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Online Node-Weighted Problems

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Years and Authors of Summarized Original Work

2011; Naor, Panigrahi, Singh
2013; Hajiaghayi, Liaghat, Panigrahi
2014; Hajiaghayi, Liaghat, Panigrahi

Problem Definition

We are given an undirected graph $G = (V, E)$ offline, where node v has a given weight w_v. Initially, the output graph $H \subseteq G$ is the empty graph. In the generic online Steiner network design problem, each online step has a connectivity request C_i and the online algorithm must augment the output graph H to meet the new request. We will consider the following problems in this domain:

- *Steiner tree.* Each connectivity request C_i comprises a new vertex $t_i \in V$ (called a *terminal*) that must be connected in H to all previous terminals. (The first terminal t_0 is often called the *root* and the constraint C_i can then be restated as connecting terminal t_i to the root.)
- *Steiner forest.* Each connectivity request C_i comprises a new vertex pair (s_i, t_i) (called a *terminal pair*) that must be connected in H.
- *Group Steiner tree.* Each connectivity request C_i comprises a new set (group) of vertices $T_i \subseteq V$ (called a *terminal group*). The first terminal group T_0 is a single vertex r called the *root*. At least one vertex in each terminal group must be connected in H to the root.
- *Group Steiner forest.* Each connectivity request C_i comprises a new pair of sets (groups) of vertices (S_i, T_i) (called a *terminal group pair*). For each terminal group pair, at least one vertex in S_i must be connected in H to at least one vertex in T_i.
- *Prize-collecting Steiner tree (resp., Prize-collecting Steiner forest).* Each connectivity request comprises a new terminal t_i (resp., a new terminal pair (s_i, t_i)) and a penalty $\pi_i > 0$; the algorithm must either *pay the penalty* π_i or augment graph H to connect terminal t_i to the root (resp., augment graph H to connect the terminal pair (s_i, t_i)).

In the (group) Steiner tree and (group) Steiner forest problems, the objective is to minimize the total weight (i.e., sum of weights of vertices) of graph H. In the prize-collecting versions of these problems, the objective is to minimize the sum of the total weight of H and the sum of penalties paid by the algorithm.

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Key Results

The following theorem was obtained by Naor et al. [7] for the online node-weighted Steiner tree problem.

Theorem 1. *There is a randomized online algorithm for the node-weighted Steiner tree problem that has a competitive ratio of $O(\log^2 k \log n)$ and runs in polynomial time.*

This was the first result to obtain a polylogarithmic competitive ratio for the online node-weighted Steiner tree problem. The competitive ratio for this problem was later improved to $O(\log k \log n)$ (see [5]), which is tight up to constants.

The lower bound follows from the observation that the online set cover problem is a special case of the online node-weighted Steiner tree problem. For the online set cover problem, a lower bound of $\Omega\left(\frac{\log m \log n}{\log\log m + \log\log n}\right)$ for deterministic algorithms was obtained by Alon et al. [1], where m is the number of sets and n is the number of elements. This was later improved and extended to a lower bound of $\Omega(\log m \log n)$ for randomized algorithms by Korman [6]. An online set cover instance can be encoded as an online node-weighted Steiner tree instance where the terminals are the elements and the nonterminals are the sets. This encoding yields a lower bound of $\Omega(\log k \log n)$ for the online node-weighted Steiner tree problem and its generalizations discussed below.

In addition to the Steiner tree problem, Naor et al. [7] also considered the online node-weighted Steiner forest problem and the online node-weighted group Steiner tree problem. In fact, they obtained the following theorem for the online node-weighted group Steiner forest problem which generalizes both these problems.

Theorem 2. *There is a randomized online algorithm for the node-weighted group Steiner forest problem that has a competitive ratio polylogarithmic in n and k and runs in quasi-polynomial time.*

For edge-weighted graphs, the same competitive ratio was obtained with a polynomial-time algorithm.

Subsequent to this work, Hajiaghayi et al. [4] investigated the online node-weighted Steiner forest problem and obtained the first polynomial-time algorithm with a polylogarithmic competitive ratio.

Theorem 3. *There is a randomized online algorithm for the node-weighted Steiner forest problem that has a competitive ratio of $O(\log^2 k \log n)$ and runs in polynomial time.*

The competitive ratio is tight up to a logarithmic factor owing to the online set cover lower bound described above. For graphs with an excluded minor (such as planar graphs), they gave an improved competitive ratio of $O(\log n)$ for this problem, which is tight up to constants. Moreover, the result can be extended to all $\{0, 1\}$-proper functions which were introduced by Goemans and Williamson [3] to capture a broad range of connectivity problems and extended later to node-weighted graphs by Demaine et al. [2].

For the prize-collecting variants of the online node-weighted Steiner tree and Steiner forest problems, Hajiaghayi et al. [5] gave the first algorithms with a polylogarithmic competitive ratio by showing that these problems can be reduced to the fractional versions of their non prize-collecting

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variants while losing only a logarithmic factor in the competitive ratio. This led to the following results.

Theorem 4. *There is a randomized online algorithm for the prize-collecting node-weighted Steiner tree problem that has a competitive ratio of* $O(\log k \log^2 n)$. *For the node-weighted prize-collecting Steiner forest problem, there is a randomized online algorithm that has a competitive ratio of* $O(\log^2 k \log^2 n)$. *Both these algorithms run in polynomial time.*

Corresponding results for edge-weighted graphs were previously known [8].

Applications

Online node-weighted Steiner problems have broad applications in designing communication networks where the clientele grows over time.

Open Problems

Suppose we are given a node-weighted undirected graph $G = (V, E)$. In the online edge-connectivity (resp., vertex connectivity) version of the survivable network design problem (SNDP), the online connectivity requirement C_i comprises a pair of terminals (s_i, t_i) and an integer *requirement* $r_i > 0$. The online algorithm must augment the output graph H so that there are r_i edge-disjoint (resp., node-disjoint) paths between s_i and t_i in H. The objective is to minimize the total weight of H.

An interesting open problem is to obtain an algorithm with competitive ratio $O(r_{\max}^{\alpha} \log^{\beta} n)$ for any constants α, β for the online node-weighted SNDP problem with either edge or vertex connectivity requirements, where $r_{\max} = \max_i r_i$.

Experimental Results

No experimental results are known.

Cross-References

- ▸ Steiner forest
- ▸ Steiner trees
- ▸ Generalized Steiner network

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Recommended Reading

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