Failures of the VCG Mechanism in Combinatorial Auctions and Exchanges*

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ABSTRACT

The *VCG mechanism* is the canonical method for motivating bidders in combinatorial auctions and exchanges to bid truthfully. We study two related problems concerning the VCG mechanism: the problem of revenue guarantees, and that of collusion. The existence of these problems even in one-item settings is well-known; in this paper, we lay out their full extent in multi-item settings. We study four settings: combinatorial forward auctions with free disposal, combinatorial reverse auctions with free disposal, combinatorial forward (or reverse) auctions without free disposal, and combinatorial exchanges. In each setting, we give an example of how additional bidders (colluders) can make the outcome much worse (less revenue or higher cost) under the VCG mechanism (but not under a first price mechanism); derive necessary and sufficient conditions for such an effective collusion to be possible under the VCG mechanism; and (when nontrivial) study the computational complexity of deciding whether these conditions hold.

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Algorithms, Economics, Theory

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1. INTRODUCTION

Combinatorial markets are important mechanisms for allocating tasks and resources in multiagent systems. In a *combinatorial auction*, there are multiple items for sale, and bidders are allowed to place a bid on a *bundle* of these items rather than just on the individual items. A rapidly growing body of literature is devoted to the study of combinatorial auctions, and, to a lesser extent, variations of it, such as combinatorial reverse auctions (where the auctioneer seeks to procure certain items) and combinatorial exchanges (where bidders can offer goods for sale as well as express demand for goods—even within the same bid). One important problem is the *winner determination problem* [16, 6, 19], which is to label bids as accepted or rejected to maximize the total value of the bids accepted (or, in the case of a reverse auction, to minimize their total value), under the constraint that the corresponding allocation of items does not require more items than are available (or, in the case of a reverse auction, under the constraint that all the desired items are procured).

Another key problem in auctions and exchanges (combinatorial or not) is that in general, the bidders may not bid their true valuations for the goods. The problem with untruthful bidding is that the winner determination algorithm can only base the final allocation of the goods on the reported valuations, and thus the final allocation may not be optimal relative to the bidders' true valuations. Thus, economic efficiency may be lost. However, by changing the payment rule, it is possible to motivate bidders to bid their true valuations. The best-known such payment rule is the Vickrey-Clarke-Groves (VCG) mechanism [20, 5, 9]. If the setting is general enough, given certain requirements, the VCG mechanism (or its generalization to *Groves mechanisms*) is in fact the only mechanism promoting truthful bidding [8, 15, 10, 21].[1] Because of this, the VCG mechanism constitutes the standard approach to promoting truthful bidding, and variations such as anytime VCG mechanisms [13] as well as distributed computation of VCG mechanisms [14] have been proposed.

Unfortunately, there are also many problems with the

[1]It should be noted, however, that when the agents' valuations are more restricted, the characterization of truthful mechanisms becomes less stringent [1, 11, 12, 10, 17, 4].

VCG mechanism [18, 3]. We will focus on two related problems: the VCG mechanism is vulnerable to collusion, and may lead to low revenue/high payment for the auctioneer. It is well-known that these problems occur even in single-item auctions (where the VCG mechanism specializes to the *Vickrey* or second-price sealed-bid auction). However, in the single-item setting, these problems are not as severe. For example, in a Vickrey auction, it is not possible for colluders to obtain the item at a price less than the bid of any other bidder. Additionally, in a Vickrey auction, various types of revenue equivalence with (for example) first-price sealed-bid auctions hold. As we will show, in the multi-item setting these properties do not hold and can be violated to an arbitrary extent. Some isolated examples of such problems with the VCG mechanism in multi-item settings have already been noted in the literature [3, 22, 2] (these will be discussed later in the paper). In contrast, our goal in this paper is to give a *comprehensive* characterization of how severe these problems can be and when these severe problems can occur. For the various variants of combinatorial auctions and exchanges, we study the following single problem that relates both issues under consideration: *Given some of the bids, how bad can the remaining bidders make the outcome?* Informally, "bad" here means that the remaining bidders are paid an inordinately large amount, or pay an inordinately small amount, relative to the goods they receive and/or provide. This is closely related to the problem of making revenue guarantees to the auctioneer. But it is also the collusion problem, if we conceive of the remaining bidders as colluders. (The collusion problem can become more difficult if the collusion is required to be *self-enforcing*. A collusion is self-enforcing when none of the colluders have an incentive to unilaterally deviate from the collusion. We will also study how this extra requirement affects our results.)

As it turns out, our fundamental problem is often computationally hard. Computational hardness here is a double-edged sword. On the one hand, if the problem is hard, collusion may not occur (or to a lesser extent) because the colluders cannot find a beneficial collusion. On the other hand, if the problem is hard, it is difficult to make strong revenue guarantees to the auctioneer. Of course, in either case, the computational hardness may be overcome in practice if the stakes are high enough.

2. THE VCG MECHANISM—REVIEW

All the results in this paper hold even when all bidders are *single-minded*, that is, they bid only on a single bundle of items. (Hence, we do not need to discuss bidding languages.) The VCG mechanism proceeds as follows. First, solve the winner determination problem to maximize the sum of the agents' utilities before payments (the sum of the values of the accepted bids, or, in the case of a reverse auction, the negative thereof). Call this sum of utilities a. Then, to determine winning bidder i's payment, remove that bidder's bid, and see what the maximum sum of the utilities before payments would have been with only the remaining bids. Call this sum of utilities b_i. Winning bidder i must pay $b_i - a + v_i$ where v_i is the value of winning bidder i's bid. Effectively, it is the externality the bidder imposed on the other bidders (before payments). We observe that this payment is negative if the bidder's presence makes the other bidders better off (before payments)—for example, in reverse auctions.

3. COMBINATORIAL (FORWARD) AUCTIONS

In a *combinatorial auction*, there is a set of items $I = \{s_1, s_2, \ldots, s_m\}$ for sale. A bid takes the form $b = (B, v)$, where $B \subseteq I$ and $v \in \mathbb{R}$. The winner determination problem is to label bids as accepted or rejected, to maximize the sum of the values of the accepted bids, under the constraint that no item occurs in more than one accepted bid. (This is assuming *free disposal*: items do not have to be allocated to anyone.)

3.1 Motivating example

(A similar example to the one described in this subsection has been given before [3], and examples of vulnerability to false-name bidding in combinatorial auctions [22] can in fact also be used to demonstrate the basic point. We include this subsection for completeness.) Consider an auction with two items, s_1 and s_2. Suppose we have collected two bids (from different bidders), both $(\{s_1, s_2\}, N)$. If these are the only two bids, one of the bidders will be awarded both the items and, under the VCG mechanism, will have to pay N. However, suppose two more bids (by different bidders) come in: $(\{s_1\}, N+1)$ and $(\{s_2\}, N+1)$. Then these bids will win. Moreover, neither winning bidder will have to pay anything! (This is because a winning bidder's item would simply be thrown away if that winning bidder were removed.)

This example demonstrates a number of issues. First, the addition of more bidders can actually decrease the auctioneer's revenue from an arbitrary amount to 0. Second, the VCG mechanism is not revenue-equivalent to the sealed-bid first-price mechanism in combinatorial auctions, even when all bidders' true valuations are common knowledge[2]—unlike in the single-item case. Third, even when the other bidders by themselves would generate nonnegative revenue for the auctioneer under the VCG mechanism, it is possible that two colluders can bid so as to receive all the items without paying anything.

The following sums up the properties of this example.

PROPOSITION 1. *In a forward auction (even with only 2 items), the following can hold simultaneously: 1. The winning bidders pay nothing under the VCG mechanism; 2. If the winning bids are removed, the remaining bids generate revenue N under the VCG mechanism; 3. If these bids were*

[2]Consider the above example with $N \geq 9$ and suppose that the four bids reflect the bidders' true valuations—since bidding truthfully is a weakly dominant strategy in the VCG mechanism. Running a first-price sealed bid auction in this setting, when all bidders' valuations are common knowledge, will not generate expected revenue less than $\frac{N}{8}$. For suppose the expected revenue is less than this. Then the probability that the revenue is at least $\frac{N}{4}$ must be less than $\frac{1}{2}$ by Markov's inequality. So, bidding $(\{A, B\}, \frac{N}{4})$ will win any bidder both items with probability at least $\frac{1}{2}$, leading to an expected utility of at least $\frac{1}{2}(N - \frac{N}{4}) = \frac{3N}{8}$. Because at most one of the three bidders with valuations $(\{A\}, N+1)$ or $(\{A, B\}, N)$ can win its desired bundle, it follows that at least one of these bidders has a probability of at most $\frac{1}{3}$ of winning its desired bundle, and thus has an expected utility of at most $\frac{N+1}{3}$. Because $N \geq 9$, $\frac{3N}{8} > \frac{N+1}{3}$, so this bidder would be better off bidding $(\{A, B\}, \frac{N}{4})$—contradicting the assumption that we are in equilibrium.

truthful (as we would expect under VCG), then if we had run a first-price sealed-bid auction instead (and the bidders' valuations were common knowledge), any equilibrium would have generated revenue $\Theta(N)$.

3.2 Characterization

We now characterize the settings where, given the noncolluders' bids, the colluders can receive all the items for free.

LEMMA 1. *If the colluders receive all the items at cost 0, then for any positive bid on a bundle B of items by a noncolluder, at least two of the colluders receive an item from B.*

PROOF. Suppose that for some positive bid b on a bundle B by a noncolluder i, one of the colluders c receives all the items in B (and possibly others). Then, in the auction where we remove that colluder's bids, one possible allocation gives every remaining bidder all the goods that bidder received in the original auction; additionally, it gives i all the items in bundle B; and it disposes of all the other items c received in the original auction. With this allocation, the total value of the accepted bids by bidders other than c is at least $v(b)$ more than in the original auction. Because the total value obtained in the new auction is at least the value of this particular allocation, it follows that c imposes a negative externality of at least $v(b)$ on the other bidders, and will pay at least $v(b)$. □

LEMMA 2. *Suppose all the items in the auction can be divided among the colluders in such a way that, for any positive bid on a bundle of items B by a noncolluder, at least two of the colluders receive an item from B. Then the colluders can receive all the items at cost 0.*

PROOF. For the given partition of items among the noncolluders, let each colluder place a bid with an extremely large value on the bundle consisting of the items assigned to him in the partition. (For instance, twice the sum of the values of all noncolluders' bids.) Then, the auction will clear awarding each colluder the items assigned to him by the partition. Moreover, if we remove the bids of one of the colluders, all the remaining colluders' bids will still win—and thus none of the noncolluders' bids will win, because each such bid requires items assigned to at least two colluders by the partition (and at least one of them is still in the auction and wins these items). Thus, each colluder (individually) imposes no externality on the other bidders. □

Combining these two lemmas, we get:

THEOREM 1. *The colluders can receive all the items at cost* 0 *if and only if it is possible to divide the items among the colluders in such a way that, for any positive bid B by a noncolluder, at least two colluders receive an item from B.*

3.3 Self-enforcing collusion

It turns out that requiring that the collusion is self-enforcing (i.e., no colluder has an incentive to unilaterally deviate) is no harder for the colluders:

THEOREM 2. *Whenever the colluders can receive all the items for free, they can also receive them all for free in a self-enforcing way.*

PROOF. Let each colluder bid on the same bundle as before; but, increase the bid value of each colluder by an amount that exceeds the utility that any colluder can get from any bundle of items. The colluders will continue to receive all the items at a cost of 0. Now, the only reason that a colluder may wish to deviate from this is that the colluder wishes to obtain items outside of the colluder's assigned bundle. However, doing so would prevent one of the other bundles from being awarded to its designated colluder. This would cause a decrease in the total value of bids awarded to bidders other than the deviating colluder that exceeds the utility of the deviating colluder for any bundle, and the deviating colluder would have to pay for this decrease under the VCG mechanism. Therefore, there is no incentive for the colluder to deviate. □

3.4 Complexity

In order to collude in the manner described above, the n colluders must solve the following computational problem.

DEFINITION 1 (DIVIDE-SUBSETS). *Suppose we are given a set I, as well as a collection $R = \{S_1, \ldots, S_q\}$ of subsets of it. We are asked whether I can be partitioned into n parts $T_1, T_2, \ldots, T_n$ so that no subset $S_i \in R$ is contained in one of these parts.*

THEOREM 3. *DIVIDE-SUBSETS is NP-complete, even when $n = 2$.*

PROOF. The problem is technically identical to HYPERGRAPH-2-COLORABILITY, which is known to be NP-complete [7]. □

This hardness result only states that it is hard to identify the *most* beneficial collusion, and one may wonder whether it is perhaps easier to find *some* beneficial collusion. It turns out that the hardness of the former problem implies the hardness of the latter problem: the utility functions of the colluders can always be such that only the most beneficial collusion actually benefits them, in which case the two problems are the same. This observation can also be applied to hardness results presented later in the paper.

4. COMBINATORIAL REVERSE AUCTIONS

In a *combinatorial reverse auction*, there is a set of items $I = \{s_1, s_2, \ldots, s_m\}$ to be procured. A bid takes the form $b = (B, v)$, where $B \subseteq I$ and $v \in \mathbb{R}$. (Here, v represents the value that the bidder must be compensated by in order to provide the goods B.) The winner determination problem is to label bids as accepted or rejected, to minimize the sum of the values of the accepted bids, under the constraint that each item occurs in at least one accepted bid. (This is assuming free disposal.)

4.1 Motivating example

Consider a reverse auction with m items, $s_1, s_2, \ldots, s_m$. Suppose we have collected two bids (from different bidders), both $(\{s_1, s_2, \ldots, s_m\}, N)$. If these are the only two bids, one of the bidders will be chosen to provide all the goods, and, under the VCG mechanism, will be paid N. However, suppose m more bids (by different bidders) come in: $(\{s_1\}, 0), (\{s_2\}, 0), \ldots, (\{s_m\}, 0)$. Then, these m bids will win. Moreover, each bidder will be paid N under the VCG mechanism. (This is because without this bidder, we would have had to accept one of the original bids.) Thus, the total payment that needs to be made is mN.[3]

Again, this example demonstrates a number of issues. First, the addition of more bidders may actually increase the total amount that the auctioneer needs to pay. Second, the VCG mechanism requires much larger payments than a first-price auction in the case where all bidders' valuations are common knowledge. (The first-price mechanism will not require a total payment of more than N for these valuations in any pure-strategy equilibrium.[4][5]) Third, even when the other bidders by themselves would allow the auctioneer to procure the items at a low cost under the VCG mechanism, it is possible for m colluders to get paid m times as much for all the items.

The following sums up the properties of this example.

PROPOSITION 2. *In a reverse auction, the following can hold simultaneously: 1. The winning bidders are paid mN under the VCG mechanism; 2. If the winning bids are removed, the remaining bids allow the auctioneer to procure everything at a cost of only N under the VCG mechanism; 3. If these bids were truthful (as we would expect under VCG), then if we had run a first-price sealed-bid reverse auction instead (and the bidders' valuations were common knowledge), any equilibrium in pure strategies would have required total payment of at most N. (However, there are also mixed-strategy equilibria with arbitrarily large expected total payment.)*

4.2 Characterization

Letting N be the sum of the values of the accepted bids when all the colluders' bids are taken out, it is clear that no colluder can be paid more than N. (With the colluder's bid, the sum of the values of others' accepted bids is still at least 0; without it, it can be at most N, because in the worst case the auctioneer can accept the bids that would be accepted if none of the colluders are present.) In this subsection, we will identify a necessary and sufficient condition for the colluders to be able to each receive N.

LEMMA 3. *If a colluder receives N, then the items that it has to provide cannot be covered by a subset of the noncolluders' bids with cost less than N.*

PROOF. If they could be covered by such a set, we could simply accept this set of bids (including those that were accepted already) rather than the colluder's bid, and increase the total cost by less than N. Thus, the colluder's VCG payment is less than N. □

Thus, in order for each of the n colluders to be able to receive N, it is necessary that there exist n disjoint subsets of the items, each of which cannot be covered with a subset of the noncolluders' bids with total value less than N. The next lemma shows that this condition is also sufficient.

LEMMA 4. *If there are n disjoint sets of items $R_1, \ldots, R_n$, each of which cannot be covered by a subset of the noncolluders' bids with cost less than N, then n colluders can be paid N each.*

PROOF. Let colluder i (for $i < n$) bid $(R_i, 0)$, and let colluder n bid $(R_n \cup (S - \bigcup_i R_i), 0)$. Then the total cost of all accepted bids with all the colluders is 0; but when one colluder is omitted, the items it won cannot be covered at a cost less than N (because its bid contained one of the R_i). Thus, each colluder's VCG payment is N. □

The next lemma shows that the necessary and sufficient condition above is equivalent to being able to *partition* all the items into n sets, so that no element of the partition can be covered by a subset of the noncolluders' bids with total value less than N. That is, we can restrict our attention to the case where the subsets exhaust all the items.

LEMMA 5. *The condition of Lemma 4 is satisfied if and only if it is possible to partition the items into $T_1, \ldots, T_n$ such that no T_i can be covered by a subset of the noncolluders' bids with cost less than N.*

PROOF. The "if" part is trivial: given T_i that satisfy the condition of this lemma, simply let $R_i = T_i$. For the "only if" part, given R_i that satisfy the condition of Lemma 4, let $T_i = R_i$ for $i < n$, and $T_n = R_n \cup (S - \bigcup_i R_i)$. We observe that this last set can also not be covered at a cost of less than N because it contains R_n. □

Combining all the lemmas, we get:

THEOREM 4. *The n colluders can receive a payment of N each (simultaneously), where N is the sum of the values of the accepted bids when all the colluders' bids are removed, if and only if it is possible to partition the items into $T_1, \ldots, T_n$ such that no T_i can be covered by a subset of the noncolluders' bids with cost less than N.*

4.3 Self-enforcing collusion

Unlike the case of combinatorial forward auctions, in reverse auctions, a stronger condition is required if the collusion is also required to be self-enforcing.

[3]Similar examples have been discovered in the context of purchasing paths in a graph [2]. However, in that setting, the buyer does not seek to procure all of the items, and hence the examples cannot be applied directly to combinatorial reverse auctions.

[4]Consider the above example and suppose that the $n+2$ bids reflect the bidders' true valuations—since bidding truthfully is a weakly dominant strategy in the VCG mechanism. Supposing that a pure-strategy equilibrium is being played, let the total payment to be made in this equilibrium be π. (We observe that the final allocation can still be uncertain, e.g. if there is a random tie-breaking rule.) Suppose $\pi > N$. Then, the expected utility for either one of the bidders interested in providing the whole bundle can never exceed $\pi - N$ (because the bidder will be paid 0 whenever none of its bids are accepted, and providing any items at all will cost it N). Moreover, it is not possible for both of these bidders to simultaneously have an expected utility of $\pi - N$ (as this would mean that both are paid π with certainty, contrary to the fact that the total payment is π). It follows at least one has an expected utility of $\pi - N - \epsilon$ for some $\epsilon > 0$. But then this bidder would be better off bidding $\pi - \frac{\epsilon}{2}$ for the whole bundle, which would be accepted with certainty and give an expected utility of $\pi - N - \frac{\epsilon}{2}$. It follows that the total payment in a pure-strategy equilibrium cannot exceed N.

[5]Perhaps surprisingly, the first-price combinatorial reverse auction for this example (with commonly known true valuations corresponding to the given bids) actually has mixed-strategy equilibria with arbitrarily high expected payments. We omit the proof because of space constraint.

THEOREM 5. *The n colluders can receive a payment of N each (simultaneously), where N is the sum of the values of the accepted bids when all the colluders' bids are removed, if and only if it is possible to partition the items into $T_1, \ldots, T_n$ such that 1) no T_i can be covered by a subset of the noncolluders' bids with cost less than N, 2) for no colluder i, the following holds: there exists a subset $T_i' \subseteq T_i$ such that T_i' can be covered by a set of noncolluders' bids with total cost less than $v_i(T_i) - v_i(T_i - T_i')$ (the marginal savings to agent i of not having to provide T_i').*

PROOF. Omitted due to space constraint. □

4.4 Complexity

In order to collude in the manner described above, the n colluders must solve the following computational problem.

DEFINITION 2 (CRITICAL-PARTITION). *We are given a set of items I, a collection of bids (S_i, v_i) where $S_i \subseteq I$ and $v_i \in \mathbb{R}$, and a number n. Say that the* cost *of a subset of these bids is the sum of their v_i; and that the cost $c(T)$ of a subset $T \subseteq I$ is the lowest cost of any subset of the bids whose S_i cover T. We are asked whether there exists a partition of I into n disjoint subsets $T_1, T_2, \ldots, T_n$, such that for any $1 \leq i \leq n$, $c(T_i) = c(I)$.*

THEOREM 6. *Even when the bids are so that a partition $T_1, \ldots, T_n$ is a solution if and only if no set $I - T_i$ covers all items in a bid, CRITICAL-PARTITION is NP-complete (even with $n = 2$).*

PROOF. The problem is in NP in this case because given a partition $T_1, \ldots, T_n$, it is easy to check if any set $I - T_i$ covers all items in a bid.

To show NP-hardness, we reduce an arbitrary NAESAT[6] instance (given by a set of clauses C over a set of variables V, with each variable occurring at most once in any clause) to the following CRITICAL-PARTITION instance with $n = 2$ (where we are trying to partition into T_1 and T_2). Let I be as follows. For every variable $v \in V$, there are two items labeled s_{+v} and s_{-v}. Let the bids be as follows. For every variable $v \in V$, there is a bid $(\{s_{+v}, s_{-v}\}, 2)$. For every clause $c \in C$, there are two bids $(\{s_l : l \in c\}, 2m_c - 1)$ and $(\{s_l : -l \in c\}, 2m_c - 1)$ where m_c is the number of literals occurring in c.

First we show that this instance satisfies the condition that a partition $T_1, \ldots, T_n$ is a solution if and only if no set $I - T_i$ covers all items in a bid. First, we observe that $c(I) = |I|$ (we can use all the bids of the form $(\{s_{+v}, s_{-v}\}, 2)$, getting a per-item cost of 1; no other bid gives a lower per-item cost).

Now, if some set $I - T_i$ covers all the items in a bid of the form $(\{s_{+v}, s_{-v}\}, 2)$, then $c(T_i) \leq 2|I| - 2$ (because we can simply omit this bid from the solution for all the items). If some set $I - T_i$ covers all the items in a bid of the form $(\{s_l : l \in c\}, 2m_c - 1)$, then $c(T_i) = |I| - 1$. (This is because we can now accept the "complement" bid $(\{s_l : -l \in c\}, 2m_c - 1)$, and we will have covered all the items s_{+v} and s_{-v} in T_i such that v occurs in c (precisely $2m_c$ items, because variables do not reoccur within a clause); for any other item s_{+v} or s_{-v}, we can accept the bid $(\{s_{+v}, s_{-v}\}, 2)$, and we need to accept at most $|V| - m_c$ such bids, leading to a total cost of $2m_c - 1 + 2(|V| - m_c) = |I| - 1$.)

On the other hand, suppose there is no set $I - T_i$ that covers all the items in a bid. Then, either T_i must include at precisely one of s_v and s_{-v}. (Otherwise one T_i would include neither and $I - T_i$ would cover all items in the bid $(\{s_{+v}, s_{-v}\}, 2)$.) Thus, when we are trying to cover T_i, covering items in it with bids of the form $(\{s_{+v}, s_{-v}\}, 2)$ would result in a per-item cost of 2. On the other hand, covering items in it with bids of the form $(\{s_l : l \in c\}, 2m_c - 1)$ or $(\{s_l : -l \in c\}, 2m_c - 1)$ would result in a per-item cost of at least $\frac{2m_c - 1}{m_c - 1} > 2$ (because at most $m_c - 1$ of the m_c items in the bid can be in T_i, otherwise T_i would cover all the items in the bid; but $T_i = I - T_{3-i}$ which by assumption does not cover all the items in any bid). It follows that $C(T_i) = 2|V| = |I| = c(I)$.

Now we show that the two instances are equivalent. First suppose there exists a solution to the NAESAT instance. Then partition the elements as $T_1 = \{s_l : l = true\}$ and $T_2 = \{s_l : l = false\}$, according to this solution. Clearly neither of $I - T_i = T_{3-i}$ covers a bid of the form $(\{s_{+v}, s_{-v}\}, 2)$. Also, because no clause has all its literals set to the same value (we have a NAESAT solution), the items in a corresponding bid $(\{s_l : l \in c\}, 2m_c - 1)$ or $(\{s_l : -l \in c\}, 2m_c - 1)$ are not all in the same set. By the previously proved property, it follows that this partition is a solution to the CRITICAL-PARTITION instance.

On the other hand, suppose that there exists a solution to the CRITICAL-PARTITION instance. Then label a literal *true* if $s_l \in T_1$, and *false* otherwise. By the previously proved property, because $(\{s_{+v}, s_{-v}\}, 2)$ is a bid, only one of s_{+v} and s_{-v} can be in $T_1 = I - T_2$, so this provides a consistent setting of the literals. Additionally, because $(\{s_l : l \in c\}, 2m_c - 1)$ is a bid, not all the s_l in that bid can be in $T_1 = I - T_2$. It follows that some of the literals $l \in c$ are set to *false*. Similarly, not all the s_l in that bid can be in $T_2 = I - T_1$, so some of the literals $l \in c$ are set to *true*. It follows that this assignment of truth values to variables is a solution to the NAESAT instance. □

[6]The goal in NAESAT is to assign truth values to all variables in such a way that there is no clause with all its literals set to *true*, and no clause with all its literals set to *false*.

5. COMBINATORIAL FORWARD (OR REVERSE) AUCTIONS WITHOUT FREE DISPOSAL

A combinatorial forward auction *without free disposal* is exactly the same as one with free disposal, with the exception that every item must be allocated to some bidder. Here, bids with a *negative* value may still be useful, as they allow us to remove some of the items—which may allow us to accept better bids for the remaining items.

Similarly, a combinatorial reverse auction without free disposal is exactly the same as one with free disposal, with the exception that no additional items can be procured. Here, bids with a *negative* value may occur—the (nondisposable) item may be a liability to the bidder.

In both cases, we seek to identify a subset of the bids that constitutes an exact cover of the items (no item covered more than once), and to maximize the bidders' total utility under this constraint. Therefore, the settings are technically identical, and in the rest of this section, we can restrict our attention to forward auctions without free disposal.

5.1 Motivating example

Consider a forward auction with two nondisposable items, s_1 and s_2. Suppose we have collected two bids (from different bidders), both $(\{s_1, s_2\}, N)$. If these are the only two bids, one of the bidders will be awarded both the items and, under the VCG mechanism, will have to pay N. However, suppose two more bids (by different bidders) come in: $(\{s_1\}, N+M)$ and $(\{s_2\}, N+M)$, with $M > 0$. Then these bids will win. Moreover, because without free disposal, we cannot accept either of these bids without the other, each of these bidders will be *paid* M under the VCG mechanism!

Again, this example demonstrates a number of issues. First, additional bidders may change the auctioneer's revenue from an arbitrarily large positive amount to an arbitrarily large negative amount (an arbitrarily large cost). Second, the VCG mechanism may require arbitrarily large payments from the auctioneer even in cases where a first-price auction would actually generate revenue for the auctioneer, in the case where all bidders' valuations are common knowledge. (The first-price mechanism will generate a revenue of at least N for these valuations in any pure-strategy equilibrium.[7,8]) Third, even when the other bidders by themselves would generate positive revenue for the auctioneer under the VCG mechanism, it is possible that two colluders can make the auctioneer pay each of them an arbitrarily large amount.

The following sums up the properties of this example.

PROPOSITION 3. *In a forward auction without free disposal (even with only two items), the following can hold simultaneously: 1. Each winning bidder is paid an arbitrary amount M under the VCG mechanism (where M depends only on the winners' bids); 2. If the winning bids are removed, the remaining bids actually generate revenue N to the auctioneer under the VCG mechanism; 3. If these bids were truthful (as we would expect under VCG), then if we had run a first-price sealed-bid auction instead (and the bidders' valuations were common knowledge), any equilibrium in pure strategies would have generated revenue N. (However, there are mixed-strategy equilibria with arbitrarily large cost to the auctioneer.)*

[7] Consider the above example and suppose that the four bids reflect the bidders' true valuations—since bidding truthfully is a weakly dominant strategy in the VCG mechanism. Supposing that a pure-strategy equilibrium is being played, let the total revenue to the auctioneer be ρ, where ρ is possibly negative. (We observe that the final allocation can still be uncertain, e.g. if there is a random tie-breaking rule.) Suppose $\rho < N$. Then the expected utility for either of the bidders interested in providing the whole bundle is at most $N - \rho$. (If the bidder receives a singleton item, its utility is $-\infty$; if it receives nothing, its utility is 0; if it receives both items, its utility is $N - \rho$.) Moreover, it is not possible for both of these bidders to both have an expected utility of $N - \rho$, as this would mean they both receive both items with probability 1. It follows that at least one of them has an expected utility of $N - \rho - \epsilon$ where $\epsilon > 0$. But then this bidder would be better off bidding $\rho + \frac{\epsilon}{2}$, as this bid would be accepted with certainty and give an expected utility of $N - \rho - \frac{\epsilon}{2}$. It follows that the expected revenue in a pure-strategy equilibrium cannot be less than N.

[8] Similarly to the case of the combinatorial reverse auction with free disposal, there are mixed-strategy equilibria in the first-price auction where the auctioneer is forced to make arbitrarily large payments—we omit the proof because of space constraint.

5.2 Characterization

In this subsection, we will identify a necessary and sufficient condition for the colluders to be able to each receive an arbitrary amount. Let $v(b)$ denote the value of bid b.

LEMMA 6. *If each colluder receives a payment of more than $2\sum_d |v(b_d)|$ (where d ranges over the noncolluders), then for each colluder c, the set of all items awarded to either that colluder or a noncolluder (that is, $s_c \cup \bigcup_d s_d$, where s_b is the set of items awarded to bidder b and d ranges over the noncolluders) cannot be covered exactly with bids from the noncolluders.*

PROOF. Say that the sum of the values of accepted noncolluder bids is D (which may be negative). Suppose that for one colluder c, the set of all items awarded to either her or a noncolluder (that is, $s_c \cup \bigcup_d s_d$) can be covered by a set of noncolluder bids of combined value C (which may be negative). Then removing colluder c can make the allocation at most $D - C$ worse to the other bidders (relative to their reported valuations), because we could simply accept the bids of combined value C and no longer accept the bids of combined value D, and keep the rest of the allocation the same. Thus, under VCG, that colluder should be rewarded at most $D - C \leq 2\sum_d |v(b_d)|$. □

Thus, in order for each colluder to be able to receive an arbitrarily large payment, it is necessary that there are n disjoint subsets of the items such that no such subset taken together with the remaining items can be covered exactly by the noncolluders' bids. Also, the set of remaining items must be exactly coverable by the noncolluders' bids (otherwise we cannot accept all the colluders' bids). The next lemma shows that this condition is also sufficient.

LEMMA 7. *If it is possible to partition the items into $R_1, \ldots, R_n, R_{n+1}$ such that for no $1 \leq i \leq n$, $R_i \cup R_{n+1}$ can be covered exactly with bids from the noncolluders; and such that R_{n+1} can be covered exactly with bids from the noncolluders; then for any $M > 0$, n colluders can place additional bids such that each of them receives at least M.*

PROOF. Let colluder i place a bid $(R_i, M + 3\sum_d |v(b_d)|)$ (where d ranges over the noncolluders). All these bids will be accepted, because it is possible to do so by also accepting the noncolluder bids that cover R_{n+1} exactly; and these noncolluder bids will have a combined value of at least $-\sum_d |v(b_d)|$, so that the sum of the values of all accepted bids is at least $(3n-1)\sum_d |v(b_d)| + nM$. (We observe that if we do not accept all of the colluder bids, the sum of the values of all accepted bids is at most $(3(n-1)+1)\sum_d |v(b_d)| + (n-1)M = (3n-2)\sum_d |v(b_d)| + (n-1)M$, which is less.) Now, if the bid of colluder i is removed, it is no longer possible to accept all the remaining $n-1$ colluder bids, because $R_i \cup R_{n+1}$ cannot be covered exactly with noncolluder bids. It follows that the total value of all accepted bids when i's bid is removed can be at most $(3(n-2)+1)\sum_d |v(b_d)| + (n-2)M$. When i's bid is not omitted, the sum of the values of all accepted bids other than i's is at least $(3(n-1)-1)\sum_d |v(b_d)| + (n-1)M$. Subtracting the former quantity from this, we get that the VCG payment to i is at least $\sum_d |v(b_d)| + M$. □

The next lemma shows that the necessary and sufficient condition above is equivalent to being able to partition all the items into n sets, so that no element of the partition can be covered exactly by a subset of the noncolluders' bids. That is, we can restrict our attention to the case where $R_{n+1} = \emptyset$.

LEMMA 8. *The condition of Lemma 7 is satisfied if and only if the items can be partitioned into $T_1, \dots, T_n$ such that no T_i can be covered exactly with bids from the noncolluders.*

PROOF. For the "if" part: given T_i that satisfy the condition of this lemma, let $R_i = T_i$ for $i \leq n$, and $R_{n+1} = \emptyset$. Then no $R_i \cup R_{n+1} = T_i$ can be covered exactly with bids from the noncolluders, and $R_{n+1} = \emptyset$ can trivially be covered exactly with noncolluder bids. For the "only if" part: given R_i that satisfy the condition of Lemma 7, let $T_i = R_i$ for $i < n$, and let $T_n = R_n \cup R_{n+1}$. That T_n cannot be covered exactly by noncolluder bids now follows directly from the conditions of Lemma 7. But also, no T_i with $i < n$ can be covered exactly: because if it could, then we could cover $R_i \cup R_{n+1} = T_i \cup R_{n+1}$ using the bids that cover T_i exactly together with the bids that cover R_{n+1} exactly (which exist by the conditions of Lemma 7). □

Combining all the lemmas, we get:

THEOREM 7. *The n colluders can receive a payment of at least M each (simultaneously), where M is an arbitrarily large number, if and only if it is possible to partition the items into $T_1, \dots, T_n$ such that no T_i can be covered exactly with bids from the noncolluders.*

5.3 Self-enforcing collusion

Again, a stronger condition is required if the collusion is also required to be self-enforcing.

THEOREM 8. *The n colluders can receive a payment of at least M each (simultaneously), where M is an arbitrarily large number, if and only if it is possible to partition the items into $T_1, \dots, T_n$ such that 1) no T_i can be covered exactly with noncolluder bids, 2) for no colluder i, the following holds: there exists a subset $T_i' \subseteq T_i$ such that T_i' can be covered by a set of noncolluders' bids with total value greater than $v_i(T_i) - v_i(T_i - T_i')$ (the marginal value to agent i of receiving T_i').*

PROOF. Omitted due to space constraint. □

5.4 Complexity

In order to collude in the manner described above, the n colluders must solve the following computational problem.

DEFINITION 3 (COVERLESS-PARTITION). *We are given a set I and a collection of subsets $S_1, S_2, \dots, S_q \subseteq I$. We are asked whether there is a partition of I into subsets $T_1, T_2, \dots T_n \subseteq I$ such that no T_i can be covered exactly by some of the S_i.*

THEOREM 9. *Even if there is a singleton S_i for all but two elements a and b, and $n = 2$, COVERLESS-PARTITION is NP-complete.*

PROOF. Omitted due to space constraint. □

5.5 An easier collusion problem

So far in this section, we have formulated the collusion problem so that *each* colluder should receive M, where M is an arbitrary amount. An easier problem for the colluders is to make sure that *together*, they receive M, where M is an arbitrary amount. Such a collusion may be less stable (because some of the colluders may be receiving very little). Nevertheless, as we will show, this type of collusion is possible whenever a weak (and easily verified, given the noncolluders' bids) condition holds: at least one item has no singleton bid on it. (A singleton bid is a bid on only one item.) We first show that this condition is necessary.

LEMMA 9. *If at least one colluder receives a payment of more than $\sum_d |v(b_d)|$ (where d ranges over the noncolluders), then there is at least one item s on which no noncolluder places a singleton bid.*

PROOF. If each item has a singleton noncolluder bid placed on it, then when we remove a colluder's bid, we can simply cover all the items in it with singleton bids (with a combined value of at least $-\sum_d |v(b_d)|$), and leave the rest of the allocation unchanged. It follows that the VCG payment to the colluder can be at most $\sum_d |v(b_d)|$). □

We now show that the condition is sufficient.

LEMMA 10. *If there is at least one item s on which no noncolluder places a singleton bid, then if one colluder bids $(\{s\}, 0)$, and the other colluder bids $(I - \{s\}, M + 2\sum_d |v(b_d)|)$ (for $M > 0$), then the total payment to the colluders is at least M.*

PROOF. The colluders' bids will be the only accepted ones (because colluder 2's bid has a greater value than all other bids combined). If we removed colluder 2's bid, the total value of the accepted bids would be at most $\sum_d |v(b_d)|$), so colluder 2 will pay at most this much under the VCG mechanism. If we removed colluder 1's bid, colluder 2's bid could no longer be accepted (because $\{s\}$ cannot be covered by itself), and thus the total value of the accepted bids could be at most $\sum_d |v(b_d)|$). It follows that colluder 1 is paid at least $M + \sum_d |v(b_d)|$). So the total payment to the colluders is at least M □

Combining the two lemmas, we get the desired result:

THEOREM 10. *Two (or more) colluders can receive a total payment of M, where M is an arbitrarily large number, if and only if there is at least one item that has no singleton bid placed on it by a noncolluder.*

6. COMBINATORIAL EXCHANGES

In a *combinatorial exchange*, there is a set of items $I = \{s_1, s_2, \dots, s_m\}$ that can be traded. A bid takes the form $b = (\lambda_1, \dots, \lambda_m, v)$, where $\lambda_1, \dots, \lambda_m, v \in \mathbb{R}$ (possibly negative). (Each λ_i is the number of units of the ith item that the bidder seeks to procure, and v is how much the bidder is willing to pay.) The winner determination problem is to label bids as accepted or rejected, under the constraint that the sum of the accepted vectors has its first m entries ≤ 0, to maximize the last entry of the sum of the accepted vectors. (This is assuming free disposal.) We will also use the notation $(\{(s_{i_1}, \lambda_{i_1}), (s_{i_2}, \lambda_{i_2}), \ldots, (s_{i_k}, \lambda_{i_k})\}, v)$ for representing a bid in which λ_{i_j} units of item s_{i_j} are demanded (and 0 units of each item that is not mentioned).

6.1 Characterization

In a combinatorial exchange with at least two items s_1 and s_2, let q_1 (respectively, q_2) be the total number of units of s_1 (respectively, s_2) offered for sale in bids so far (by noncolluders). Now consider the following two bids (by colluders): $(\{(s_1, q_1 + 1), (s_2, -q_2 - 1)\}, M + \sum_d |v(b_d)|)$ and $(\{(s_1, -q_1 - 1), (s_2, q_2 + 1)\}, M + \sum_d |v(b_d)|)$, where $M > 0$ and d ranges over the original (noncolluding) bids. Both these bids will be accepted (for otherwise, the total value of the accepted bids could be at most $M + 2\sum_d |v(b_d)| < 2(M + \sum_d |v(b_d)|)$). Moreover, if we remove one of these two bids, the other cannot be accepted (because its demand cannot be met), so the total value of the accepted bids can be at most $\sum_d |v(b_d)|$). It follows that the VCG payment to each of these two bidders is at least M. This proves the following:

THEOREM 11. *In a combinatorial exchange with at least two items, for any set of bids by noncolluders, two colluders can place bids so that each of them will receive at least M, where M is an arbitrary amount. Moreover, each one receives exactly the items that the other provides, so that their net contribution in terms of items is nothing.*

7. CONCLUSIONS

The VCG mechanism is the canonical mechanism for motivating the bidders to bid truthfully in combinatorial auctions and exchanges; if the setting is general enough, under some requirements, it is the only one. Unfortunately, it also introduces many problems. In this paper, we focused on the related problems of revenue guarantees and bidder collusion. We showed that these problems can be much worse in combinatorial auctions and exchanges than in single-item settings. We gave necessary and sufficient conditions for when these highly undesirable scenarios can occur, and showed how computationally hard it is to verify these conditions. One direction for future research is to characterize when similar but less severe failures of the VCG mechanism can occur. A more interesting (and difficult) direction is to find (possibly untruthful) mechanisms (perhaps for restricted settings) that do not suffer from these problems.

8. REFERENCES

[1] A. Archer and E. Tardos. Truthful mechanisms for one-parameter agents. In *FOCS*, 2001.

[2] A. Archer and E. Tardos. Frugal path mechanisms. In *SODA*, 991–999, 2002.

[3] L. M. Ausubel and P. Milgrom. The lovely but lonely Vickrey auction. In P. Cramton, Y. Shoham, and R. Steinberg, editors, *Combinatorial Auctions*, chapter 1. MIT Press, 2006.

[4] M. Babaioff, R. Lavi, and E. Pavlov. Mechanism design for single-value domains. In *AAAI*, 2005.

[5] E. H. Clarke. Multipart pricing of public goods. *Public Choice*, 11:17–33, 1971.

[6] Y. Fujishima, K. Leyton-Brown, and Y. Shoham. Taming the computational complexity of combinatorial auctions: Optimal and approximate approaches. In *IJCAI*, 548–553, 1999.

[7] M. Garey and D. Johnson. *Computers and Intractability*. W. H. Freeman and Company, 1979.

[8] J. Green and J.-J. Laffont. Characterization of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica*, 45:427–438, 1977.

[9] T. Groves. Incentives in teams. *Econometrica*, 41:617–631, 1973.

[10] R. Lavi, A. Mu'Alem, and N. Nisan. Towards a characterization of truthful combinatorial auctions. In *FOCS*, 574–583, 2003.

[11] D. Lehmann, L. I. O'Callaghan, and Y. Shoham. Truth revelation in rapid, approximately efficient combinatorial auctions. *Journal of the ACM*, 49(5):577–602, 2002.

[12] A. Mu'alem and N. Nisan. Truthful approximate mechanisms for restricted combinatorial auctions. In *AAAI*, 379–384, 2002.

[13] D. Parkes and G. Schoenebeck. GROWRANGE: Anytime VCG-based mechanisms. In *AAAI*, 34–41, 2004.

[14] D. Parkes and J. Shneidman. Distributed implementations of generalized Vickrey-Clarke-Groves auctions. In *AAMAS*, 261–268, 2004.

[15] K. Roberts. The characterization of implementable social choice rules. In J.-J. Laffont, editor, *Aggregation and Revelation of Preferences*. North-Holland Publishing Company, 1979.

[16] M. Rothkopf, A. Pekeč, and R. Harstad. Computationally manageable combinatorial auctions. *Management Science*, 44(8):1131–1147, 1998.

[17] M. E. Saks and L. Yu. Weak monotonicity suffices for truthfulness on convex domains. In *ACM-EC*, 286–293, 2005.

[18] T. Sandholm. Issues in computational Vickrey auctions. *International Journal of Electronic Commerce*, 4(3):107–129, 2000. Early version appeared in ICMAS-96.

[19] T. Sandholm. Algorithm for optimal winner determination in combinatorial auctions. *Artificial Intelligence*, 135:1–54, Jan. 2002.

[20] W. Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance*, 16:8–37, 1961.

[21] M. Yokoo. The characterization of strategy/false-name proof combinatorial auction protocols: Price-oriented, rationing-free protocol. In *IJCAI*, 733–742, 2003.

[22] M. Yokoo, Y. Sakurai, and S. Matsubara. Robust combinatorial auction protocol against false-name bids. *Artificial Intelligence*, 130(2), 2004.