5.4. For $r = r' + 1$, it follows from Lemma B.1, which says that $a_{j,r'} \geq a'_{j,r'}$. For all subsequent rounds, up to and including $r = r'_i$, it follows inductively from Lemma B.6: $t_{j,r} \leq t'_{j,r}$ implies that either $a_{j,r} = t_{j,r}$ (in which case $t_{j,r+1} = 0 \leq t'_{j,r+1}$), or $a_{j,r} \geq a'_{j,r}$ (in which case $t_{j,r+1} = t_{j,r} - a_{j,r} \leq t'_{j,r} - a'_{j,r} = t'_{j,r+1}$).

Consider an arbitrary round $r \neq r'$, with $r \leq r'_i$. By the above argument, we know that $t_{j,r} \leq t'_{j,r}$ for all $j \neq i$. Further, because reports in our two instances are identical on all rounds $r \neq r'$, we have that $d_{k,r} = d'_{k,r}$ for all $k \in [n]$. Therefore, by Lemma B.7, $a_{i,r} \geq a'_{i,r}$.

For rounds $r > r'_i$, it is also true that $a_{i,r} \geq a'_{i,r}$, since $a'_{i,r} = 0$ for these rounds by the definition of r'_i.

Finally,

$$\begin{aligned} u_{i,R}(\mathbf{a_i}) - u_{i,R}(\mathbf{a'_i}) &= \sum_{r=1}^{R} (u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})) \\ &= \sum_{r \neq r'} (u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})) + (u_{i,r'}(a_{i,r'}) - u_{i,r'}(a'_{i,r'})) \\ &= \sum_{r \neq r'} (u_{i,r}(a_{i,r}) - u_{i,r}(a'_{i,r})) - L(a'_{i,r'} - a_{i,r'}) \\ &\geq L(a'_{i,r'} - a_{i,r'}) - L(a'_{i,r'} - a_{i,r'}) = 0 \end{aligned}$$

Where the third transition follows from Lemma B.3, and the final transition because $\sum_{r \neq r'} (a'_{i,r} - a_{i,r}) = a_{i,r'} - a'_{i,r'}$, and every term in the sum is positive. $\square$