Algebraic Complexity and Algorithms: Recent Advances and New Open Problems

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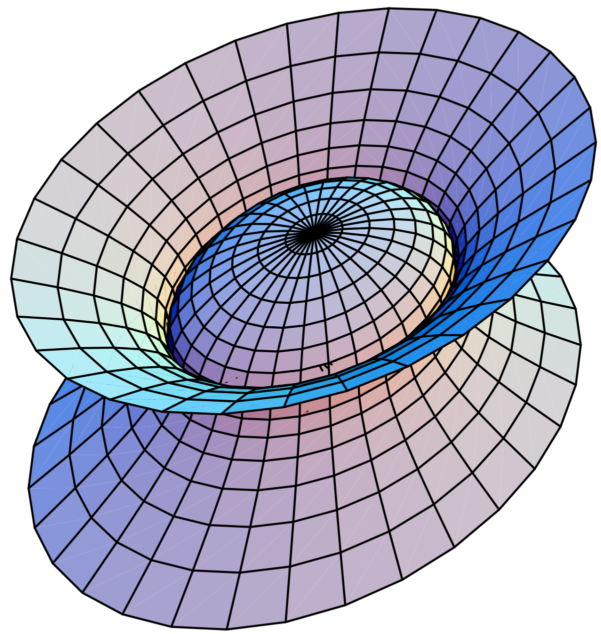
1. Nearest Singular Problems

Factorization of nearby polynomials over the complex numbers

$$81x^4 + 16y^4 - 648z^4 + 72x^2y^2 - 648x^2 - 288y^2 + 1296 = 0$$



$$(9x^2 + 4y^2 + 18\sqrt{2}z^2 - 36)(9x^2 + 4y^2 - 18\sqrt{2}z^2 - 36) = 0$$

$$81x^4 + 16y^4 - 648.003z^4 + 72x^2y^2 + .002x^2z^2 + .001y^2z^2 - 648x^2 - 288y^2 - .007z^2 + 1296 = 0$$

Open Problem 1

Given is a polynomial $f(x,y) \in \mathbb{Q}[x,y]$ *and* $\varepsilon \in \mathbb{Q}$.

Decide in polynomial time in the degree and coefficient size if there is a factorizable $\hat{f}(x,y) \in \mathbb{C}[x,y]$ *with* $\|f - \hat{f}\| \leq \varepsilon$,

for a reasonable coefficient vector norm $\|\cdot\|$.

Sensitivity analysis: approximate consistent linear system

Suppose the linear system $Ax = b$ is unsolvable.
Find $\hat{b}$ "nearest to" b that makes it solvable.

Minimizing Euclidean distance: $\min_{\hat{x}} \|A\hat{x} - b\|_2$ (least squares)

Minimizing component-wise distance: $\min_{\hat{x}} \left(\max_{1 \leq i \leq m} |b_i - \sum_{j=1}^{n} a_{i,j}\hat{x}_j| \right)$

Introduce new variable y and solve the linear program

minimize: y

linear constraints: $y \geq b_i - \sum_{j=1}^{n} a_{i,j}\hat{x}_j \quad (1 \leq i \leq m)$

$y \geq -b_i + \sum_{j=1}^{n} a_{i,j}\hat{x}_j \quad (1 \leq i \leq m)$

Sensitivity analysis: nearest singular matrix

Given are $2n^2$ rational numbers $\underline{a}_{i,j}, \bar{a}_{i,j}$.
Let $\mathcal{A}$ be the *interval* matrix

$$\mathcal{A} = \left\{ \begin{bmatrix} a_{1,1} & \dots & a_{n,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \mid \underline{a}_{i,j} \leq a_{i,j} \leq \bar{a}_{i,j} \text{ for all } 1 \leq i, j \leq n \right\}.$$

Does $\mathcal{A}$ contain a singular matrix?

This problem is *NP-complete* [Poljak&Rohn 1990].

Sensitivity analysis: nearest singular matrix using matrix norms

$\|A\|_p = \max_{x \neq 0} \|Ax\|_p / \|x\|_p$ where $\|\cdot\|$ is a vector norm

$\|A\|_\infty = \max_i \sum_j |a_{i,j}|$ (max row-sum)

$\|A\|_1 = \max_j \sum_i |a_{i,j}|$ (max column-sum)

Theorem [Eckart&Young 1936] Suppose A is non-singular.

$$\min_{\tilde{A} \text{ singular}} \|A - \tilde{A}\| = 1/\|A^{-1}\|.$$

Note: one can also compute $\tilde{A}$ efficiently.

Sensitivity analysis: nearest matrix with a given eigenvalue

Given a complex matrix A and a complex value μ (exactly or parametrically), one can efficiently compute

$$\min_{\tilde{A}:\ \mu \text{ is an eigenvalue of } \tilde{A}} \|A-\tilde{A}\| = 1/\|(A-\mu I)^{-1}\|$$

where $\|\cdot\|$ is either the ∞- or 1-matrix norm.

Sensitivity analysis: approximate greatest common divisor

Suppose $f = x^m + a_{m-1}x^{m-1} + \cdots + a_0$, $g = x^n + b_{n-1}x^{n-1} + \cdots + b_0$
have no common divisor.
Find $\hat{f}, \hat{g}$ "nearest to" f, g that have a common root.

Karmarkar&Lakshman [1996] minimize

$$\sqrt{|a_m - \hat{a}_m|^2 + \cdots + |a_0 - \hat{a}_0|^2 + |b_n - \hat{b}_n|^2 + \cdots + |b_0 - \hat{b}_0|^2}.$$

Equivalent formulation:
Compute the nearest singular *Sylvester matrix* to the Sylvester matrix

$$\begin{bmatrix} a_m & a_{m-1} & \ldots\ldots & a_0 & & & \\ & a_m & \ldots\ldots & a_1 & a_0 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & a_m & \ldots\ldots\ldots & & a_0 \\ b_n & b_{n-1} & \ldots\ldots & b_0 & & & \\ & b_n & \ldots\ldots & b_1 & b_0 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & b_n & \ldots\ldots\ldots & & b_0 \end{bmatrix}$$

Sensitivity analysis: Kharitonov [1978] theorem

Given are $2n$ rational numbers $\underline{a}_i, \bar{a}_i$.
Let P be the *interval* polynomial

$$P = \{x^n + a_{n-1}x^{n-1} + \cdots + a_0 \mid \underline{a}_i \le a_i \le \bar{a}_i \text{ for all } 0 \le i < n\}.$$

Then every polynomial in P is *Hurwitz* (all roots have negative real parts), if and only if the four "corner" polynomials

$$g_k(x) + h_l(x) \in P, \quad \text{where } k = 1,2 \text{ and } l = 1,2,$$

with

$$g_1(x) = \underline{a}_0 + \bar{a}_2x^2 + \underline{a}_4x^4 + \cdots, \quad h_1(x) = \underline{a}_1 + \bar{a}_3x^3 + \underline{a}_5x^5 + \cdots,$$
$$g_2(x) = \bar{a}_0 + \underline{a}_2x^2 + \bar{a}_4x^4 + \cdots, \quad h_2(x) = \bar{a}_1 + \underline{a}_3x^3 + \bar{a}_5x^5 + \cdots$$

are Hurwitz.

Sensitivity analysis: constraint root problem

Given is a real or complex polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

and a root $\alpha \in \mathbb{C}$.

Compute $\hat{f}$ "nearest to" f such that $\hat{f}(\alpha) = 0$.

Hitz and K [1998] solve this problem efficiently for

- parametric α (root stability) and Euclidean distance
- explicit roots $\alpha_1, \alpha_2, \ldots$ and coefficient-wise distance
- with linear coefficient constraints, e.g., $a_n = 1$.

Theorem [Hitz and K 1998] Given is the real polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{R},$$

with no real root. Then

$$\min_{\substack{\hat{a}_0, \ldots, \hat{a}_{n-1} \text{ such that} \\ \exists \alpha \in \mathbb{R}:\ \alpha^n + \hat{a}_{n-1}\alpha^{n-1} + \cdots + \hat{a}_0 = 0}} \left(\max_{0 \le i < n} |a_i - \hat{a}_i| \right)$$

$$= \min_{\alpha \in \mathbb{R}} \left| \frac{f(\alpha)}{\sum_{k=0}^{n-1} |\alpha^k|} \right|$$

(which one can compute in polynomial time in n and the size of the coefficients a_i).

2. Black Box Linear Algebra

The black box model of a matrix

$A \in \mathbb{K}^{n \times n}$ singular

$\mathbb{K}$ an arbitrary, e.g., finite field

Perform linear algebra operations, e.g., $A^{-1}b$ [Wiedemann 86] with

$$\begin{array}{rl} O(n) & \text{black box calls and} \\ n^2(\log n)^{O(1)} & \text{arithmetic operations in } \mathbb{K} \text{ and} \\ O(n) & \text{intermediate storage for field elements} \end{array}$$

Flurry of recent results

Lambert [1996], Teitelbaum [1997], Eberly & K [1997]	relationship of Wiedemann and Lanczos approach
Villard [1997]	analysis of *block* Wiedemann algorithm
Giesbrecht [1997]	computation of integral solutions
Giesbrecht & Lobo & Saunders [1997]	certificates for inconsistency

Open Problem 2
Within the resource limitations stated above, compute the characteristic polynomial of a black box matrix. Randomization is allowed (of course!), as is a "Monte Carlo" solution.

Classes of randomized algorithms

Monte Carlo	$\equiv$	always fast, probably correct
Las Vegas	$\equiv$	always correct, probably fast
BPP	$\equiv$	probably correct, probably fast

Why Las Vegas algorithms may be bad for you

```
repeat
        pick random numbers
        compute candidate answer
until check if a solution succeeds
```

A programming bug leads to an infinite loop!

Diophantine solutions
by Giesbrecht:
Find several rational solutions.

$$A(\tfrac{1}{2}x^{[1]}) = b, \quad x^{[1]} \in \mathbb{Z}^n$$
$$A(\tfrac{1}{3}x^{[2]}) = b, \quad x^{[2]} \in \mathbb{Z}^n$$

$$\gcd(2,3) = 1 = 2 \cdot 2 - 1 \cdot 3$$
$$A(2x^{[1]} - x^{[2]}) = 4b - 3b = b$$

What if there is not solution? Certify that!

Let $r = \text{rank}(A)$.
By preconditioning [K&Saunders 1991], one may assume that top-left $r \times r$ submatrix of A is non-singular of determinant d.

Compute integers y_i such that

$$[y_1, \ldots, y_r, 0, \ldots, 0]A = [\delta_1 d, \ldots, \delta_r d, a_{r+1} d, \ldots, a_n d],$$

where δ_i are random chosen bits, until

$$[y_1, \ldots y_r, 0, \ldots, 0]b \not\equiv 0 \pmod{d}$$

LinBox project

U. Calgary:	Wayne Eberly
U. Delaware:	David Saunders
IMAG Grenoble:	Jean-Guillaume Dumas, Jean-Louis Roch, Gilles Villard
NC State U.:	Erich Kaltofen, Wen-shin Lee, Will Turner
Washington College:	Austin Lobo
U. Western Ontario:	Mark Giesbrecht

Design, analyze and implement black box matrix algorithms

- early termination criterion for block Wiedemann
- rank certificates over characteristic 0 fields
- new Smith normal form algorithm
- “plug-and-play” generic design for library

3. Lattice Reduction

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$

Derivation by lattice reduction [Bailey&Borwein&Plouffe 1995]

$$\int_0^1 \frac{y^{k-1}}{1-\frac{y^8}{16}} dy = \int_0^1 \sum_{i=0}^{\infty} y^{k-1} \left(\frac{y^8}{16}\right)^i dy = \sum_{i=0}^{\infty} \frac{1}{16^i} \int_0^1 y^{8i+k-1} dy$$

$$= \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}$$

Maple takes over

```
> latt := proc(digits)
> local k, j, v, saved_Digits, ltt;
> saved_Digits := Digits; Digits := digits;
> for k from 1 to 8 do
>   v[k] := [];
>   for j from 1 to 10 do v[k] := [op(v[k]), 0]; od;
>   v[k][k] := 1;
>   v[k][10] := trunc(10^digits *
>                   evalf(Int(y^(k-1)/(1-y^8/16),
>                          y=0..1, digits), digits));
> od;
> v[9] := [0,0,0,0,0,0,0,0,1,
>             trunc(evalf(Pi*10^digits,digits+1))];
> ltt := [];
> for k from 1 to 9 do ltt:=[op(ltt),evalm(v[k])];od;
> Digits := saved_Digits;
> RETURN(ltt);
> end:
```

```
> L := latt(25);
```

$$
\begin{aligned}
L := [&[1, 0, 0, 0, 0, 0, 0, 0, 0, 1007184476414676228644760\,0],\\
&[0, 1, 0, 0, 0, 0, 0, 0, 0, 5064768766674304809559394],\\
&[0, 0, 1, 0, 0, 0, 0, 0, 0, 3392302452451990725155853],\\
&[0, 0, 0, 1, 0, 0, 0, 0, 0, 2554128118829953416027570],\\
&[0, 0, 0, 0, 1, 0, 0, 0, 0, 2050025576364235339441503],\\
&[0, 0, 0, 0, 0, 1, 0, 0, 0, 1713170706664974589667328],\\
&[0, 0, 0, 0, 0, 0, 1, 0, 0, 1472019346726350271955981],\\
&[0, 0, 0, 0, 0, 0, 0, 1, 0, 1290770422751423433458478],\\
&[0, 0, 0, 0, 0, 0, 0, 0, 1, 3141592653589793238462643\,4]]
\end{aligned}
$$

```
> readlib(lattice):
> lattice(L);
```

$$[[-4,0,0,2,1,1,0,0,1,5],[0,-8,-4,-4,0,0,1,0,2,5],$$
$$[-61,582,697,-1253,453,-1003,-347,-396,10,559],$$
$$[-333,966,324,-1656,-56,784,1131,-351,-27,255],$$
$$[429,714,-1591,778,-517,-1215,598,362,-87,398],$$
$$[-1046,-259,-295,-260,1286,393,851,800,252,-1120],$$
$$[494,906,-380,-1389,1120,1845,-1454,-926,-218,400],$$
$$[1001,-1099,422,1766,1405,-376,905,-1277,-394,-30],$$
$$[-1144,491,-637,-736,-1261,-680,-1062,-1257,637,-360]]$$

```
> g := (8*y + 4*y^2 + 4*y^3 - y^6)/(1-y^8/16);
```

$$g := \frac{8y+4y^2+4y^3-y^6}{1-\frac{1}{16}y^8}$$

```
> int(g, y=0..1);
```

$$2\pi$$

Open Problem 3

Compute the the n-th digit of π *in radix* $b = 10$ *(more precisely, of an approximation of* π *within precision* $10^{-n-1000}$*— there could be a very long sequence of nines or zeros in the decimal expansion of* π*) in* $n(\log n)^{O(1)}$ *time and simultaneously* $(\log n)^{O(1)}$ *space.*

4. Algorithm Synthesis

Let $\sigma \in \mathbb{K}[\alpha, \beta]/(f, g)$ where $f(\alpha, \beta) = 0$ and $g(\beta) = 0$.
E.g., $\sigma = \sqrt{1+\sqrt{2}} - \sqrt{2} = \alpha - \beta$, $f = \alpha^2 - \beta - 1$, and $g = \beta^2 - 2$.
Task: Compute the minimum polynomial $h(\sigma) = 0$:

$$h(x) = x^m - c_{m-1}x^{m-1} - \cdots - c_0 \in \mathbb{K}[x], \quad m \le \deg(f) \cdot \deg(g)$$

The coefficient vectors $\overrightarrow{\sigma^i}$ of σ^i mod $(f(\alpha, \beta), g(\beta))$ satisfy

$$\forall\, j \ge 0\colon\ \overrightarrow{\sigma^{m+j}} = c_{m-1}\overrightarrow{\sigma^{m-1+j}} + \cdots + c_0\overrightarrow{\sigma^{j}}$$

Any non-trivial linear projection $\mathcal{L}(\overrightarrow{\sigma^i})$ preserves the linear recursion because h is irreducible.

Power Projections = Transposed Modular Polyn Composition

Linear projections of powers

$$\left[\mathcal{L}(\overrightarrow{\sigma^0})\ \ \mathcal{L}(\overrightarrow{\sigma^1})\mathcal{L}(\overrightarrow{\sigma^2})\ \ \ldots\right] = \left[u_0\ u_1\ \ \ldots\ \ u_{n-1}\right] \cdot \underbrace{\left[\overrightarrow{\sigma^0} \mid \overrightarrow{\sigma^1} \mid \overrightarrow{\sigma^2} \mid \ldots\right]}_{A}$$

Modular polynomial composition

$$w(z) = w_0 + w_1 z + w_2 z^2 + \cdots \longmapsto w(\sigma) \bmod (f(\alpha,\beta), g(\beta))$$

$$\overrightarrow{w(\sigma)} = \underbrace{\left[\overrightarrow{\sigma^0} \mid \overrightarrow{\sigma^1} \mid \overrightarrow{\sigma^2} \mid \ldots\right]}_{A} \cdot \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \end{bmatrix}$$

By Tellegen's Theorem [1960] the problems can be solved equally fast

Transposed Modular Polynomial Multiplication in NTL

1. $T_1 \leftarrow \mathrm{FFT}^{-1}(\mathrm{RED}_k(g))$
2. $T_2 \leftarrow T_1 \cdot S_2$
3. $v \leftarrow -\mathrm{CRT}_{0\ldots n-2}(\mathrm{FFT}(T_2))$
4. $T_2 \leftarrow \mathrm{FFT}^{-1}(\mathrm{RED}_{k+1}(x^{n-1} \cdot v))$
5. $T_2 \leftarrow T_2 \cdot S_3$
6. $T_1 \leftarrow T_1 \cdot S_4$
7. Replace T_1 by the 2^{k+1}-point residue table whose j-th column ($0 \leq j < 2^{k+1}$) is 0 if j is odd, and is column number $j/2$ of T_1 if j is even.
8. $T_2 \leftarrow T_2 + T_1$
9. $u \leftarrow \mathrm{CRT}_{0\ldots n-1}(\mathrm{FFT}(T_2))$

"we offer no other proof of correctness other than the validity of this transformation technique (and the fact that it does indeed work in practice)" [Shoup 1994]

Open Problem 4

With inputs $A \in \mathbb{K}^{m \times n}$ and $y \in \mathbb{K}^n$ you are given an algorithm for $A \cdot y$ that uses $T(m,n)$ arithmetic field operations and $S(m,n)$ auxiliary space.

Show how to construct an algorithm for $A^T \cdot z$ where $z \in \mathbb{K}^m$ that uses $O(T(m,n))$ time and $O(S(m,n))$ space.

Your construction must be applicable to practical problems.