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On Stackelberg Mixed Strategies

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Abstract It is sometimes the case that one solution concept in game theory is equivalent to applying another solution concept to a modified version of the game. In such cases, does it make sense to study the former separately (as it applies to the original representation of the game), or should we entirely subordinate it to the latter? The answer probably depends on the particular circumstances, and indeed the literature takes different approaches in different cases. In this article, I consider the specific example of Stackelberg mixed strategies. I argue that, even though a Stackelberg mixed strategy can also be seen as a subgame perfect Nash equilibrium of a corresponding extensive-form game, there remains significant value in studying it separately. The analysis of this special case may have implications for other solution concepts.

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	L	R
U	1,1	3,0
D	0,0	2,1

Fig. 1 A game that illustrates the advantage of commitment.

1 Introduction

Game theory provides ways of representing strategic situations, as well as *solution concepts* indicating what it means to "solve" the resulting games. These are intertwined: a solution concept may be meaningfully defined only for some ways of representing games. Moreover, sometimes, a solution concept is equivalent to the application of another solution concept to a transformation of the original game. In this case, one may wonder whether it is sensible to study the former concept separately. One might well argue that we should only define the latter concept, and see the former as just an application of it, for the sake of parsimony. Entities should not be multiplied unnecessarily!

In this article, I consider the case of *Stackelberg mixed strategies*, which are optimal mixed strategies to commit to. It will be helpful to first review Stackelberg models in general. A (two-player) Stackelberg model involves one player being able to act (or commit to a course of action) before the other player moves. The standard example is that of two firms competing on quantity. If one firm is able to commit to a quantity before the other moves (Stackelberg competition), the committing firm can benefit significantly from this in comparison to the model where both firms move simultaneously (Cournot competition). (For more detail, see, e.g., Fudenberg and Tirole (1991).) A Stackelberg model requires that the commitment is absolute: the Stackelberg leader cannot backtrack on her commitment. It also requires that the other player, the Stackelberg follower, sees what the leader committed to before he himself moves.

Of course, we can consider what happens if one player obtains a commitment advantage in other games as well.[1] We can take any two-player game represented in *normal form* (i.e., a bimatrix game), and give one player a commitment advantage. The game in Figure 1 is often used as an example. In this game, if neither player has a commitment advantage (and so they make their choices simultaneously), then player 1 (the row player) has a *strictly dominant strategy*: regardless of player 2's choice, U gives player 1 higher utility than D. Realizing that player 1 is best off playing U, player 2 is better off playing L and getting 1, rather than playing R and getting 0. Hence, (U, L) is the solution of the game by *iterated strict dominance* (also implying that it is the only equilibrium of the game), resulting in utilities $(1, 1)$ for the players.

[1] One line of work concerns settings where there are many selfish followers and a single benevolent leader, for example a party that "owns" the system and controls part of the activity in it, who acts to optimize some system-wide objective. See, e.g., Roughgarden (2004). In this article I will not assume that the leader is benevolent.

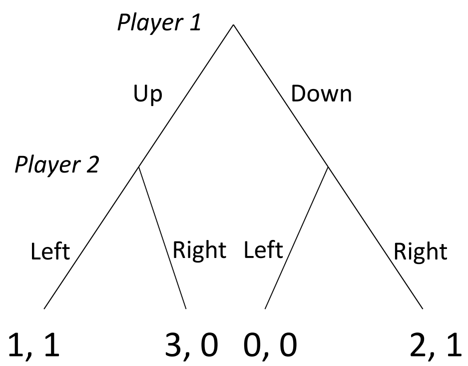

Fig. 2 The extensive-form representation of the pure Stackelberg version of the game in Figure 1.

Now suppose that player 1 can *commit* to an action (and credibly communicate the action to which she has committed to player 2) before player 2 moves. If she commits to playing U, player 2 will again play L. On the other hand, if she commits to playing D, player 2 will realize he is better off playing R. This would result in utilities $(2, 1)$ for the players. Hence, player 1 is now better off than in the version of the game without commitment.[2]

While in this example, the Stackelberg outcome of the game is different from the simultaneous-move outcome, my impression is that most game theorists would not consider the Stackelberg outcome to correspond to a different *solution concept*. Rather, they would see it simply as a different *game*. Specifically, the time and information structure of the game—who moves when knowing what—is different. The *extensive form* provides a natural representation scheme to model the time and information structure of games. For example, the Stackelberg version of the game in Figure 1 can be represented as the extensive-form game in Figure 2. This game is easily solved by backward induction: if player 1 has committed to Up, then it is better to move Left for player 2, resulting in utilities $(1, 1)$; on the other hand, if player 1 has committed to Down, then it is better to move Right for player 2, resulting in utilities $(2, 1)$. Hence, player 1 is best off moving Down. Thus, solving the extensive-form game by backward induction gives us the Stackelberg solution.

So far, so reasonable. Now, let us turn to Stackelberg *mixed* strategies. Here, one of the players has an even stronger commitment advantage: not only is she able to commit to a course of action, she is able to commit to a mixed strategy, that is, a *distribution* over the actions that she can take. Consider again the game from Figure 1, and now suppose that player 1 can commit to a mixed strategy. She could commit to the distribution $(0, 1)$, i.e.,

[2] Note that player 1 merely *stating* that she will play D, without any commitment, will not work: she would always have an incentive to back out and play U after all, to collect an additional 1 unit of utility, regardless of what player 2 plays. Player 2 will anticipate this and play L anyway.

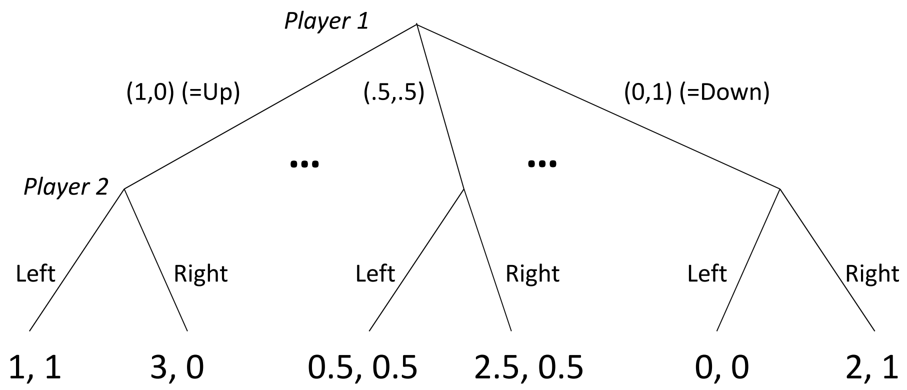

Fig. 3 The extensive-form representation of the mixed Stackelberg version of the game in Figure 1.

putting all the probability on D, and again obtain 2. However, she can do even better: if she commits to $(0.49, 0.51)$, i.e., putting slightly more than half of the probability mass on D, player 2 will still be better off playing R (which would give him 1 slightly more than half the time) than playing L (which would give him 1 slightly less than half the time). This results in an expected utility of $0.49 \cdot 3 + 0.51 \cdot 2 = 2.49 > 2$ for player 1. Of course, player 1 can also commit to $(0.499, 0.501)$, and so on; in the limit, player 1 can obtain 2.5. Stackelberg mixed strategies have recently received significant attention due to their direct application in a number of real security domains (Pita et al. 2009; Tsai et al. 2009; An et al. 2012; Yin et al. 2012).

Again, it is possible to capture the commitment advantage that player 1 has using the extensive form, as illustrated in Figure 3. Note that player 1 has a continuum of moves in the first round, as indicated by the ellipses. Each of the (infinitely many) subgames has a straightforward solution, with the exception of the one where player 1 has committed to $(0.5, 0.5)$, in which player 2 is indifferent between his choices. If player 2 responds by playing Right in this case, then it is optimal for player 1 to in fact commit to $(0.5, 0.5)$; and this constitutes the unique subgame perfect Nash equilibrium of the game.

Again, this way of representing the game in extensive form and solving it is entirely correct. However, it appears more awkward than it did in the case of committing to a pure strategy. For one, the first node in the game tree now has infinitely many children. This is due to the fact that committing to a mixed strategy is *not* the same as randomizing over which pure strategy to commit to. The reason that they are not the same is that if player 1 randomizes over which pure strategy to commit to, then player 2 sees the *realization* of that random process, i.e., the realized pure strategy, before acting. Because of this, there is indeed no reason to randomize over which pure strategy to commit to, as this could not lead to a higher utility than simply committing (deterministically) to whichever pure strategy maximizes player 1's utility.

Consequently, randomizing over which pure strategy to commit to could not result in a utility greater than 2 for player 1 in the game above.

Because the game tree has infinitely many nodes, it cannot be explicitly written down on paper or—perhaps more importantly—in computer memory. An algorithm for computing the optimal mixed strategy to commit to must operate on a different representation of the game—most naturally, the original normal form from which the game was obtained. Of course, an alternative is to discretize the space of mixed strategies, choosing only a finite subset of them to stand in as "representatives" in the hope of getting a reasonable approximation. This, however, gives up on exactly representing the game, and moreover is not even a computationally efficient way of solving the game, as we will discuss in more detail later. A closely related issue is that this infinitely-sized extensive-form representation does little to facilitate seeing the underlying structure of the game.[3] From seeing it (or a finite approximation of it), the viewer may not even realize that player 1's actions in the game correspond to the set of all mixed strategies of an original normal-form game.[4]

Still, one may argue that, while it may be true that the extensive form obscures some of the structure of the game, this is not sufficient reason to study Stackelberg mixed strategies separately (i.e., as directly providing a solution for a game represented in normal form). After all, it is often the case that, when we consider a solution concept in the special context of some specific family of games, additional structure appears that was not there in the general case. However, in what follows, we will see that there are other reasons to study Stackelberg mixed strategies separately.

2 Von Neumann's heritage: Zero-sum games

If there is one class of games that game theory can be said to truly *solve* (other than games solvable by iterated dominance), it is that of two-player zero-sum games. This is due to von Neumann's famous *minimax theorem* (von Neumann 1928). In such games, there are two players with pure strategy sets S_1 and S_2, respectively, and for all $s_1 \in S_1, s_2 \in S_2$, we have $u_1(s_1, s_2) +$

[3] Schelling (1960) similarly suggests that, by the time we have incorporated aspects such as commitment moves into a standard game-theoretic representation of the game at hand, we have abstracted away these issues and are at some level not really studying them anymore.

[4] It is easy to be misled by Figure 3 into thinking that it does make this fairly obvious, due to the natural ordering of the mixed strategies from left to right. However, this is an artifact of the fact that there are only two pure strategies for player 1 in the original game. If there were three pure strategies, it would not be possible to order the mixed strategies so naturally from left to right. We could in principle visualize the resulting tree in three dimensions instead of two. For more pure strategies, this of course becomes problematic. More importantly, such visualizations are technically *not* part of the extensive form. The extensive form only specifies a *set* of actions for each node, and no ordering, distance function, or topology on them. Such are only added when we draw the tree on a piece of paper (or in three dimensions, etc.).

$u_2(s_1, s_2) = 0$.[5] Consider the scenario where player 1 is extremely conservative and assumes that, no matter which mixed strategy she chooses, player 2 will manage to choose the strategy that is worst for player 1. Under this pessimistic assumption, player 1 can still guarantee herself

$$\max_{\sigma_1 \in \Sigma_1} \min_{s_2 \in S_2} u_1(\sigma_1, s_2)$$

where $\Sigma_1 = \Delta(S_1)$ is the set of player 1's mixed strategies. Strategies σ_1 that achieve this maximum are known as *maximin* strategies. If player 2 were to make a similar pessimistic assumption, he could guarantee himself

$$\max_{\sigma_2 \in \Sigma_2} \min_{s_1 \in S_1} u_2(s_1, \sigma_2)$$

Because the game is zero-sum, instead of trying to maximize his own utility, player 2 could equivalently try to minimize player 1's utility. Then, a pessimistic player 2 could guarantee that player 1 gets no more than

$$\min_{\sigma_2 \in \Sigma_2} \max_{s_1 \in S_1} u_1(s_1, \sigma_2)$$

Strategies σ_2 that achieve this minimum are known as *minimax* strategies. Indeed, note that

$$\min_{\sigma_2 \in \Sigma_2} \max_{s_1 \in S_1} u_1(s_1, \sigma_2) = - \max_{\sigma_2 \in \Sigma_2} \min_{s_1 \in S_1} u_2(s_1, \sigma_2)$$

but this is not yet the minimax theorem. Rather, the minimax theorem states that

$$\max_{\sigma_1 \in \Sigma_1} \min_{s_2 \in S_2} u_1(\sigma_1, s_2) = \min_{\sigma_2 \in \Sigma_2} \max_{s_1 \in S_1} u_1(s_1, \sigma_2)$$

This quantity is known as the *value* of the game. If (for example) the game is played repeatedly by sophisticated players, it seems very reasonable to expect that this is the average value that player 1 will obtain from a round of play over time. If she were getting less, she should just switch to a strategy that guarantees at least the value. If she were getting more, then player 2 should switch to a strategy that guarantees that player 1 gets at most the value.

From the minimax theorem, it is straighforward to deduce that a strategy profile is a Nash equilibrium of a two-player zero-sum game if and only if player 1 plays a maximin strategy and player 2 plays a minimax strategy. Hence, the concept of Nash equilibrium provides a generalization of these strategies to general-sum games. On the other hand, it is even easier to see that the Stackelberg mixed strategies for player 1 coincide with her maximin strategies in a two-player zero-sum game; the definition of a Stackelberg mixed strategy is a straightforward generalization of that of a maximin strategy in such games. Hence, Stackelberg mixed strategies and Nash equilibrium strategies coincide in two-player zero-sum games. This should not be surprising, because any solution concept that does not coincide with (or refine) maximin/minimax

[5] *Constant-sum* games, in which $u_1(s_1, s_2) + u_2(s_1, s_2) = c$ for some constant c, are effectively equivalent.

maximize v_1
subject to
$(\forall s_2 \in S_2)\ v_1 - \sum_{s_1 \in S_1} u_1(s_1, s_2) p_{s_1} \leq 0$
$\sum_{s_1 \in S_1} p_{s_1} = 1$
$(\forall s_1 \in S_1)\ p_{s_1} \geq 0$

Fig. 4 Linear program formulation for computing a maximin strategy for player 1. p_{s_1} is a variable indicating the probability placed on player 1's pure strategy s_1. The first constraint requires that v_1 be at most the utility that player 1 gets when player 2 best-responds, and the goal is to maximize this minimum utility for player 1.

strategies in two-player zero-sum games would seem suspect given the minimax theorem. Nevertheless, Stackelberg mixed strategies and Nash equilibrium strategies generalize to general-sum games in different ways, and arguments can be given both ways as to which is more natural. But viewing Stackelberg mixed strategies (only) as the solution to an extensive-form game obscures this and would appear to leave Nash equilibrium (or related equilibrium concepts) as the only generalization.

We will return to properties that are obscured by not studying Stackelberg mixed strategies directly on the normal form in Section 4. First, however, we will consider computational aspects.

3 The computational angle

Historically, the development of our understanding of the minimax theorem was tied up with the development of *linear programming*. A linear program describes an optimization problem over multiple variables, with multiple linear inequality constraints on these variables as well as an objective to be minimized or maximized. Figure 4 shows how the problem of finding a maximin strategy can be formulated as a linear program (as is well known). Dantzig (1951) showed that, from a computational viewpoint, the two problems are equivalent.[6]

Besides providing a mathematically elegant way to model many optimization problems, linear programs are useful for determining the *computational complexity* of problems. An example of a computational problem is that of finding maximin strategies of two-player zero-sum games, represented in normal form (and any specific two-player zero-sum game would be an *instance* of this problem). Computer scientists design *algorithms* for solving such problems. Such an algorithm is generally required to provide the correct output for *any* input—e.g., any two-player zero-sum game. With some training, designing correct algorithms is usually not that hard; however, for many problems, designing *fast* algorithms is challenging. One may wonder why we should really

[6] In fact, he pointed out that there was a case in which his reduction from linear programs to zero-sum games does not work; this gap was later filled by Adler (2013).

care whether algorithms are fast. So what if my computer needs to work a little harder? I can wait a few seconds if needed. The flaw in this reasoning is that for many problems, the runtime of the obvious algorithms scales *exponentially* in the size of the input, so that as we increase the size of the problem instances, rather quickly we find instances that would take the algorithm more than the lifetime of the universe to solve, even on the fastest computer available. In contrast, other algorithms have the property that their runtime scales only as a *polynomial* function in the size of the input. Problems for which such algorithms exist are generally considered *tractable*, and the algorithm is said to be *efficient*. Note that the same problem may have two correct algorithms, one of which scales exponentially and one of which (perhaps one that requires more design effort) scales polynomially; in this case, still, the problem is considered tractable. (It is always possible to find a slow algorithm for a problem; the question is whether fast ones exist.)

It is known that linear programs can be solved in polynomial time (Khachiyan 1979). That means that any problem that can be rephrased as (or, technically, *reduced to*) a linear program can also be solved in polynomial time. (Note that this does require that the linear program itself can be obtained in polynomial time, and *a fortiori* that the linear program has polynomial size—an exponentially sized linear program could not be written down in polynomial time.) In particular, this implies that the problem of computing a maximin strategy of a two-player zero-sum game can be solved in polynomial time.

Now, what about the more general problem of computing a Nash equilibrium of a two-player (general-sum) game represented in normal form? This one turns out to be significantly trickier. There is no known linear program formulation for this problem, and more generally, no polynomial-time algorithms are known. Perhaps the best-known algorithm—the *Lemke-Howson* algorithm (Lemke and Howson 1964)—is known to require exponential time on some families of games (Savani and von Stengel 2006). (Other algorithms more obviously require exponential time in some cases (Dickhaut and Kaplan 1991; Porter et al. 2008; Sandholm et al. 2005).) Can we prove it is in fact impossible to design a polynomial-time algorithm for this problem? As is the case for many other computational problems, we do not currently have the techniques to unconditionally prove this. What computer scientists often can do in these cases is to prove the following type of result: "If this problem can be solved in polynomial time, then so can any problem in the class C of problems." In this case, the original problem is said to be C-hard (and, if the problem additionally is itself a member of C, it is said to be C-complete). The most famous such class is NP. Indeed, problems such as the following turn out to be NP-complete: "Given a two-player game in normal form, determine whether it has a Nash equilibrium in which pure strategy s_1 receives positive probability," or "Given a two-player game in normal form, determine whether it has a Nash equilibrium in which the sum of the players' expected utilities exceeds a threshold ϵ" (Gilboa and Zemel 1989; Conitzer and Sandholm 2008). For the problem of computing just one Nash equilibrium of a two-player game in normal form—i.e., any one Nash equilibrium will do—the problem is known to be PPAD-complete (Daskalakis et al. 2009; Chen et al. 2009).[7] The precise definition of these classes is not of importance here; suffice it to say that computer scientists generally give up on designing an efficient algorithm for the problem when such a complexity result is found for it.

Then, what about computing a Stackelberg mixed strategy for a two-player game represented in normal form? One approach—arguably the most natural one when we do not study Stackelberg mixed strategies separately—would be to convert the game to the extensive-form representation of the leadership model, and solve the resulting game for a subgame perfect Nash equilibrium. As discussed before, one problem with this approach is that the extensive form of such a game in fact has infinite size, and can therefore not be (directly) represented in computer memory. A natural (though only approximate) approach is to discretize the space of distributions to which player 1 can commit. For any N, we can restrict our attention to the finitely many distributions that only use probabilities that are multiples of $1/N$. However, this still results in $\binom{N+|S_1|-1}{|S_1|-1}$ different distributions. (This is equal to the number of ways in which N indistinguishable balls—corresponding to the N atomic units of $1/N$ probability mass—can be placed in S_1 distinguishable bins—corresponding to the different pure strategies for player 1.) This number is exponential in the number of pure strategies for player 1, so this approach cannot lead us to a polynomial-time algorithm (even ignoring the fact that it in general will not provide an exact solution).

As it turns out, though, it is in fact possible to solve this problem in polynomial time, if we avoid converting the game into extensive form first. Because computing a Stackelberg mixed strategy is a generalization of computing a maximin strategy in a two-player zero-sum game, it should not come as a surprise that this algorithm relies on linear programming. The algorithm uses a divide-and-conquer approach, as follows. For every pure strategy $s_2^* \in S_2$ for player 2, we ask the following question: (Q) what is the highest utility that player 1 can obtain, *under the condition that player* 1 *plays a mixed strategy* σ_1 *to which* s_2^* *is a best response* (and assuming that player 2 in fact responds with s_2^*)? For some strategies s_2^*, it may be the case that there is *no* σ_1 to which s_2^* is the best response, and this will correspond to the linear program having no feasible solutions—but this obviously cannot be the case for *all* of player 2's strategies. Among the ones that do have feasible solutions, we choose one that gives the highest objective value, and the corresponding mixed strategy σ_1 is an (optimal) Stackelberg mixed strategy for player 1. It remains to be shown how to formulate (Q) as a linear program. This is shown in Figure 5. (Later on, I will discuss another formulation for the problem, as a single linear program (Figure 7).) The main point to take away is that the extensive-form view of Stackelberg mixed strategies does little to lead us to an efficient al-

[7] Papadimitriou (1994) introduced the class PPAD. Daskalakis and Papadimitriou (2005) showed that the problem is PPAD-hard for three players; Chen and Deng (2005) then obtained the stronger result that it is PPAD-hard even for two players. Etessami and Yannakakis (2010) proved that with three or more players, the problem of computing an exact Nash equilibrium, rather than an ϵ-equilibrium, is FIXP-complete.

maximize $\sum_{s_1 \in S_1} u_1(s_1, s_2^*) p_{s_1}$
subject to
$(\forall s_2 \in S_2)\ \sum_{s_1 \in S_1} (u_2(s_1, s_2^*) - u_2(s_1, s_2)) p_{s_1} \geq 0$
$\sum_{s_1 \in S_1} p_{s_1} = 1$
$(\forall s_1 \in S_1)\ p_{s_1} \geq 0$

Fig. 5 Linear program formulation for computing a Stackelberg mixed strategy (more precisely, an optimal strategy for player 1 that induces s_2^* as a best response) (Conitzer and Sandholm 2006; von Stengel and Zamir 2010). p_{s_1} is a variable indicating the probability placed on player 1's pure strategy s_1. The objective gives player 1's expected utility given that player 2 responds with s_2^*, and the first constraint ensures that s_2^* is in fact a best response for player 2.

gorithm for computing them, whereas studying these strategies separately, as providing a solution for games represented in normal form, suggests that a linear programming approach may succeed, which in fact it does.

4 Other properties that are easier to interpret when studying Stackelberg mixed strategies separately

As discussed in Section 2, if we do not separately study how Stackelberg mixed strategies provide solutions to 2-player normal-form games, this obscures that they are a generalization of maximin strategies. In this section, I discuss some other properties of Stackelberg mixed strategies that involve comparisons to Nash equilibria of the simultaneous-move game. I argue that it is easier to get insight into these properties if we do study Stackelberg mixed strategies separately.

One may wonder about the following: is commitment always advantageous, relative to, say, playing a Nash equilibrium of the simultaneous-move game? It is clear that committing to a *pure* strategy is not always a good idea. For example, when playing Rock-Paper-Scissors, presumably it is not a good idea to commit to playing Rock and make this clear to your opponent. On the other hand, committing to the mixed strategy $(1/3, 1/3, 1/3)$ does not hurt one bit.[8] More generally, in any two-player zero-sum game, (optimally) committing to a mixed strategy beforehand does not hurt (or help) one bit: this is exactly what the minimax theorem tells us. But what about in general-sum games? We have already seen that it can (strictly) help there,[9] but does it ever hurt? It turns out that it does not, and it is not hard to get some intuition why. Consider any Nash equilibrium (σ_1, σ_2) of the simultaneous-move game—for example, one that is optimal for player 1. Then, if player 1 commits to playing σ_1, then any one of the pure strategies in σ_2's support is a best response. If we assume that player 2 breaks ties in player 1's favor—or if the game is such

[8] An exception is, of course, if we play against an exploitable non-game-theoretic player, such as one who always plays Scissors.

[9] For a study of *how much* it can help, see Letchford et al. (2014).

that player 1 can, by adjusting her strategy slightly, make the most favorable strategy in σ_2's support the unique best response for player 2—then player 1 must be at least as well off as in the case where player 2 responds with σ_2, which is the Nash equilibrium case. Without these assumptions, things become significantly hairier, because depending on how player 2 breaks ties, player 1 may end up with any utility in an interval—but it can be shown that this interval is still more favorable than the corresponding interval for Nash equilibrium (von Stengel and Zamir 2010). At least in my view, comparisons such as these are more natural when we study Stackelberg mixed strategies separately, so that we are comparing player 1's utility in two different solutions of the same game, rather than comparing player 1's utility across two different games.

As another example, Kiekintveld et al. (2009) introduce a class of games called *security games*, which involve a defender and an attacker. In these games, the defender chooses how to allocate its resources to (subsets of) the targets, and the attacker chooses a target to attack. Both players' utilities are a function of (1) which target is attacked and (2) whether that target is defended by some resource(s). Holding the attacked target fixed, the defender prefers for it to be defended, and the attacker prefers for it not to be defended. Korzhyk et al. (2011) show that, under a minor assumption—namely, that if a resource can (simultaneously) defend a given set of targets, then it can also defend any subset of that set—every Stackelberg mixed strategy for the defender is also a Nash equilibrium strategy for the defender (in the simultaneous-move version of the game). (Moreover, it is shown that the Nash equilibria of the simultaneous-move game satisfy *interchangeability*: if (σ_1, σ_2) and (σ_1', σ_2') are equilibria, then necessarily so are (σ_1, σ_2') and (σ_1', σ_2).) Hence, in a sense, for these games, Stackelberg mixed strategies are a refinement of Nash equilibrium strategies (for the defender). Now, the point here is not to place undue emphasis on security games. Rather, the point is, again, that this type of refinement property is very cumbersome to state if we strictly hold to the view that Stackelberg mixed strategies are just subgame perfect Nash equilibria of a different game. We would have to make a statement about how the solutions to two different games relate to each other, rather than just being able to state that one concept is a refinement of the other.

5 The analogous (and related) case of correlated equilibrium

Unlike Stackelberg mixed strategies, *correlated equilibrium* (Aumann 1974) is commonly considered a solution concept in its own right. Roger Myerson has been quoted as saying that: "If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium." In fact, I personally believe that most of them would have discovered Nash equilibrium before correlated equilibrium, like we did, but this will be a difficult one to settle. The point, anyway, is well taken: correlated equilibrium is a natural solution concept that is technically more elegant than

Nash equilibrium in a number of ways. Having thus built up the suspense, let us now define the correlated equilibrium concept.

In a correlated equilibrium of a 2-player[10] game, an ordered pair of signals $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 = \Theta$ is drawn according to some distribution $p : \Theta \to [0, 1]$. (Note that the θ_i need not be independent or identically distributed.) Each player i receives her signal θ_i, and based on this takes an action in the game. That is, player i has a strategy $\tau_i : \Theta_i \to \Sigma_i$, where S_i is the set of actions for player i in the game and $\Sigma_i = \Delta(S_i)$ is the set of probability distributions over these actions.[11] All of this collectively constitutes a correlated equilibrium if it is optimal for each player to follow her strategy assuming that the other does so as well. That is, for every player i and every signal θ_i that i receives with positive probability ($p(\theta_i) = \sum_{\theta_{-i}} p(\theta_i, \theta_{-i}) > 0$), and for every action s_i that player i might take, we have

$$\sum_{\theta_{-i}} p(\theta_{-i}|\theta_i)(u_i(\tau_i(\theta_i), \tau_{-i}(\theta_{-i})) - u_i(s_i, \tau_{-i}(\theta_{-i}))) \geq 0$$

That is, the strategies τ_i are an equilibrium of the Bayesian game defined by the distribution over the signals. (Note, however, that this distribution is considered part of the solution.)

It is well known and straightforward to show that, if all we care about is the resulting distribution over outcomes S—where $S = S_1 \times S_2$ and the probability of an outcome $s = (s_1, s_2)$ is

$$P(s_1, s_2) = \sum_{(\theta_1, \theta_2) \in \Theta} p(\theta_1, \theta_2)\tau_1(\theta_1)(s_1)\tau_2(\theta_2)(s_2)$$

where $\tau_i(\theta_i)(s_i)$ is the probability that the distribution $\tau_i(\theta_i)$ places on s_i—then it is without loss of generality to

- let each player's signal space coincide with that player's action space, i.e., $\Theta_i = S_i$,
- consider the strategies where players simply follow their signals, i.e., if $\theta_i = s_i$, then $\tau_i(\theta_i)$ is the distribution that places probability 1 on s_i, and
- (consequently) for $(\theta_1, \theta_2) = (s_1, s_2)$, we have $P(s_1, s_2) = p(\theta_1, \theta_2)$.

That is, if a correlated equilibrium resulting in probability distribution P over outcomes S exists, then it is also a correlated equilibrium to draw the outcome directly according to that distribution and signal to each player the action that she is supposed to play in this outcome (but nothing more). Hence, we may dispense with the θ_i notation. It also allows us to describe the set of correlated equilibria with the set of inequalities in Figure 6.

[10] All of this is easily generalized to n players, but for simplicity I will stick to two players here.

[11] The notation here is a bit nonstandard: in isolation, it would be more natural to use A_i to denote the set of actions and S_i to denote the set of pure strategies, i.e., mappings from signals to actions. However, in order to make the comparison to other concepts easier, it will help to stick to using s_i for the rows and columns of the game.

$$(\forall s_1, s_1' \in S_1)\ \sum_{s_2 \in S_2}(u_1(s_1, s_2) - u_1(s_1', s_2))p_{s_1,s_2} \geq 0$$
$$(\forall s_2, s_2' \in S_2)\ \sum_{s_1 \in S_1}(u_2(s_1, s_2) - u_2(s_1, s_2'))p_{s_1,s_2} \geq 0$$
$$\sum_{s_1 \in S_1} \sum_{s_2 \in S_2} p_{s_1,s_2} = 1$$
$$(\forall s_1 \in S_1, s_2 \in S_2)\ p_{s_1,s_2} \geq 0$$

Fig. 6 Linear inequalities specifying the set of correlated equilibria of a 2-player game (this can easily be generalized to n-player games). p_{s_1,s_2} is a variable representing the probability of the profile (s_1, s_2) being played. The first constraint says that player 1, upon receiving a signal to play s_1, should not be better off playing another s_1' instead. The second constraint, for player 2, is similar.

Now let us consider the question of whether correlated equilibrium "deserves" to be considered a solution concept in its own right. One might argue that it does not, in a way that is analogous to the argument against studying Stackelberg mixed strategies separately, as follows. We can define the set of correlated equilibria of a game G simply as the set of all Nash equilibria[12] of all games that result from extending G with signals to the players, as described at the beginning of this section. Hence, the concept can be seen as derivative rather than primitive. Instead of thinking about it as a separate solution concept, we can simply think of it as the application of the Nash equilibrium concept to a modified game (the game extended with signals).[13]

Of course, my aim here is not to *actually* argue that correlated equilibrium should not be considered a separate solution concept. Correlated equilibria have many elegant and useful properties that would be obscured by thinking of them merely as the application of the Nash equilibrium concept to an enriched game. The fact that correlated equilibria can be computed in polynomial time using the linear feasibility formulation in Figure 6 is one example of this: an explicit Bayesian game formulation (with signals) would presumably not be helpful for gaining insight into this polynomial-time computability, as such a formulation would involve exponentially many strategies. Rather, the point is that the case for studying correlated equilibrium separately is, in my view, quite similar to the case for studying Stackelberg mixed strategies separately.

Indeed, returning to the line of reasoning from Section 4, when *both* correlated equilibria and Stackelberg mixed strategies are studied in their own

[12] Nash equilibria of a game with private information are often referred to as *Bayes-Nash* equilibria.

[13] It could be argued that the analogy is imperfect because in the Stackelberg version of the argument, the game is modified to a *single* (two-stage) game, whereas in the correlated equilibrium version of the argument, two different correlated equilibria potentially require different ways of modifying the game, extending them with different signaling schemes. It is not entirely clear to me how significant this distinction is. In any case, if two correlated equilibria require different signaling schemes, then consider a new, joint signaling scheme where each player receives the signals from *both* signaling schemes, with the signals drawn independently across the two schemes. Then, both correlated equilibria are (Bayes-)Nash equilibria of the game with the joint signaling scheme (with the players simply ignoring the part of the signal that corresponds to the other equilibrium). Taking this to the limit, we may imagine a single, universal signaling scheme such that all correlated equilibria of interest are Nash equilibria of the game with this universal signaling scheme.

maximize $\sum_{s_1 \in S_1} \sum_{s_2 \in S_2} u_1(s_1, s_2) p_{s_1,s_2}$
subject to
$(\forall s_2, s_2' \in S_2)\ \sum_{s_1 \in S_1} (u_2(s_1, s_2) - u_2(s_1, s_2')) p_{s_1,s_2} \geq 0$
$\sum_{s_1 \in S_1} \sum_{s_2 \in S_2} p_{s_1,s_2} = 1$
$(\forall s_1 \in S_1, s_2 \in S_2)\ p_{s_1,s_2} \geq 0$

Fig. 7 A single linear program for computing a Stackelberg mixed strategy (Conitzer and Korzhyk 2011). This linear program can be obtained by combining the linear programs (one for each s_2) from Figure 5 and renaming the variable p_{s_1} from the linear program corresponding to s_2 to p_{s_1,s_2}. An optimal solution for which there exists some s_2^* such that $p_{s_1,s_2} = 0$ whenever $s_2 \neq s_2^*$ is guaranteed to exist.

right—as applying directly to the normal form of the game, rather than being the solutions to two different games—it becomes apparent that they are in fact closely related. Consider again the linear program in Figure 5, which is used to compute an optimal strategy for the leader under the constraint that a particular pure strategy for the follower must be optimal, so that solving this linear program for every pure follower strategy gives an optimal solution. Conitzer and Korzhyk (2011) observe that we can combine all these linear programs (one for each pure follower strategy) into a larger single linear program. The resulting linear program is given in Figure 7. The constraints of this linear program are exactly the set of linear inequalities above for correlated equilibrium (Figure 6), except that only the constraints for player 2 appear. Moreover, the objective is to maximize player 1's utility. An immediate corollary of this is a result (which was earlier proved directly by von Stengel and Zamir (2010)) that a Stackelberg mixed strategy is at least as good for the leader as any correlated equilibrium, because if we add the constraints for player 1 we get a linear program for finding the best correlated equilibrium for player 1—and adding constraints can never improve the optimal value of a linear program. One way to interpret the linear program in Figure 7 is as follows: player 1 now gets to commit to a *correlated* strategy, where she chooses a profile (s_1, s_2) according to some distribution, signals to player 2 which action s_2 he should play (where there is a constraint on the distribution such that player 2 is in fact best off listening to this advice), and plays s_1 herself. Conitzer and Korzhyk (2011) prove that there always exists an optimal solution where player 1 always sends the same signal s_2^* to player 2, so that effectively player 1 is just committing to a mixed strategy. (When there are 3 or more players, then the optimal solution may require true correlation.) Again, the main point is that this close relationship between Stackelberg mixed strategies and correlated equilibrium is obscured if we think of Stackelberg mixed strategies in terms of extensive-form games (or, for that matter, if we think of correlated equilibrium in terms of Bayesian games).

6 Conclusion

In game theory, sometimes one solution concept is equivalent to the application of another solution concept to a modified representation of the game. In such cases, is it worthwhile to study the former in its own right, as it applies to the original representation? It appears difficult to answer this question in general, without knowing either what the solution concept is or what the context is in which we are attempting to answer the question. In this article, I have investigated this question for the specific concept of Stackelberg mixed strategies. Often, game theorists think of Stackelberg models as just that—a different model of how the game is to be played, rather than a different way of solving the game. There are certainly good reasons for this view. However, my overall conclusion is that, in the context of Stackelberg *mixed* strategies, limiting oneself to this view comes at too great a cost. Studying them in their own right, as providing solutions of normal-form games, often facilitates mathematical analysis—making connections to other concepts such as minimax strategies, Nash equilibrium, and correlated equilibrium more apparent—as well as computational analysis, allowing one to find efficient direct algorithms rather than attempting to work with discretizations of infinitely sized objects.

I should emphasize, however, that the possibility of viewing these strategies as solutions of an extensive-form game surely remains valuable too. For example, from the perspective of epistemic game theory, Stackelberg mixed strategies may be easiest to justify via this interpretation. Similarly, I would argue that both views are valuable for correlated equilibrium, which I have argued is an analogous case: while it is extremely useful for mathematical and computational purposes to study correlated equilibrium as a solution concept for normal-form games in its own right, as indeed it usually is viewed, seeing it as a (Nash) equilibrium of an enriched game has its own benefits—not the least of which is that this is a common and natural way to introduce the concept. Hence, I believe that the choice between the two views is much like the choice between seeing the young woman and the old woman in the famous ambiguous image. While we generally cannot hold both views simultaneously, if we do not allow our minds to switch from one view to the other, we miss out on much of what is there.

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