Hybrid Symbolic-Numeric Computation
♥ *A Marriage Made in Heaven* ♥

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寒爐 

Deep are the roots

First approximate GCD paper:

Donna K. Dunaway, "Calculation of Zeros of a Real Polynomial Through Factorization Using Euclid's Algorithm," SIAM J. Numer. Anal. vol. 11 (1974)

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Recommendations in Boyle/Caviness Report 1988:
Stimulate developments at the interface of symbolic and numeric computation by:

- *Funding research in defining the interface and on algorithms that employ both symbolic and numeric methods*
- *Funding course development that incorporates symbolic and numeric computing*
- *Funding workshops to attack a particular problem using symbolic and numeric methods*

What's in a Name?

— Integrated Symbolic-Numeric Computing [ISSAC 1992]

— Symbolic-Numeric Algebra for Polynomials [SNAP'96, JSC special issue]

— Symbolic and Numerical Scientific Computation [SNSC'99]

— Hybrid Symbolic-Numeric Computation [Computer Algebra Handbook 2002]

— Symbolic-Numeric Computation [SNC 2005]

— Approximate Algebraic Computation [AAC@ACA'05]

— Approximate Commutative Algebra [ApCoA'06]

— Numerical Algebraic Geometry [Advances in ...@AAG'11]

Famous Hybrids

— *Qilin* @ Summer Palace: lion + deer (announces the arrival of a sage)

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— Dhala ("white") Swarna [Swarna-Sub1] rice:
High yield Nali ("red") Swarna rice [IR64]
+ low yield F(lood)R(esistant)13A rice
Can survive 15 days underwater

— Toyota *Prius:* electro + gasoline engine

Selected Hybrid Symbolic-Numeric Algorithms

Part floating point ("high yield"), part exact symbolic ("flood resistant")

♥ ♥

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— Nearest polynomial with a given root, approximate GCD [Corless, Gianni, Trager, Watt '95; Lakshman, Karmarkar '96; Zhi, Wu '98; Hitz, Kaltofen '99; Stetter '99,...]
$\longrightarrow$ Hiroshi Sekigawa's talk

— Approximate factorization [Sasaki, Suzuki, Kolář, Sasaki '91; Nagasaka '02; Gao, Kaltofen, May, Zhi '04]

— Sparse interpolation [Giesbrecht, Labahn, Lee '03; Kaltofen, Yang, Zhi '07; Giesbrecht, Roche '11; Comer, Kaltofen, Pernet '12; Kaltofen, Yang '13, '14]
$\longrightarrow$ Bin Han's,Wen-shin Lee's,Dan Roche's,Zhengfeng Yang's talks

— Sum-of-squares certificates [Harrison '07; Peyrl, Parrilo '07; Kaltofen, Li, Yang, Zhi '08, Guo, Kaltofen, Zhi '12]
$\longrightarrow$ Jean B. Lasserre's and Bican Xia's talks

Selected Hybrid Symbolic-Numeric Algorithms Continued...

— Linear algebra, lattices [Eckart, Young '36; Saunders, Wan '04; Dumas, Giorgi, Pernet '08; Morel, Stehlé, Villard '09]
⟶ Daniel Lichtblau's and Dan Steffy's talks

— Polynomial systems [many results, not all hybrid, including our chair's Lihong Zhi]
⟶ Jin-San Cheng, Anton Leykin's, Pablo Parrilo's, and Jan Verschelde's talks

— Integrals and differential equations [Geddes '01; *MapleSim*,...]
⟶ Mark Giesbrecht's talk

A Note on Sum-Of-Squares Proofs for Polynomial Global Minima

Notation: $f(x_1, x_2, \ldots) \succeq 0 \Longleftrightarrow \forall \xi \in \mathbb{R}^n\colon f(\xi_1, \xi_2, \ldots) \geq 0$

Richard Askey, June 8, 2015: $f(x,y,z) = 2(x^4+y^4+z^4) - (x^3y+x^3z+y^3x+y^3z+z^3x+z^3y) - 2(x^2y^2+x^2z^2+y^2z^2) + 2(x^2yz+y^2xz+z^2xy)$
Is $f(x^2,y^2,z^2) \succeq 0$?

With *ArtinProver* [Kaltofen, Li, Yang, Zhi '09]
$f(x^2,y^2,z^2) = g_1(x,y,z)^2 + \cdots + g_{11}(x,y,z)^2$ **exactly**

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Note: all $f(x) - \mu \succeq 0$ are numerically ill-posed at their optima

$$\mu = \inf_\xi f(\xi)\colon \ f(x) - \mu - \varepsilon \not\succeq 0$$

But it's worse: $\inf_{\xi,\eta} \quad \xi^2 - 2\xi\eta + \eta^2 = 0$, but
$\inf_{\xi,\eta} (1-\varepsilon)\xi^2 - 2\xi\eta + \eta^2 = -\infty$

Hutton, Kaltofen, Zhi 2010: *radius of positive semidefiniteness*

An Example of a Hybrid Algorithm With Randomization: Numerically Sparse Interpolation

Definition: $f(x) \in \mathbb{C}[x]$ is T-sparse if

$$f(x) = \sum_{j=1}^{T} c_j x^{e_j}, \quad 0 \le e_1 < e_2 < \cdots < e_T, \quad \forall j\colon c_j \neq 0.$$

We want to interpolate f from $2T+1$ values, **not** knowing T

Exact Algorithm: Early Termination [Kaltofen & Lee '03] in Prony 1796 / Blahut 1979 sparse interpolation

— Pick a **random** element $\omega \in S$

Evaluate T-sparse $f(x)$ at ω^k: $h_0 = f(\omega), \ldots, h_{k-1} = f(\omega^k), \ldots$

— Consider the $k \times k$ Hankel matrices:

$$H^{[k]} = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & \ldots & h_{k-1} \\ h_1 & h_2 & h_3 & h_4 & \iddots & h_k \\ h_2 & h_3 & h_4 & h_5 & \iddots & h_{k+1} \\ h_3 & h_4 & h_5 & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \iddots & \\ h_{k-1} & h_k & h_{k+1} & \ldots & & h_{2k-2} \end{bmatrix}$$

— **Theorem:** $\mathrm{Prob}(\forall 1 \le k \le T\colon \det(H^{[k]}) \ne 0) \le 1 - \dfrac{O(T^3 \deg(f))}{|S|}$

Note: $H^{[k]}$ is singular for $k > T$

Term Locator Polynomial

For the largest non-singular Hankel system solution

$$\begin{bmatrix} h_0 & h_1 & h_2 & h_3 & \dots & h_{T-1} \\ h_1 & h_2 & h_3 & h_4 & \iddots & h_T \\ h_2 & h_3 & h_4 & h_5 & \iddots & h_{T+1} \\ h_3 & h_4 & h_5 & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \iddots & \\ h_{T-1} & h_T & h_{T+1} & \dots & & h_{2T-2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_{T-1} \end{bmatrix} = \begin{bmatrix} h_T \\ h_{T+1} \\ h_{T+2} \\ \vdots \\ \vdots \\ h_{2T-1} \end{bmatrix}$$

we have

$$z^T - c_{T-1} z^{T-1} - \cdots - c_0 = (z - \omega^{e_1})(z - \omega^{e_2}) \cdots (z - \omega^{e_T})$$

One obtains the term degrees by log's of polynomial roots

Numeric Zippel/Schwartz Lemma [Kaltofen,Yang,Zhi'07]

Let $\Delta(z_1,\ldots,z_s) \in \mathbb{Z}[\boldsymbol{i}][z_1,\ldots,z_s], \quad \Delta \neq 0, \quad \boldsymbol{i} = \sqrt{-1},$

$\zeta_j = \exp(\frac{2\pi \boldsymbol{i}}{p_j}) \in \mathbb{C}$, $p_j \in \mathbb{Z}_{\geq 3}$ *distinct prime numbers* $\forall 1 \leq j \leq s$

Suppose $\Delta(\zeta_1,\ldots,\zeta_s) \neq 0$ *(use exact Zi / Schw Lemma ♥ to enforce)*

Then for random integers r_j *with* $1 \leq r_j < p_j$

$$\textit{Expected value}\{ \quad \left|\Delta(\zeta_1^{r_1},\ldots,\zeta_s^{r_s})\right| \quad \} \quad \geq \quad 1. \quad ♥$$

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Can justify identification of those Δ with $\Delta \neq 0$

For Prony/Blahut there are explict upper bounds for the expectancy of the condition number [Giesbrecht, Labahn, Lee 2006]

Problem with Numeric Zippel Approach: Identifying 0

$H^{[t+1]}$ is singular + noise: ill-conditioned

Rump 2003: distance to nearest singular Hankel matrix

$$= \|(H^{[T+1]})^{-1}\|_{\mathbf{2}}^{-1}$$

Main Question: how well-conditioned is $\det(H^{[T+1]}), \kappa_{\text{struct}}(H^{[T+1]})$?

What is the condition number *of the condition number?*

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My moment of epiphany

One can tolerate a lousy estimate of the condition number: wrong by a factor of 1000: big, big$/1000$, $1000 \times$ big is big $\Rightarrow$ ill-conditioned

Randomized Blahut algorithm uses ill-conditioned value only for early termination and is correct with high probability [Kaltofen,Lee,Yang'11]

Who invented interpolation?

Sun Tzu ($\approx$300): Chinese remaindering

Liu Zhuo (544–610): quadratic polynomial interpolation

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Interpolation tolerating errors in values by oversampling:
Irving Reed & Gustave Solomon 1960

Decoding Reed-Solomon Error-Correcting Codes'60

Problem: Communicate $c_0, \ldots, c_{d-1} \in \mathsf{K}$ with $\leq E$ errors/transmission

Idea: Choose

$$n = d + 2E \text{ elements } \xi_i \in \mathsf{K}, 1 \leq i \leq n, \xi_{i_1} \neq \xi_{i_2} \text{ for } i_1 \neq i_2$$

and transmit

$$\gamma_i = f(\xi_i) = c_0 + c_1\xi_i + \cdots + c_{d-1}\xi_i^{d-1}, \quad i = 1, \ldots, n$$

Recover f from $\beta_i = \gamma_i + \gamma_i'$, where $\gamma_{\ell_j}' \neq 0$ ("errors") exactly at $k \leq E$ **unknown** indices $1 \leq \ell_1 < \cdots < \ell_k \leq n$ $(\forall i \notin \{\ell_1, \ldots, \ell_k\}\colon \gamma_i' = 0)$

i.e., interpolate with erroneous values by oversampling

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Welch's & Berlekamp's 1986(!) solution [Kaltofen & Pernet 2013]

Cauchy interpolate (recover the "rational" function)

$$\frac{f(x)\Lambda(x)}{\Lambda(x)}, \Lambda(x) = (x - \xi_{\ell_1}) \cdots (x - \xi_{\ell_k}) \text{ ("error locator polyn.")}$$

Note: $\Lambda(\xi_{\ell_j})/\Lambda(\xi_{\ell_j}) = 0/0$

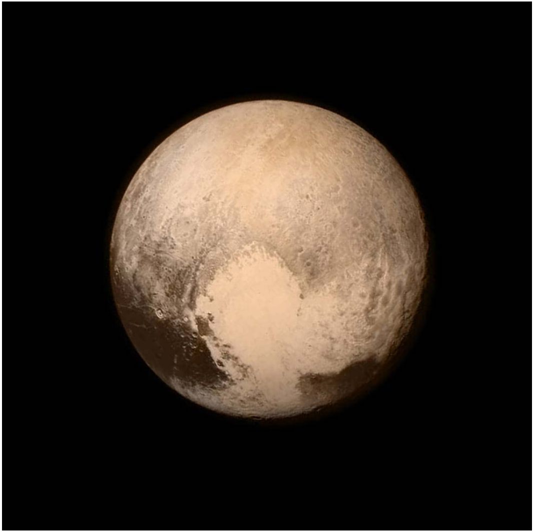
Sidebar: rational vector-function codes [Kaltofen, Pernet, Storjohann, Waddell ISSAC 2015]

Recover vector of rational functions

$$\begin{bmatrix} \vdots \\ \dfrac{f^{[i]}(x)\Lambda(x)}{g(x)\Lambda(x)} \\ \vdots \end{bmatrix}$$

Higher error capacity for "bursts of errors"
for Stanley Cabay's 1971 $f^{[i]}, g$

Application of digital error-correcting codes
New Horizons’s images of Pluto



Numerical R / S decoding [Olshevsky-Shokrollahi'03]

Let $\Phi(x), \Psi(x) \in \mathbb{C}[x], \quad \deg(\Phi) \leq d+E, \quad \deg(\Psi) \leq E$
have **unknown** coefficients
Consider the structured homogeneous linear system

$$\Phi(\xi_i) = \beta_i \Psi(\xi_i), \quad i = 1, \ldots, n$$

Note: $(f\Lambda, \Lambda)$ is a non-zero solution.

Theorem: *The* ***lowest degree*** *nullspace pair* $(\Phi_{\min}, \Psi_{\min})$ *has*

$$\Phi_{\min} = f\,\Psi_{\min}, \quad \Psi_{\min}(\xi_{\ell_1}) = \cdots = \Psi_{\min}(\xi_{\ell_k}) = 0, \quad \deg(\Psi_{\min}) = k$$

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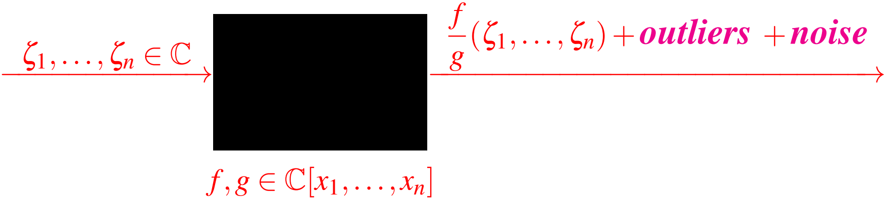
Kaltofen and Yang [2013] compute **outlier** locations from matching roots of $\Psi_{\min}$ and least-squares fit f to **good but noisy** $\beta_i \approx \gamma_i$ ♥

Multivariate Sparse Rational Function Fitting with Outliers and Possibly with Noise [Kaltofen,Yang'14]



By sampling black box, compute sparse representation

$$\frac{\sum_{j=1}^{T_f} \tilde{a}_j x_1^{d_{j,1}} \cdots x_n^{d_{j,n}}}{\sum_{m=1}^{T_g} \tilde{b}_m x_1^{e_{m,1}} \cdots x_n^{e_{m,n}}} = \frac{\tilde{f}}{\tilde{g}}, \quad \tilde{a}_j \neq 0, \tilde{b}_m \neq 0$$

Note 1: Terms and outlier locations are **not** known.

Note 2: Error rate **can be high:** 25% outliers

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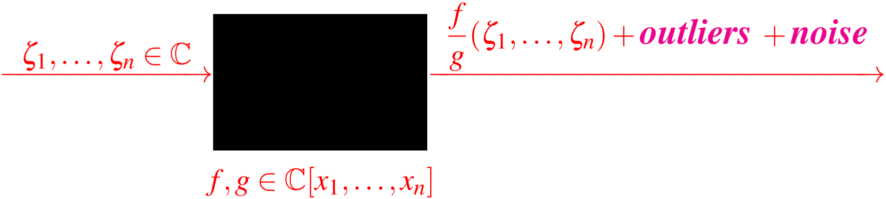

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We use Zippel [1979] variable-by-variable lifting

Condition Numbers of Sparse Fourier Matrices [Kaltofen, Yang 2014]

Lemma: *Let*

$0 \le d_1 < d_2 < \cdots < d_T \le D-1,$

r_k *be **uniformly random** with* $0 \le r_k \le D-1,\ 1 \le k \le T$

Then the matrix
$$\begin{bmatrix} \omega_D^{d_1 r_1} & \cdots & \omega_D^{d_1 r_T} \\ \omega_D^{d_2 r_1} & \cdots & \omega_D^{d_2 r_T} \\ \vdots & & \vdots \\ \omega_D^{d_T r_1} & \cdots & \omega_D^{d_T r_T} \end{bmatrix}, \quad \omega_D = e^{2\pi \mathbf{i}/D}$$

is non-singular with probability $\ge 1 - \dfrac{d_1 + \cdots + d_T}{D}$

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The Lemma is true for **prime** T without randomization [Gauss]

For **composite** T a consequence of our estimates on the minimum number of samples [thanks Terence Tao!]

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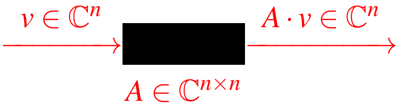
Rules of Thumb: 1. oversampling improves conditioning

2. a high error rate worsens conditioning

An Application: Sparse Matrix Reconstruction

Input: a black box for a matrix $A \in \mathbb{C}^{n\times n}$:



Output: the sparse representation of A [Jianlin Xia ILAS'13]

Solution: Interpolate $A \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ y^2 \\ xy \\ \vdots \end{bmatrix} \in \mathbb{C}[x,y]^n$

If each row of A has $O(T)$ non-zero entries,
one needs $O(T)$ matrix-times-vector products for reconstruction;
the maximum degrees are $O(\sqrt{n})$ (for 2 variables)

Return to **Univariate** Sparse Interpolation
Sparse Chebyshev Representation [Arnold, Kaltofen'15]

Use Chebyshev polynomials (of any kind) as basis

$$f(x) = \sum_{j=1}^{T} c_j U_{\delta_j}(x) \in \mathsf{K}[x], \quad c_j \neq 0, \quad 0 \leq \delta_1 < \delta_2 < \cdots < \delta_T,$$

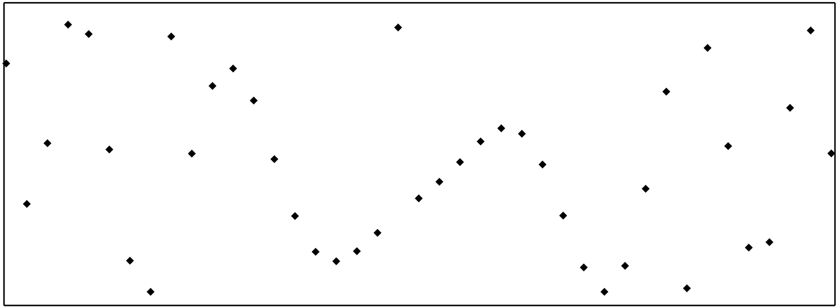
where

$$\begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix}^n \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} U_n(x) \\ U_{n+1}(x) \end{bmatrix} \quad \text{for } n \in \mathbb{Z} \text{ (first kind)}$$

Prony/Blahut-like sparse interpolation algorithm by Lakshman and Saunders 1995 uses $f(U_i(\xi))$ for $\xi > 1,\ 0 \leq i < 2T$

[For second kind $\longrightarrow$ Zhengfeng Yang's talk]

Interpolating with Errors - Example



E.g. $\deg(f) \leq 19$, $\quad T \leq 3$ terms, $\quad k \leq 5$ errors

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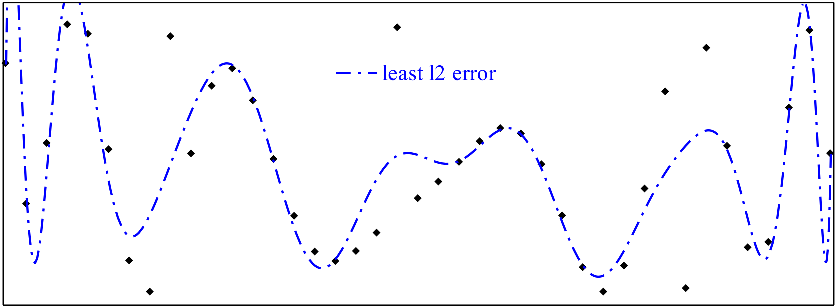

E.g. $\deg(f) \le 19$, $T \le 3$ terms, $k \le 5$ errors

— Minimizing ℓ_2-error gives a **dense approximation** for the model,

$0.786462U_{19} - 0.253808U_{19} - 0.270838U_{18} + 0.101009U_{16} + 0.206344U_{15} - 0.135857U_{15} - 0.076361U_{14} + 0.051550U_{12} - 0.699793U_{12} + 0.003612U_{10} - 0.473865U_{10} + 0.352537U_{8} - 0.307681U_{8} - 1.054240U_{7} + 0.753950U_{5} - 0.112232U_{5} - 1.388821U_{4} + 1.025795U_{2} + 1.364547U_{1} + 3.325460U_{0}$

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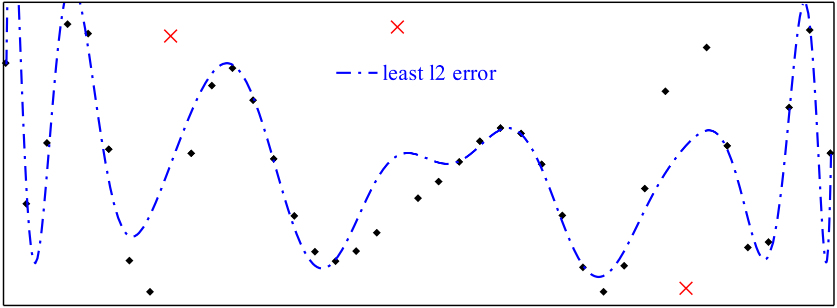

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— But if we identify 3 **outliers**...

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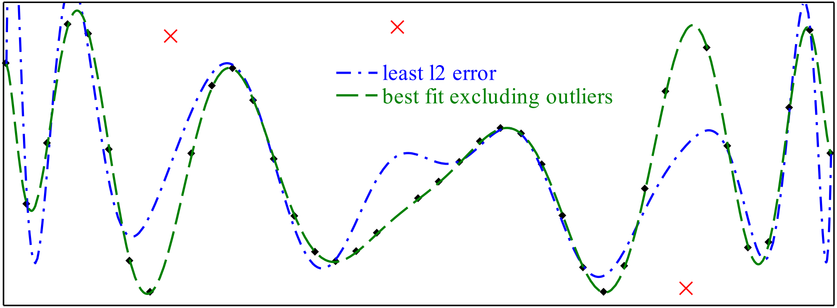

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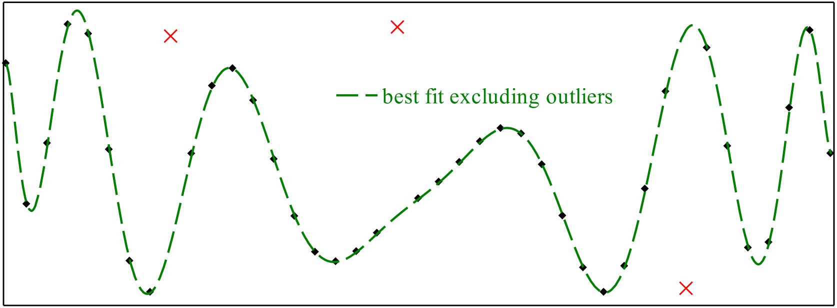
$1.388821U_{4} + 1.025795U_{2} + 1.364547U_{1} + 3.325460U_{0}$

— But if we identify 3 **outliers** we get the sparse fit $U_{15} - 2U_{11} + U_{2}$

Interpolating with Errors - Example



E.g. $\deg(f) \leq 19$, $T \leq 3$ terms, $k \leq 5$ errors

— Minimizing ℓ_2-error gives a **dense approximation** for the model,

$0.786462U_{19} - 0.253808U_{19} - 0.270838U_{18} + 0.101009U_{16} + 0.206344U_{15} - 0.135857U_{15} - 0.076361U_{14} + 0.051550U_{12} - 0.699793U_{12} + 0.003612U_{10} - 0.473865U_{10} + 0.352537U_{8} - 0.307681U_{8} - 1.054240U_{7} + 0.753950U_{5} - 0.112232U_{5} - 1.388821U_{4} + 1.025795U_{2} + 1.364547U_{1} + 3.325460U_{0}$

— But if we identify 3 **outliers** we get the sparse fit $U_{15} - 2U_{11} + U_{2}$

First error decoder: evaluate for $\xi_1 > 1, \xi_2 > 1, \ldots$

$$f(U_0(\xi_1)), f(U_1(\xi_1)), \ldots, f(U_{2T-1}(\xi_1)),$$
$$f(U_0(\xi_2)), f(U_1(\xi_2)), \ldots, f(U_{2T-1}(\xi_2)),$$
$$\vdots$$

with $U_{i_1}(\xi_{\ell_1}) \neq U_{i_2}(\xi_{\ell_2})$ until a line has no error

If $k \leq E$ errors in $(E+1)2T$ evaluations, one block has no error.

Theorem [Kaltofen & Pernet '14]: the **list** of T-sparse interpolants (with $\leq E$ errors) contains **no second** solution.

Let $f, f^* \in \mathbb{R}[x]$.

f and f^* both have Chebyshev-sparsity $\leq T$.

$(\xi_1, \beta_1), \ldots (\xi_{2T+2E}, \beta_{2T+2E})$ be distinct interpolation points

with $\boxed{\forall i\colon\ \xi_i > 1}$

Suppose $f(\xi_i) = \beta_i$ for all $i \notin \ell_1, \ldots, \ell_k$, $k \leq E$,

$f^*(\xi_i) = \beta_i$ for all $i \notin \{\ell_1^*, \ldots, \ell_{k^*}^*\}$, $k^* \leq E$:

no more than E interpolation errors for f and f^*

Then $f = f^*$: $f - f^*$ has sparsity $\leq 2T$ and is zero at $2T$ distinct reals>1 [Obrechkoff's 1918 Descartes's-sign-rule for orthogonal polynomials].

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Note: $(E+1)2T \geq 2T + 2E$ for $T \geq 1$.

Better decoder for $T = 3\colon\ 74\lfloor (E+13)/13 \rfloor$ evaluations suffice, and $74\lfloor (E+13)/13 \rfloor < 6(E+1)$ for all $E \geq 222$ [Arnold, Kaltofen '15]

No (polynomial-time) decoder is known for $2T + 2E$ evaluations!

Conclusion

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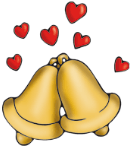
$\Longrightarrow$  ???

Thank you! xièxie!

”End Key” wrong!