Deterministic Irreducibility Testing of Polynomials over Large Finite Fields*

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Abstract

We present a sequential deterministic polynomial-time algorithm for testing dense multivariate polynomials over a large finite field for irreducibility. All previously known algorithms were of a probabilistic nature. Our deterministic solution is based on our algorithm for absolute irreducibility testing combined with Berlekamp's algorithm.

1. Introduction

Berlekamp (1970)[2] first showed how the factoring problem for univariate polynomials over large finite fields could be solved in polynomial-time be introducing random choices. However, already Butler (1954)[3] had established that the determination of the number of factors in polynomial-time does not require random choices. Although great effort has been spent in the last fifteen years to remove the necessity for random choices for the factoring problem (cf. Zassenhaus 1969[21], Shanks 1972[19], Moenck 1977[16], Cantor & Zassenhaus 1981[5], Camion 1983[4], Schoof 1985[18], Huang 1985[10], von zur Gathen 1985[8], and Adleman & H. Lenstra 1986)[1] the problem remains in general unresolved. Only within the last five years has it been shown that for multivariate polynomials probabilistic polynomial-time solutions exist as well (cf. Chistov & Grigoryev 1982[6], von zur Gathen & Kaltofen 1985[9], and A. K. Lenstra 1985)[15] However, in the dense representation case these results did not quite parallel the univariate factorization theory. The reason was that all the algorithms known needed to factor a univariate polynomial in order to determine irreducibility and therefore were not deterministic. Here we present an algorithm that tests dense multivariate polynomials over

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large finite fields for irreducibility in deterministic polynomial-time. Contrary to most univariate deterministic factoring results our solution is not subject to any unproven mathematical conjecture such as the Riemann hypothesis.

We have observed in (Kaltofen 1985a)[11] that absolute irreducibility of multivariate polynomials over large finite fields could be decided in polynomial-time. Here we essentially modify the algorithm presented there to solve the problem of irreducibility over the field itself. It comes as a small surprise that irreducibility can be related to absolute irreducibility. Absolute irreducibility is a purely rational question, that is it can be decided by field arithmetic alone (Noether 1922)[17] whereas irreducibility over certain constructive fields can be shown undecidable (Fröhlich & Shepherdson 1955)[7] Our solution, which makes use of the Butler-Berlekamp Q-matrix construction seems to establish this relationship only for finite fields. It is therefore very special and does not contradict the differences of the problems known for arbitrary fields.

In this paper we restrict ourselves to bivariate polynomials. It is fairly easy to generalize our algorithms to dense multivariate polynomials, see e.g. Algorithm 2 in (Kaltofen 1985b)[12]

Notation: $\mathbf{F}_q$ denotes a finite field with q elements; $\deg_x(f)$ denotes the highest degree of x in $f \in \mathbf{F}_q[y, x]$ and $\deg(f)$ the total degree of f. The coefficient of the highest power of x in f, a polynomial in y, is referred to as the leading coefficient of f in x and will be denoted by $\mathrm{ldcf}_x(f)$. We call f monic in x if $\mathrm{ldcf}_x(f)$ is the one of $\mathbf{F}_q$.

2. Previous Results Needed

We now discuss several facts needed in the deterministic irreducibility test. First we observe that the input polynomial $f \in \mathbf{F}_q[y, x]$ can be assumed monic in x and $f(0, x)$ can be assumed squarefree. The preprocessing necessary to enforce these conditions is discussed, e.g., in (Kaltofen 1985b)[12], §4, or in (Kaltofen 1985a)[11], §2. Notice that the translation necessary to make $f(0, x)$ squarefree requires

$$q \geq 2 \deg_x(f) \deg_y(f).$$

We can also assume this because otherwise even the factorization problem in $\mathbf{F}_q[y, x]$ can be solved in deterministic polynomial-time, cf. (von zur Gathen & Kaltofen 1985)[9], §4.2. It should be also noted that the monicity requirement can be at all avoided by slightly changing the algorithm along the lines of (von zur Gathen & Kaltofen 1985)[9], Remark 2.4. An even simpler way to get monicity than the methods refered to above would be to translate the original polynomial as $f(x, y+bx)$ for a suitable $b \in \mathbf{F}_q$, see (Kaltofen 1985c)[13], Lemma 6.1. We could also have restricted ourselves to q being a prime since the algorithm in (Trager 1976)[20] can reduce the problem of irreducibility testing over algebraic extensions to that of irreducibility testing over the base field in deterministic polynomial-time. However, this restriction does not simplify our proofs but would drastically increase the complexity of the complete algorithm.

We now outline the basic algorithm from (Kaltofen 1985b)[12] for testing multivariate polynomials for irreducibility. We will not prove the correctness of this algorithm here but refer the reader to (Kaltofen 1985b)[12], §5, for more details on the algorithm and the necessary arguments.

Algorithm 1:
[Given $f(y,x) \in F[y,x]$ monic in x, $f(0,x)$ squarefree, F an arbitrary field, and given an irreducible factor $t(z)$ of $f(0, z)$ in $F[z]$, this algorithm determines irreducibility of f over F:]

(N) [Compute approximation of root in $G[[y]]$, where $G = F[z]/(t(z))$:]
$n \leftarrow \deg_x(f)$; $d \leftarrow \deg_y(f)$; $k \leftarrow (2n-1)d$; $a_0 \leftarrow (z \bmod t(z)) \in G$.
By Newton iteration, calculate $a_1, \ldots, a_k \in G$ such that

$$f(y, a_0 + a_1 y + \cdots + a_k y^k) \equiv 0 \bmod y^{k+1}.$$

FOR $i \leftarrow 0, \ldots, n-1$ DO $\alpha^{(i)} \leftarrow (a_0 + \cdots + a_k y^k)^i \bmod y^{k+1} \in G[y]$.

(L) [Try to find a polynomial of degree $n-1$ in $F[y, x]$ for which $\alpha^{(1)}$ is the approximation for one of its roots:]
Try to solve the equation

$$\alpha^{(n-1)} + \sum_{i=0}^{n-2} u_i(y)\alpha^{(i)} \equiv 0 \bmod y^{k+1} \tag{1}$$

for polynomials $u_i \in F[y]$ with $\deg(u_i) \le d$. This equation leads to a linear system over F in $(k+1)\deg(t)$ equations and $(n-1)(d+1)$ unknown coefficients of u_i. If there exists a solution then RETURN (‘‘reducible’’). Otherwise RETURN (‘‘irreducible’’). □

The problem is that $t(z)$ cannot be found in deterministic polynomial-time. It turns out that in the absolute irreducibility test we can work with $f(0, z)$ instead. The following theorem establishes the connection between working with any irreducible factor of $f(0, z)$, as we may, and working with $f(0, z)$.

Theorem 1 (Butler 1954)[3]: Let $f(z) \in \mathbf{F}_q[z]$ be monic and squarefree of degree n, $f = f_1 \cdots f_r$ be its factorization into monic irreducible polynomials. Consider the sub-algebra of $\mathbf{F}_q[z]/(f(z))$,

$$V(f(z)) := \{v(z) \mid \deg(v) < n,\ v \bmod f_j \in \mathbf{F}_q \text{ for all } 1 \le j \le r\},$$

and the matrix

$$Q(f) := [a_{i,j}]_{0 \le i,j \le n-1}, \quad \text{where} \quad a_{i,0} + a_{i,1}z + \cdots + a_{i,n-1}z^{n-1} :\equiv (z^{iq} \bmod f(z)).$$

Then

$$v_0 + v_1 z + \cdots + v_{n-1}z^{n-1} \in V(f(z))$$

if and only if

$$(v_0, v_1, \ldots, v_{n-1})\, Q(f) = (v_0, v_1, \ldots, v_{n-1}). \quad \square$$

The importance of this theorem to our irreducibility test is that membership of v in $V(f(z))$ can be enforced by linear relations on the coefficients of v. Let $v^{[1]}, \ldots, v^{[r]}$ be a basis for the left null-space of $Q(f)-I_n$, where I_n is the $n \times n$ identity matrix. Then $v \in V(f(z))$ if and only if

$$(w_1, \ldots, w_r) \begin{bmatrix} v_0^{[1]} & \cdots & v_{n-1}^{[1]} \\ \vdots & & \vdots \\ v_0^{[r]} & \cdots & v_{n-1}^{[r]} \end{bmatrix} = (v_0, \ldots, v_{n-1})$$

is solvable for w_i over $\mathbf{F}_q$.

3. Deterministic Irreducibility Testing

We now present the deterministic irreducibility test in $\mathbf{F}_q[y, x]$. The algorithm is very similar to Algorithm 1, but instead of working in G we work in $\mathbf{F}_q[z]/(f(0, z))$. This leads to an algorithm like Algorithm 2 of (Kaltofen 1985a)[11] except that the final linear solution is restricted further.

Algorithm 2:
[Given $f(y,x) \in \mathbf{F}_q[y,x]$ monic in x, $f_0(x) := f(0,x)$ squarefree, this algorithm determines whether f is irreducible.]

(N) [Approximate a root of $f(y,x)$ in $R[[y]]$, where $R = \mathbf{F}_q[z]/(f_0(z))$:]
$n \leftarrow \deg_x(f)$; $d \leftarrow \deg_y(f)$; $k \leftarrow (2n-1)d$; $a_0 \leftarrow z \bmod f_0(z) \in R$.
By Newton iteration (cf. Kaltofen 1985a[11], Algorithm 2, Steps I and N), calculate $a_1, \ldots, a_k \in R$ such that

$$f(y, a_0 + a_1 y + \cdots + a_k y^k) \equiv 0 \bmod y^{k+1}.$$

FOR $i \leftarrow 0, \ldots, n-1$ DO $\alpha^{(i)} \leftarrow (a_0 + \cdots + a_k y^k)^i \bmod y^{k+1}$.

(Q) Find a basis $\{v^{[1]}, \ldots, v^{[r]}\}$ for the left null-space of $Q(f_0) - I_n$, see Theorem 1. [More details for this step can be found in (Knuth 1981)[14], §4.6.2. Note that $z^q \bmod f_0(z)$ is computed by binary exponentiation.]

(L) [Try to find a polynomial of degree $n-1$ in $V(f_0(z))[y,x]$, $V(f_0(z))$ as defined in Theorem 1, for which $\alpha^{(1)}$ is the approximation for one of its roots:]
Examine whether the equation

$$\alpha^{(n-1)} + \sum_{i=0}^{n-2} u_i(y)\alpha^{(i)} \equiv 0 \bmod y^{k+1} \tag{2}$$

is solvable for polynomials $u_i(y) \in V(f_0(z))[y]$ such that $\deg(u_i) \le d$. Let $u_i(y) = \sum_{\delta=0}^{d} u_{i,\delta} y^{\delta}$ and let

$$\alpha^{(i)} = \sum_{\kappa=0}^{k} a_\kappa^{(i)} y^\kappa, \quad a_\kappa^{(i)} \in R.$$

Then (2) leads to the linear system for the coefficients of y^κ

$$a_\kappa^{(n-1)} + \sum_{i=0}^{n-2} \sum_{\delta=0}^{d} a_{\kappa-\delta}^{(i)} u_{i,\delta} = 0 \tag{3}$$

for $\kappa = 0, \ldots, k$ in the variables $u_{i,\delta} \in V(f_0(z))$, $i = 0, \ldots, n-2$, $\delta = 0, \ldots, d$.

Let

$$u_{i,\delta} = \sum_{j=0}^{n-1} u_{i,\delta,j} z^j, \quad a_\kappa^{(i)} = \sum_{j=0}^{n-1} a_{\kappa,j}^{(i)} z^j$$

and let

$$z^\lambda \equiv \sum_{j=0}^{n-1} c_{\lambda,j} z^j \mod f_0(z), \quad \lambda = n, \ldots, 2n-2,\ c_{\lambda,j} \in \mathbf{F}_q.$$

Then the coefficient of z^l, $0 \le l \le n-1$, for each equation in (3) is, setting $a_{\delta,j}^{(i)}$ and $u_{i,\delta,j}$ to 0 for $j \ge n$, $\delta < 0$,

$$a_{\kappa,l}^{(n-1)} + \sum_{i=0}^{n-2} \sum_{\delta=0}^{d} \left[\sum_{j=0}^{l} a_{\kappa-\delta,l-j}^{(i)} u_{i,\delta,j} + \sum_{\lambda=n}^{2n-2} \sum_{j=0}^{\lambda} c_{\lambda,l}\, a_{\kappa-\delta,\lambda-j}^{(i)} u_{i,\delta,j} \right], \tag{4}$$

which is a linear expression in $u_{i,\delta,j}$ and which must vanish on a solution of (3).

Furthermore $u_{i,\delta}$ must be an element in $V(f_0(z))$. We introduce new unknows $w_{i,\delta,\rho}$, $1 \le \rho \le r$, and require that

$$(w_{i,\delta,1}, \ldots, w_{i,\delta,r})[v_j^{[i]}]_{\substack{1 \le i \le r \\ 0 \le j \le n-1}} = (u_{i,\delta,0}, \ldots, u_{i,\delta,n-1}) \tag{5}$$

be solvable for all $0 \le i \le n-2$, $0 \le \delta \le d$. Equation (5) leads to the linear equations

$$u_{i,\delta,j} - \sum_{\rho=1}^{r} v_j^{[\rho]} w_{i,\delta,\rho}, \quad 0 \le j \le n-1. \tag{6}$$

Equations (4) and (6) determine a linear system over $\mathbf{F}_q$ in

$$n(k+1) + (n-1)n(d+1)$$

equations and

$$(n+r)(n-1)(d+1)$$

unknowns. If this system has a solution, we return ‘‘f is reducible in $\mathbf{F}_q$’’, otherwise, we return ‘‘f is irreducible’’. □

We will not fully analyze this algorithm because its running time is inferior to that of Algorithm 1 in conjunction with finding $t(z)$ probabilistically. The algorithm is clearly polynomial in $\log(q)$ and does not require random choices. However, its correctness needs explanation. First let us formulate our main result in a theorem.

Theorem 2: Algorithm 2 correctly decides irreducibility of f in $\mathbf{F}_q[y, x]$ in $(\log(q) \deg(f))^{O(1)}$ sequential deterministic steps.

Proof: Solving the linear system determined by (4) and (6) is by theorem 1 equivalent to solving (2) for $u_i(y) \in V(f_0(z))[y]$. If (2) has a solution then for an irreducible factor $t(z)$ of $f_0(z)$, $u_i(y) \bmod t(z) \in \mathbf{F}_q[y]$. Thus by applying Algorithm 1 to f and t, f must be composite. On the other hand, if f is composite, Algorithm 1 will find a solution to (1) for <u>all</u> irreducible factors $t_\rho(z)$ of $f_0(z)$. By the Chinese remainder theorem (2) now becomes solvable for $u_i(y) \in V(f_0(z))[y]$. Therefore the algorithm will correctly determine the compositeness of f. □

We remark that Algorithm 3 of (Kaltofen 1985a)[11] applies to the solution of (4) and (6) as well. Depending on f that algorithm may split $f_0(z)$.

4. Conclusion

We have resolved one problem left open during the polynomial-time polynomial factorization tempest of 1982, namely that random choices are not needed to test multivariate polynomials over large finite fields for irreducibility. In order to completely parallel the univariate results it would be necessary to also compute the number and the degrees of all irreducible factors without probabilistic choices. Unfortunately, it is not clear to us how our algorithm could accomplish that and we must leave this question for future research.

References

1. Adleman, L. M. and Lenstra, H. W., ‘‘Finding irreducible polynomials over finite fields,’’ *Proc. 18th ACM Symp. Theory Comp.*, pp. 350-355, 1986.

2. Berlekamp, E. R., ‘‘Factoring polynomials over large finite fields,’’ *Math. Comp.*, vol. 24, pp. 713-735, 1970.

3. Butler, M. C. R., ‘‘On the reducibility of polynomials over a finite field,’’ *Quart. J. Math., Oxford Ser. (2)*, vol. 5, pp. 102-107, 1954.

4. Camion, P., ‘‘A deterministic algorithm for factorizing polynomials of $F_{p[x]}$,’’ *Ann. Discrete Math.*, vol. 17, pp. 149-157, 1983.

5. Cantor, D. G. and Zassenhaus, H., ‘‘A new algorithm for factoring polynomials over finite fields,’’ *Math. Comp.*, vol. 36, pp. 587-592, 1981.

6. Chistov, A. L. and Grigoryev, D. Yu., ‘‘Polynomial-time factoring of multivariable polynomials over a global field,’’ *LOMI preprint E-5-82*, Steklov Institute, Leningrad, 1982.

7. Fröhlich, A. and Shepherdson, J. C., ‘‘Effective procedures in field theory,’’ *Phil. Trans. Roy. Soc., Ser. A*, vol. 248, pp. 407-432, 1955/56.

8. Gathen, J. von zur, ‘‘Factoring polynomials and primitive elements for special primes,’’ *Theoretical Comput. Sci.*, vol. 52, pp. 77-89, 1987.

9. Gathen, J. von zur and Kaltofen, E., ‘‘Factoring multivariate polynomials over finite fields,’’ *Math. Comp.*, vol. 45, pp. 251-261, 1985.

10. Huang, M.-D. A., ‘‘Riemann hypothesis and finding roots over finite fields,’’ *Proc. 17th ACM Symp. Theory Comp.*, pp. 121-130, 1985.

11. Kaltofen, E., ‘‘Fast parallel absolute irreducibility testing,’’ *J. Symbolic Computation*, vol. 1, pp. 57-67, 1985.

12. Kaltofen, E., ‘‘Polynomial-time reductions from multivariate to bi- and univariate integral polynomial factorization,’’ *SIAM J. Comp.*, vol. 14, pp. 469-489, 1985.

13. Kaltofen, E., ‘‘Greatest common divisors of polynomials given by straight-line programs,’’ *J. ACM*, vol. 35, no. 1, pp. 231-264, 1988.

14. Knuth, D. E., *The Art of Programming, vol. 2, Semi-Numerical Algorithms, ed. 2,* Addison Wesley, Reading, MA, 1981.

15. Lenstra, A. K., ‘‘Factoring multivariate polynomials over finite fields,’’ *J. Comput. System Sci.*, vol. 30, pp. 235-248, 1985.

16. Moenck, R. T., ‘‘On the efficiency of algorithms for polynomial factoring,’’ *Math. Comp.*, vol. 31, pp. 235-250, 1977.

17. Noether, E., ‘‘Ein algebraisches Kriterium für absolute Irreduzibilität,’’ *Math. Ann.*, vol. 85, pp. 26-33, 1922.

18. Schoof, R. J., ‘‘Elliptic curves over finite fields and the computation of square roots mod p,’’ *Math. Comp.*, vol. 44, pp. 483-494, 1985.

19. Shanks, D., ‘‘Five number-theoretical algorithms,’’ *Proc. 2nd Manitoba Conf. Numerical Math.*, pp. 51-70, 1972.

20. Trager, B. M., ‘‘Algebraic factoring and rational function integration,’’ *Proc. 1976 ACM Symp. Symbolic Algebraic Comp.*, pp. 219-228, 1976.

21. Zassenhaus, H., ‘‘On Hensel factorization I,’’ *J. Number Theory*, vol. 1, pp. 291-311, 1969.