Computing Higher Polynomial Discriminants

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ABSTRACT
In https://arxiv.org/abs/1609.00840 (see also https://doi.org/10.1007/s11425-018-1594-2), Dongming Wang and Jing Yang in 2016 have posed the problem how to compute the “third” discriminant of a polynomial \( f(x) = (x - \alpha_1) \cdots (x - \alpha_n) \),
\[
\delta_3(f) = \prod_{1 \leq i < j < k \leq n} ((\alpha_i + \alpha_j - \alpha_k)(\alpha_i - \alpha_j + \alpha_k)(\alpha_i - \alpha_j - \alpha_k)),
\]
from the coefficients of \( f \); note that \( \delta_3 \) is a symmetric polynomial in the \( \alpha_i \). For complex roots, \( \delta_3(f) = 0 \) if the mid-point (average) of 2 roots is equal the mid-point of another 2 roots. Iterated resultant computations yield the square of the third discriminant. We apply a symbolic homotopy by Kaltofen and Trager [JSC, vol. 9, nr. 3, pp. 301–320 (1990)] to compute its squareroot. Our algorithm uses polynomially many coefficient field operations in the degree of \( f \).

CCS CONCEPTS
• Mathematics of computing → Computations on polynomials; Computations in finite fields; • Computing methodologies → Algebraic algorithms.

KEYWORDS
symmetric functions; polynomial resultant; Orlando polynomial; symbolic homotopy;

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1 INTRODUCTION
Let \( K \) be a field and consider the polynomial
\[
F(x) = (x - y_1) \cdots (x - y_n) = x^n + a_{n-1}x^{n-1} + \cdots + a_0,
\]
where \( a_i \in K[y_1, \ldots, y_n] \), (1)
and \( y_i \) are fresh variables. Note that the \( a_i \) are plus/minus the elementary symmetric functions in \( y_i \). The discriminant of \( F \) is defined as
\[
\Delta_1(y_1, \ldots, y_n) = \prod_{1 \leq i < j \leq n} ((y_i - y_j)(y_j - y_i)),
\]
which is a symmetric polynomial in the \( y_i \), that is, there is a polynomial \( D_1(z_0, \ldots, z_{n-1}) \in K[z_0, \ldots, z_{n-1}] \) (\( z_i \) fresh variables) such that \( \delta_1(F, x) = D_1(a_0, \ldots, a_{n-1}) = \Delta_1(y_1, \ldots, y_n) \). If \( K \) is a field of characteristic \( \neq 2 \), then \( D_1 \) is absolutely irreducible, that is an irreducible polynomial over the algebraic closure of \( K \) (see the paragraph below the Proof of Lemma 1.1). Note our notation: \( \Delta_1 \) is the discriminant polynomial in variables for the roots, \( D_1 \) is the discriminant polynomial in variables for the coefficients (except the leading 1), and \( \delta_1 \) is a corresponding function with a univariate polynomial argument whose variable is the second argument. For a polynomial \( f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0 \in K[x] \) we write \( \delta_1(f) = D_1(c_0, \ldots, c_{n-1}) \); because there is a single variable \( x \), we omit that second argument. Each symbol denotes a family of \( n \)-variate polynomials, where \( n \) is the degree of \( F \). All our discriminant formulas will be identities on multivariate polynomials into which actual roots and coefficients can be substituted. For instance, the formula for the discriminant as the Sylvester resultant of \( F \) and its derivative \( \partial F / \partial x \),
\[
D_1(a_0, \ldots, a_{n-1}) = \det \begin{pmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & a_{n-2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n(n-1)a_{n-1} & \cdots & 2a_2 & a_1 & 0 & \cdots & 0 \\ 0 & n(n-1)a_{n-1} & \cdots & 2a_2 & a_1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix},
\]
is an identity in \( K[y_1, \ldots, y_n] \) over any field \( K \), even those whose characteristic divides \( n \).\(^1\)

For fields \( K \) of characteristic 2, \( D_1 \) is the square of the Orlando polynomial
\[
\hat{\Delta}_1 = \prod_{1 \leq i < j \leq n} (y_i + y_j)
\]
\(^1\)Note that if the discriminant is defined as \( \prod_{i < j} (x_i - x_j)^2 \) one multiplies (\( \delta \)) by \( (-1)^{(n-1)/2} \).
whose corresponding \( \hat{D}_1 \) is again an absolutely irreducible polynomial in the coefficient variables \( z_i \) (see Lemma 1.1). A corresponding determinantial expression is Luciano Orlando’s 1911 formula (5) of Figure 1.

**Figure 1**: Luciano Orlando’s 1911 Formula [4], [2, Eq. (1.37)].

\[
\hat{D}_1(a_0, \ldots, a_{n-1}) = \begin{vmatrix}
    a_{n-1} & a_{n-3} & \cdots & a_{-(n-3)} & a_{-(n-1)} \\
    1 & a_{n-2} & \cdots & a_{-(n-2)} & a_{-(n-2)} \\
    a_{n-1} & a_{n-3} & a_{n-5} & \cdots & a_{-(n-3)} \\
    1 & a_{n-2} & a_{n-4} & \cdots & a_{-(n-4)} \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    a_{2n-1} & \cdots & \cdots & a_1 & a_0
\end{vmatrix}
\]

where \( a_i = 0 \) for all \( i > n \) and \( i < 0 \). (5)

In [5, 6] higher discriminants are introduced. One forms a product of linear combinations of the roots \( y_i \) that lead to symmetric polynomials. For fields of characteristic \( \neq 2 \) and \( \neq 3 \) [5] define

\[
\Delta_2 = \prod_{1 \leq i < j \leq n} (y_i - y_j),
\]

which is defined for any characteristic and where the corresponding \( \hat{D}_2 \) is absolutely irreducible (see Lemma 1.1).

Wang and Yang [5, Section 8] further define third discriminants. For fields \( K \) of characteristic \( \neq 2 \), they define

\[
\Delta_3 = \prod_{1 \leq i < j < k \leq n} (y_i + y_j + y_k),
\]

and the corresponding \( D_3(z_0, \ldots, z_n-1) \) as the third discriminant, which is absolutely irreducible over \( K \). Over fields of characteristic 2 we have \( \Delta_3 = \Delta_3^2 \) (see (12)). By expressing the lexicographically leading term with variable order \( y_1 > y_2 > \cdots > y_n \) in (8) as the leading term in a symmetric product,

\[
y_1^{3(n-1)/3} y_2^{3(n-2)/3} \cdots y_{n-3}^{3(2)/3} y_{n-2}^{3(1)/3} = \text{lead-term}(3^{(n-2)/2} a_{n-2}^{3(n-3)/3} a_{n-2}^{2}),
\]

where the \( a_i \)'s are defined in (1), we obtain the total degree

\[
\deg_{y_0, \ldots, y_{n-1}}(D_3) = 3^{(n-1)/3}.
\]

In Section 4 we give the polynomial for \( D_3(z_0, \ldots, z_4) \) (\( n = 5 \)) whose leading monomial is \( -z_2 z_3^2 \). Note that the variables \( z_i \) correspond to the coefficients \( c_i \) of \( f \), not the elementary symmetric functions in the roots of \( f \): In (1) we have \( a_i = (-1)^{n-i} e_{n-i} (y_1, \ldots, y_n) \) with \( e_1 (y_1, \ldots, y_n) = \sum_{1 \leq i \leq j \leq n} y_i y_j \).

Wang and Yang [2016] also consider factors of the form \( 3y_i - y_j - y_k \). The geometric interpretation of \( D_2 \) and \( D_3 \) is as follows. Let \( f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0 \in K[x] \), where \( K \) is a field of characteristic \( \neq 2 \) and \( \neq 3 \) and \( n \geq 4 \). If \( D_2(c_0, \ldots, c_{n-1}) = 0 \) then the average of 2 roots is \( \neq 3 \), and if \( D_2(c_0, \ldots, c_{n-1}) = 0 \) then the average of 2 roots is \( \neq 3 \) of another 2 roots. Over the complex numbers \( K = \mathbb{C} \) the average is the mid-point between the two complex roots on the complex plane.

One can generalize the construction to the factors being weighted linear forms \( w_1 y_1 + w_2 y_2 + \cdots + w_n y_n \) with \( v \geq 2 \) and \( n \geq v \) and weights \( w_1 = 1, w_\mu \in K \) for \( 2 \leq \mu \leq v \). Then for a set \( P \subseteq S_v \) of permutations on \( (1, \ldots, v) \), where \( S_v \) is the set of all permutations on \( (1, \ldots, v) \), we define

\[
\Delta_{v-1,S_v}(P) = \prod_{1 \leq i_1 < \cdots < i_v \leq n} (w_{\sigma(1)} y_{i_1} + w_{\sigma(2)} y_{i_2} + \cdots + w_{\sigma(v)} y_{i_v}).
\]

By reordering the weights we may assume that the identity permutation \( \in P \). We shall require from \( P \) that \( \Delta_{v-1,S_v}(P) \) in (9) is a symmetric polynomial in \( y_1, \ldots, y_n \) whose corresponding polynomial \( D_{v-1,S_v}(z_0, \ldots, z_{n-1}) \) in the coefficient variables \( F \) is absolutely irreducible over \( K \). A sufficient condition for \( P \) is the following:

**Lemma 1.1.** Suppose the following conditions are satisfied for \( P \subseteq S_v \):

1. For all permutations \( \pi \in S_v \) on \((1, \ldots, n)\) we have: if \( \{\pi(i_1), \ldots, \pi(i_v)\} = \{j_1, \ldots, j_v\} \) with \( j_1 < j_2 < \cdots < j_v \) then

\[
\prod_{\sigma \in \pi} (w_{\sigma(1)} y_{i_1} + w_{\sigma(2)} y_{i_2} + \cdots + w_{\sigma(v)} y_{i_v})
\]

\[
= \prod_{\sigma \in \pi} (w_{\sigma(1)} y_{j_1} + w_{\sigma(2)} y_{j_2} + \cdots + w_{\sigma(v)} y_{j_v}).
\]

2. For all permutations \( \sigma, \tau \in P \) with \( \sigma \tau = (w_{\sigma(1)} y_{j_1} + w_{\sigma(2)} y_{j_2} + \cdots + w_{\sigma(v)} y_{j_v}) \) is not a scalar multiple of \( (w_{\tau(1)} y_{j_1} + w_{\tau(2)} y_{j_2} + \cdots + w_{\tau(v)} y_{j_v}) \).

Then \( \Delta_{v-1,S_v}(P) \) is a symmetric polynomial and \( D_{v-1,S_v}(P) \) is absolutely irreducible over \( K \).

**Proof.** By Item 1 a permutation \( \pi \) maps an inner product polynomial of (9) to another inner product polynomial, which establishes symmetry. Item 2 establishes that all linear forms in (9) are relatively prime to one another. Now let \( E(z_0, \ldots, z_{n-1}) \) be an absolutely irreducible factor of \( D_{v-1,S_v}(P)(z_0, \ldots, z_{n-1}) \) such that \( w_1 y_1 + \cdots + w_n y_n \) divides \( E(a_0, \ldots, a_{n-1}) \). Because \( E \) on evaluation is symmetric in the \( y_i, w_1 y_{j_1} + \cdots + w_n y_{j_n} \) must also divide \( E(a_0, \ldots, a_{n-1}) \) for all \( \pi \in S_n \). Now for all \( 1 \leq i_1 < \cdots < i_v \leq n \) and all \( \sigma \in P \) let \( \pi(\mu) = \mu_{i_{\sigma(1)}}(\mu) \) for \( 1 \leq \mu \leq v \). Then \( w_1 y_{i_{\sigma(1)}} + \cdots + w_n y_{i_{\sigma(v)}} = w_{\sigma(1)} y_{j_1} + \cdots + w_{\sigma(v)} y_{j_v} \), and therefore all (relatively prime) factors of (9) divide \( E(a_0, \ldots, a_{n-1}) \), implying that \( E = D_{v-1,S_v}(P) \). □

Lemma 1.1 applies to all the above discriminants except the classical discriminant \( D_1 \). Item 2 in Lemma 1.1 is not satisfied for \( \Delta_1 \) in (2): the weight vectors of the linear forms have \((1, -1) = (-1, 1)\). In the proof of Lemma 1.1 we then get that \( \prod_{i<j}(y_i - y_j) \)
We now give algorithms that compute one can use Orlando's Formula (13) we have for monic \( f \in K[x] \) with \( f(0) \neq 0 \):

\[
\hat{\delta}_2(x f(x)) = \delta_1(f) \hat{\delta}_2(f), \quad \hat{\delta}_2(x^k f(x)) = 0 \quad \text{for} \quad k \geq 3. \tag{15}
\]

Note that \( \delta_1(f) = \prod_{i<j}(a_i + a_j) \) can be computed by Orlando's Formula (5). Then by (14) we have

\[
f[i; j](x) \overset{\text{def}}{=} \frac{\text{res}(1 - x^i f(-u + x), f(u), u)}{\prod_{i \leq j}(x - (a_i + a_j))} \in K[x]. \tag{16}
\]

For an integer \( \lambda \in \mathbb{Z} \) with \( \lambda \geq 2 \) we define

\[
f[\lambda i](x) \overset{\text{def}}{=} \prod_{i=1}^n (x - \lambda a_i) = x^n \quad \text{if} \quad \lambda \not{\text{mod char}(K)} = 0, \tag{17}
\]

We compute

\[
f[i; \sigma i](x) \overset{\text{def}}{=} \frac{f[i; j](x)}{f[2l i](x)} \overset{\text{def}}{=} \prod_{i \leq j}(x - a_i - a_j). \tag{18}
\]

We next compute, over fields of characteristic \( \neq 2 \), by power series expansion (Hensel lifting with multiplicities—see Appendix A) the squareroot of (18), which is

\[
f[i; j](x) \overset{\text{def}}{=} \prod_{i \leq j} (x - a_i - a_j) = f[i; j](x)^{1/2}. \tag{19}
\]

If \( K \) has characteristic \( = 2 \), we assume we have to compute a sqaurefree algorithm in \( K \). For example, for \( K = \mathbb{F}_2 \), we have \( (c^2)^{1/2} = c \in \mathbb{F}_2 \) (the squareroots are unique). Then the squareroot of a squared polynomial is \((x^2d + b_4 x^2 d^{(d-1)} + \cdots + b_2 x^{d-1} + \cdots + b_0)^{1/2} = x^d + b_4 x^{d-1} + \cdots + b_0\). Note that if the field \( K \) has characteristic \( \neq 2 \), one may use (5) to compute \( f[i; j](x) \): If \( f(u + x/2) = u^2 + b_{n-1} u^{n-1} + \cdots + b_0 \) with \( b_i \in K[x] \), then \( f[i; j](x) = (-1)^{(n(n-1)/2)} \frac{D_1(b_0, \ldots, b_{n-1})}{D_1(b_1, \ldots, b_{n-1})} \). Again by (14) we have

\[
f[-2i; j](x) \overset{\text{def}}{=} \frac{\text{res}(f[2i](u - x), (-1)^n f(-u), u)}{\prod_{i \leq j}(x - 2a_i + a_j)} \in K[x], \tag{20}
\]

where \( \text{deg}(f[-2i; j]) = n^2 \)

\[
f[-i; j; k](x) \overset{\text{def}}{=} \frac{\text{res}(f[i; j](u - x), (-1)^n f(-u), u)}{\prod_{i \leq j \leq k}(x - a_i + a_j + a_k)} \in K[x], \tag{21}
\]

where \( \text{deg}(f[-i; j; k]) = n^2 \).

\[
f[\text{chd}](x) \overset{\text{def}}{=} \frac{(-1)^n f[-2i; j; k](x)}{f[-2i; j; j](x) f[3b i](x) f[b i](x)} \in K[x], \tag{22}
\]
where \( \deg(f_{[\text{cbd}]}(t)) = n(n^3 - n \cdot n + n = 3n^3). \) If \( K \) has characteristic \( \neq 3, \) we compute \( f_{\text{orl}}(x) \) as the cube root of \( f_{[\text{cbd}]}(x) \) (see Appendix A). The Orlando value \( \tilde{\delta}_2(f) \) is then the constant coefficient of \( f_{\text{orl}}. \) For fields \( K \) of characteristic 3 we again must assume that one can compute cube roots of elements in \( K. \) For example, for \( K = \mathbb{F}_3^k \) we have \((c^3)^{1/k} = c \in \mathbb{F}_3^k \) (the cube root is unique).

One reviewer has pointed out that in [1] a different algorithm is presented for computing \( f_{\text{orl}}(x) \) from \( f(x), \) which is based on using the Newton-identities for the sums of the powers of the roots of a polynomial in terms of symmetric functions. That algorithm is presented to iterate to arbitrary \( v, \) which is also possible for our algorithm above. The algorithm in [1] requires the field \( K \) to have 0 or a large characteristic, as divisions by factors occur.

One may avoid the computation of the polynomial \( f^{[-, j, \ldots, k]} \) in (21) whose degree grows cubically in \( n, \) using a symbolic homotopy. We construct the symbolic homotopy polynomial (cf. [3])

\[
f_{\text{hom}}(x) = tf(x) + (1 - t)f(0) \in (K[t])[x],
\]

(23)

where \( f(0) \) is computed for such values of the roots \( r_i \in K \) that we have

\[
0 \neq \tilde{\delta}_2(f(0)) = \prod_{1 \leq i < j \leq n} (2r_i + r_j).
\]

(24)

Since \( r_n \) as a variable in (25) has degree \( n - 1 + \binom{n-1}{2}, \) this is always possible if the cardinality of \( K \) is \( \geq n + \binom{n-1}{2}. \) For smaller finite fields we can algebraically extend \( K \) to sufficiently large cardinality.

One computes \( \tilde{\delta}_2(f(0)) \) as follows (writing \( f_{\text{hom}}(x) = \prod_i (x - a_i^h) \)):

\[
d^{[i, j, k]}(t) \overset{\text{def}}{=} \text{res}(f_{\text{hom}}(u), (-1)^n f_{\text{hom}}(-u), u) = \prod_{1 \leq i \leq n \leq 1 \leq k \leq n} (a_i^h + a_j^h + a_k^h) \in K[t],
\]

(26)

\[
d^{[2i, j]}(t) \overset{\text{def}}{=} \text{res}(f_{\text{hom}}(u), (-1)^n f_{\text{hom}}(-u), u) = \prod_{1 \leq i \leq n \leq 1 \leq k \leq n} (2a_i^h + a_j^h) \in K[t],
\]

(27)

\[
d^{[2i, j, \ldots, j]}(t) \overset{\text{def}}{=} \begin{cases} 
\frac{d^{[2i, j]}(t)}{d^{[i, j, \ldots, j]}(t)} & \text{if } \text{mod char}(K) \neq 0, \\
\delta_i(f, x) \text{ (cf. (3))} & \text{if } \text{mod char}(K) = 0.
\end{cases}
\]

(28)

The denominators in (28 and 29) are non-zero polynomials because \( f_{\text{hom}}(0) \) evaluates at \( t = 1 \) to \( f(0), \) which is non-zero by an earlier assumption (15), and \( d^{[2i, j, \ldots, j]}(0) \neq 0 \) by the second factor in (25).

We first prove that if \( d^{[\text{cbd}]} \) is a constant polynomial in \( K, \) then \( \tilde{\delta}_2(f) = \tilde{\delta}_2(f_{\text{hom}}) \), which has been computed while choosing the \( r_i. \) Suppose \( \tilde{\delta}_2(f) \neq \tilde{\delta}_2(f_{\text{hom}}). \) The polynomial \( \tilde{\delta}_2(f_{\text{hom}}) \in K[t] \) then cannot be a constant polynomial because for \( t = 0 \) and \( t = 1 \) it evaluates to different values. Therefore its third power, \( d^{[\text{cbd}]}(t) \), is not constant. Note that \( d^{[\text{cbd}]} \) may be constant even if the coefficients of \( f_{\text{hom}} \) depend on \( t. \)

In case \( d^{[\text{cbd}]}(t) \) has degree \( \geq 3 \) in \( t \) (it is the third power of a non-constant polynomial) we compute the monic cube root polynomial of its monic associate, \( h(t) = \frac{1}{c} d^{[\text{cbd}]}(t), \) where \( c \) is the leading coefficient of \( d^{[\text{cbd}]}(t) \) in \( t. \) Because \( d^{[\text{cbd}]}(0) = \tilde{\delta}_2(f_{\text{hom}})^3 \neq 0 \) by (25) we must have \( h(0) \neq 0. \) Then the homotopy (25) yields \( \tilde{\delta}_2(f_{\text{hom}}) = \tilde{\delta}_2(f_{\text{hom}})/h(0)h(t) \) and \( \tilde{\delta}_2(f) = \tilde{\delta}_2(f_{\text{hom}})h(1)/h(0). \)

### 3 Computing Higher Discriminants

Let

\[
f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0 = \prod_{i=1}^{n} (x - a_i), n \geq 4, c_j \in K, a_i \in \overline{K},
\]

(30)

where \( K \) is a field and \( \overline{K} \) its algebraic closure. We can compute the second discriminant \( \tilde{\delta}_2(f) = \tilde{\delta}_2(f_{\text{hom}}) = c_{n-1} = \) the constant coefficient of (33) (see (6)) as follows:

\[
f^{[-2i, j < k]}(x) \overset{\text{def}}{=} \text{res}(f^{[2i]}(x), f^{[i < j]}(x), u) = \prod_{1 \leq i \leq n \leq 1 \leq k \leq n} (x + 2a_i - a_j - a_k),
\]

(31)

where \( f^{[2i]} \) is (17) for \( \lambda = 2 \) and \( f^{[i < j]} \) is (19),

\[
f^{[i < j, i \neq j]}(x) \overset{\text{def}}{=} \text{res}(f(x), f(u), u) = \prod_{1 \leq i \leq n \leq 1 \leq j \neq i \leq n} (x + 2a_i - a_j - a_k),
\]

(32)

\[
f^{[2\ell - j < k]}(x) \overset{\text{def}}{=} \text{res}(f^{[2\ell]}(x), f^{[\ell < j]}(x), x^n) = \prod_{1 \leq j \leq k \leq n \leq 1 \leq i \neq j \neq k \leq n} (x + 2a_i - a_j - a_k),
\]

(33)

One may continue to the next higher polynomial:

\[
f^{[i < j < k]}(x) \overset{\text{def}}{=} (-1)^n \prod_{1 \leq j \leq k \leq n} (x - a_i - a_j - a_k),
\]

(34)

where \( f_{\text{orl}} \) is defined in the text below (22),

\[
\text{res}(f^{[3i]}(x), f^{[i < j < k]}(x), u) = \prod_{1 \leq j \leq k \leq n \leq 1 \leq i \neq j \neq k \neq i} (x + 3a_i - a_j - a_k),
\]

(35)

\[
\text{res}(f^{[3i]}(x), f^{[i < j < k]}(x), u) = \prod_{1 \leq j \leq k \leq n \leq 1 \leq i \neq j \neq k \neq i} (x + 3a_i - a_j - a_k),
\]
whose constant coefficient is one version of the third discriminant in [5] (see (41) below).

One cannot avoid the nested results by computing the squarefree decomposition of \( f \). Let 
\[
 f = \prod_{i=1}^d f_i^e_i
\]
where \( f_i \) are squarefree, \( \alpha_k \) \( \equiv \deg(f_i) \geq 1 \) and \( \text{GCD}(f_i, f_j) = 1 \) for all \( 1 \leq i < j \leq k \). If \( k \geq 3 \) then \( \delta_2(f) = 0 \) because there is a root of multiplicity 3. We have for a squarefree polynomial \( f_i \):
\[
\delta_2(f_i) = \text{res}(f_i^{[2i]}(x), f_i^{[1<i<j]}(x), x)/\delta_1(f_i),
\]
where \( f_i^{[2i]} \) is (17) for \( f = f_i \) and \( \lambda = 2 \) and \( f_i^{[1<i<j]} \) is (19) for \( f = f_i \), and for the square of a squarefree polynomial \( f_2 \):
\[
\delta_2(f_2^2) = 2^{2n_2(n_2 - 1)} \delta_1(f_2^2) 6 \delta_2(f_2)^3, \quad n_2 = \deg(f_2).
\]
Now for \( g(x) = \prod_{i=1}^d (x - \beta_i) \in K[x] \) and \( h(x) = \prod_{j=1}^e (x - y_j) \in K[x] \) we define
\[
\chi_{1,1}(g, h) \equiv \text{res}(\text{res}(g(u), g^{[2i]}(u + x), u/g(x), h, x)
\]
\[
= \prod_{1 \leq i < j \leq k \leq e} (2\beta_i - \beta_j - y_k),
\]
where \( g^{[2i]} \) is (17) for \( g = h \) and \( \lambda = 2 \),
\[
\chi_{0,2}(g, h) \equiv \text{res}(g^{[2i]}(x), h^{[1<i<j]}(x), x)
\]
\[
= \prod_{1 \leq i < j \leq k < e} (2\beta_i - y_j - y_k),
\]
where \( h^{[1<i<j]} \) is (19) for \( h = g \). We then have
\[
\delta_2(f_1 f_2) = \delta_2(f_1) \delta_2(f_2) \times
\]
\[
\chi_{1,1}(f_1 f_2^2) \chi_{0,2}(f_1, f_2^2) \chi_{1,1}(f_2 f_1) \chi_{0,2}(f_1, f_2^2).
\]
Similarly, one may compute over fields of characteristic \( \neq 2 \) and \( \neq 3 \) the constant coefficient of (35),
\[
\overline{\delta}_3(f, x) \equiv \prod_{1 \leq i < j < k \leq n, 1 \leq t \neq n, \ell \neq i, j, k \neq \ell} (3\alpha_t - (\alpha_i + \alpha_j + \alpha_k))
\]
\[
\quad \times \frac{\text{res}(f^{[3i]}(x), f^{[1<i<j<k]}(x), x)}{\delta_2(f)},
\]
provided that \( \delta_2(f) \neq 0 \) (\( f^{[1<i<j<k]} \) is defined in (34)). If \( \delta_2(f) = 0 \) for an already squarefree polynomial \( f \), one computes a further refined factorization of \( f \) by
\[
f^{[2i: \gcd(i<j)]}(x) \equiv \text{GCD}(f^{[2i]}(x), f^{[1<i<j]}(x))
\]
\[
= \prod_{1 \leq i \leq n, 3 \mid j, k, \ell \neq k} (x - 2\alpha_i),
\]
where \( f^{[1<i<j]} \) is computed in (19) (the characteristic of \( K \) is \( \neq 2 \)). Then \( 2^{-d} f^{[2i: \gcd(i<j)]}(2x) (d = \deg(f^{[2i: \gcd(i<j)]})) \) divides \( f(x) \). For \( K = \mathbb{C} \) the division is non-trivial because not all roots of the squarefree \( f \) can be midpoints of roots in the complex plane. For fields of characteristic \( p \geq 5 \), the GCD in (42) can be \( f^{[2i]}(x) \), for example for \( f(x) = x^p - 1 \). Then no refinement of the squarefree factorization is possible, and (35) is used.

Because of an additional symmetry (11) the above approach does not seem to yield an algorithm for the third discriminant (8) directly. A difficulty is that the factorization \( G(x^2) = F(x)F(-x) \) is not unique, unlike the monic square-root of \( G(x) = F(x^2) \); for instance, \((x^2 + 3x + 2)(x^2 - 3x + 2) = (x^2 + x - 2)(x^2 - x - 2) \). We now show how to compute, by using a symbolic homotopy, the third discriminant \( \delta_3(f) \), which is defined as
\[
\delta_3(f) = \prod_{1 \leq i < j < k \leq n} ((a_i - a_j - a_k)(a_i - a_j + a_k - a_k) \times
\]
\[
(a_i - a_j + a_k)) \in K. \]

We construct the symbolic homotopical polynomial (cf. [3])
\[
f_{\text{hom}}(x) = tf(x) + (1 - t)f_{\text{known}}(x) \in (K[t])[x]
\]
where \( f_{\text{known}}(x) = (x - r_1) \cdots (x - r_n) \in K[x] \) is computed for such values of the roots \( r_j \in K \) that we have \( \delta_1(f_{\text{known}}) \delta_3(f_{\text{known}}) \neq 0 \).

Since \( r_n \) as a variable in (46) has degree \( n - 1 + 3(\ell - 1) \), this is always possible if the cardinality of \( K \) is \( \geq n + 3(\ell - 1) \). For smaller finite fields we can algebraically extend \( K \) to sufficiently large cardinality.

One computes \( \delta_3(f_{\text{hom}})^2 \in K[t] \) as follows (writing \( f_{\text{hom}}(x) = \prod_{t}(x - a_{t}^2) \)):
\[
d^{[i<j<k\leq t]}(t) \equiv \delta_1(f_{\text{hom}}^{[i<j\leq t]}(t), x)
\]
\[
= \prod_{1 \leq i < j \leq n, 1 \leq k \leq t} (a_i^2 + a_j^2 - a_k^2 - a_k^2) \in K[t],
\]
where \( f_{\text{hom}}^{[i<j]} \) is (19) for \( f = f_{\text{hom}} \),
\[
d^{[\text{sqd}]}(t) \equiv (-1)^{(n)} d^{[i<j<k\leq t]}(t)/\delta_3(f_{\text{hom}})^{n-2}
\]
\[
= \prod_{1 \leq i < j \leq n, 1 \leq k \leq t \leq n, i \neq j, 3 \mid k, j \neq k} (a_i^2 + a_j^2 - a_k^2 - a_k^2) \quad \in K[t].
\]
The division in (48) is by a non-zero polynomial because \( \delta_1(f_{\text{hom}}) \) evaluates at \( t = 0 \) to \( \delta_1(f_{\text{known}}) \). By (46) (46) is \( \neq 0 \). The exponent \( n - 2 \) in (48) arises because for each \((a_i^2 - a_j^2)(a_i^2 - a_k^2)\) in \( \delta_1(f_{\text{known}}) \) there are \( n - 2 \) indices for the \( a_k^2 \) which has canceled by virtue of \( q = i = k \) or \( q = i = \ell \) or \( q = j = k \) or \( q = j = \ell \) in (47). For \( 3 \mid (n) \) of the linear factors in \( d^{[i<j<k\leq t]}(t)/\delta_3(f_{\text{known}})^{n-2} \) we have \( k \neq i \).

We first prove that if \( d^{[\text{sqd}]}(t) \) is a constant polynomial \( \in K \), then \( \delta_3(f) = \delta_3(f_{\text{known}}) \), which has been computed while choosing the \( r_i \). Suppose \( \delta_3(f) \neq \delta_3(f_{\text{known}}) \). The polynomial \( \delta_3(f_{\text{hom}}) \) \( \in K[t] \) then cannot be a constant polynomial because for \( t = 0 \) and \( t = 1 \) it evaluates to different values. Therefore its square, \( d^{[\text{sqd}]}(t) \), is not constant. Note that \( d^{[\text{sqd}]}(t) \) may be constant even if the coefficients of \( f_{\text{hom}} \) depend on \( t \).

In case \( d^{[\text{sqd}]}(t) \) has degree \( \geq 2 \) in \( t \) (it is the square of a non-constant polynomial) we compute the monic square-root polynomial of its monic associate, \( h(t)^2 = \frac{1}{2} d^{[\text{sqd}]}(t) \), where \( c \) is the leading coefficient of \( d^{[\text{sqd}]}(t) \) in \( t \). Note that \( \Delta_3(\delta) \) is defined for fields \( K \) of characteristic \( \neq 2 \). Because \( d^{[\text{sqd}]}(0) = \delta_3(f_{\text{known}})^2 \neq 0 \) by (46) we must have \( h(0) \neq 0 \). Then the homotopy (44) yields
$\delta_3(f_{\text{hom}}) = \delta_3(f_{\text{known}})/h(t)$ and $\delta_3(f) = \delta_3(f_{\text{known}})h(1)/h(0)$.

4 THIRD DISCRIMINANTS FOR DEGREES $N = 4$ AND $N = 5$

The third discriminant for $n = 4$ in coefficients is $D_3(z_0, \ldots, z_3) = -z_0^3 + 4z_3 z_2 - 8z_1$, which does not depend on $z_0$, because the product (8) has 3 linear factors and therefore does not contain the constant coefficient $a_0 = y_1 \cdots y_4$ of $F(x)$ in (1).

The third discriminant for $n = 5$ in coefficients is shown in Figure 2, which can be copied-and-pasted into a worksheet, for instance, into Maple. Note that $D_3$ could still be computed from the $f_{\text{hom}}$ s $\neq 0$. In such a way, the entire computation would be highly parallelizable. We used $f_{\text{known}}(x) = (x - 11)(x - 7)x(x + 1)(x + 3)$ with $\delta_3(f_{\text{known}}) = 9608341743000$.

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REFERENCES


A APPENDIX: COMPUTING POLYNOMIAL ROOTS

We briefly describe how the polynomial $\lambda$-th root of a monic polynomial can be computed. Let $g(x) = g(x)x^m \in \mathbb{K}[x]$ where $g = \tilde{h}$. One computes for $d = \deg(\tilde{h})$ the reverse polynomial $h(x) = x^d\tilde{h}(1/x)$ with $h(0) = 1$ from the reverse $\grev(x) = x^{\lambda^d}\tilde{g}(1/x)$ with $\grev(0) = 1$. The power series expansion $h(x)$ can be computed by quadratically convergent Newton iteration: Suppose $h^{[i]}(x) = (h \bmod x^d)^{[i]}$ with $h^{[0]} = 1$. We obtain $h^{[i+1]}(x) = h^{[i]}(x) + x^d h^{[i]}(x)$ from $h^{[i]}(x)x^\lambda + \lambda h^{[i]}(x)x^{\lambda-1}x^d h^{[i]}(x) \equiv \grev(x) \bmod x^{d+i}$. (49) that is $h^{[i]}(x) = \frac{x^{d+i}}{\lambda} (h^{[i]}(x))^{-1}x^{\lambda-i}x^d h^{[i]}(x) \equiv \grev(x) - h^{[i]}(x)x^d \bmod x^{d+i}$. (50) The power series expansion of $h^{[i]}(x)$ can again be computed by Newton iteration. As $\lambda$-$\text{th}$ root, we have $g(x) = (x^{d+m}h(1/x))^\lambda$. Note that for a monic polynomial $g \in D[x]$, where $D$ is a unique factorization domain of characteristic 0, the division by $\lambda$ in (50) is exact and the root polynomial $x^{d+m}h(1/x) \in D[x]$.

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### B APPENDIX: NOTATION

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_i \in K[y_1, \ldots, y_n]$</td>
<td>$x^n + a_{n-1}x^{n-1} + \cdots + a_0 \overset{\text{def}}{=} (x-y_1) \cdots (x-y_n)$</td>
</tr>
<tr>
<td>$D_1$</td>
<td>the discriminant $= D_{1, (1, -1), S_3}$ in coefficient variables (cf. (2)).</td>
</tr>
<tr>
<td>$D_2$</td>
<td>the second discriminant (in coefficient variables) $= D_{2, (2, -1, -1), P_2}$ with $P_2 = {\text{id, (1, 2, 3)}, (1, 2, 3, 4), (1, 2, 3, 4, 5)} \subset S_5$ (cf. (6)).</td>
</tr>
<tr>
<td>$D_3$</td>
<td>the third discriminant (in coefficient variables) $= D_{3, (1, 1, -1), P_3}$ with $P_3 = {\text{id, (1, 2, 3, 4, 5), (1, 2, 3, 4)}} \subset S_3$ (cf. (8)).</td>
</tr>
<tr>
<td>$D_{\nu-1, (w_1, \ldots, w_\nu), P}$</td>
<td>the $(\nu-1)$-st discriminant with root weights $w_i \in K, 1 \leq i \leq \nu$ and permutation set $P$ (cf. (9)).</td>
</tr>
<tr>
<td>$\delta_\nu(f, x)$</td>
<td>the $\nu$-th discriminants of a univariate polynomial $f$ in the variable $x$.</td>
</tr>
<tr>
<td>$\Delta_\nu$</td>
<td>the $\nu$-th discriminants in root variables</td>
</tr>
</tbody>
</table>

**Notation continued (in alphabetic order):**

| $\hat{\Lambda}_\nu$ | the $\nu$-th Orlando polynomial (12) in the roots. |
| $\hat{\mathcal{D}}_\nu$ | the $\nu$-th Orlando polynomials in the coefficients |
| $\hat{\delta}_\nu(f, x)$ | the $\nu$-th Orlando polynomial of a univariate polynomial $f$ in the variable $x$. |
| $\delta_3(f, x)$ | $\overset{\text{def}}{=} \prod_{1 \leq i < j < k \leq n} \left(3\alpha_\ell - (\alpha_i + \alpha_j + \alpha_k)\right)$ for $f(x) = \prod_{i=1}^{n}(x - \alpha_i)$ (see (41)). |
| $\hat{f}_{\nu}(x)$ | $\overset{\text{def}}{=} \prod_{1 \leq i < j < k \leq n} (x + \alpha_i + \alpha_j + \alpha_k)$ for $f(x) = \prod_{i=1}^{n}(x - \alpha_i)$. |
| $f_{\text{sec-discr}}(x)$ | $\overset{\text{def}}{=} \prod_{1 \leq i < j < k \leq n} (x + 2\alpha_i - \alpha_j - \alpha_k)$ for $f(x) = \prod_{i=1}^{n}(x - \alpha_i)$. (see (33)). |
| $S_n$ | the set of permutations of $(1, \ldots, n)$. |
| $y_1, \ldots, y_n$ | the variables denoting the roots. |
| $Z_0, \ldots, Z_{n-1}$ | the variables denoting the coefficients. |