Efficient Algorithms for Computing the Nearest Polynomial With Parametrically Constrained Roots and Factors

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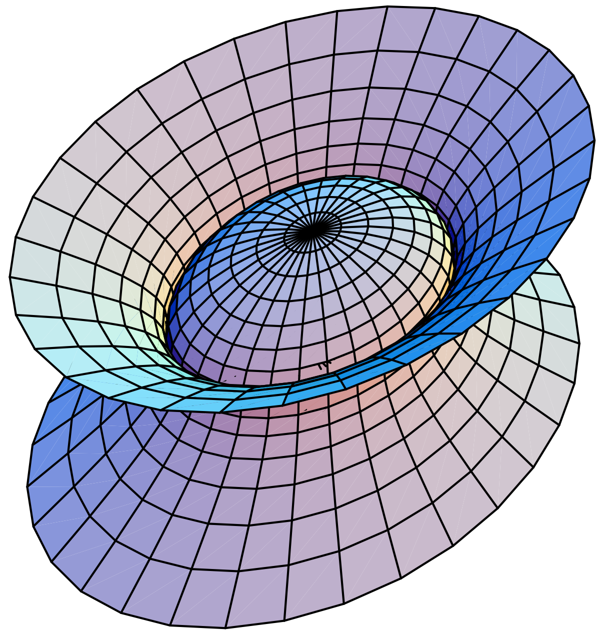
Factorization of nearby polynomials over the complex numbers

$$81x^4 + 16y^4 - 648z^4 + 72x^2y^2 - 648x^2 - 288y^2 + 1296 = 0$$



$$(9x^2 + 4y^2 + 18\sqrt{2}z^2 - 36)(9x^2 + 4y^2 - 18\sqrt{2}z^2 - 36) = 0$$

$$81x^4 + 16y^4 - 648.003z^4 + 72x^2y^2 + .002x^2z^2 + .001y^2z^2$$
$$- 648x^2 - 288y^2 - .007z^2 + 1296 = 0$$

Open Problem 1

Given is a polynomial $f(x,y) \in \mathbb{Q}[x,y]$ *and* $\varepsilon \in \mathbb{Q}$.

Decide in polynomial time in the degree and coefficient size if there is a factorizable $\hat{f}(x,y) \in \mathbb{C}[x,y]$ *with* $\|f - \hat{f}\| \leq \varepsilon$,

for a reasonable coefficient vector norm $\|\cdot\|$.

Sensitivity analysis: approximate consistent linear system

Suppose the linear system $Ax = b$ is unsolvable.
Find $\hat{b}$ "nearest to" b that makes it solvable.

Minimizing Euclidean distance: $\min_{\hat{x}} \|A\hat{x} - b\|_2$ (least squares)

Nearest singular matrix (Eckart & Young 1936, Gastinel 196?):
Let $\mathbf{A} \in \mathbb{C}^{n\times n}$:

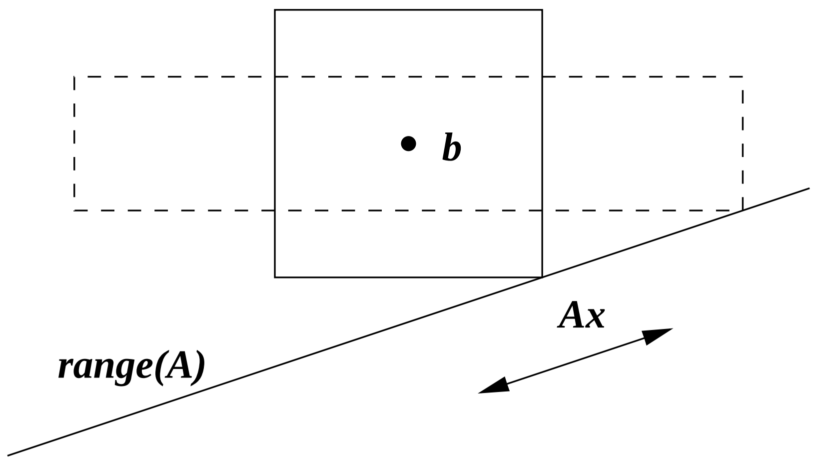
$$\delta_{\mathbf{A}} = \min_{\tilde{\mathbf{A}}:\ \det(\tilde{\mathbf{A}})=0} \|\mathbf{A} - \tilde{\mathbf{A}}\| = \frac{1}{\|\mathbf{A}^{-1}\|}$$

where $\|\cdot\| = \|\cdot\|_{p,p}$ is an **induced matrix norm**.

Example: $\|\mathbf{B}\|_{\infty,\infty} = \max_i \sum_j |b_{i,j}|, \quad \|\mathbf{B}\|_{1,1} = \max_j \sum_i |b_{i,j}|$

Tchebycheff's nearest consistency: $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m$:

$$\min_{\hat{\mathbf{x}}} \|\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}\|_\infty = \min_{\hat{\mathbf{x}}} \left(\max_{1\le i\le m} \left| b_i - \sum_{j=1}^{n} a_{i,j}\hat{x}_j \right| \right)$$



Solution by linear programmming:

minimize: δ

linear constraints: $\delta \geq \quad b_i - \sum_{j=1}^{n} a_{i,j}\hat{x}_j \;\; (1 \le i \le m)$

$\delta \geq -b_i + \sum_{j=1}^{n} a_{i,j}\hat{x}_j \;\; (1 \le i \le m)$

Backward error-analysis: Oettli & Prager (1964)

Given: $\mathbf{A}, \mathbf{b}$, an error matrix $\mathbf{E}$, an error vector δ,
$\tilde{\mathbf{x}}$, which is the approx. solution to $\mathbf{Ax} = \mathbf{b}$.

$$\left.\begin{array}{l} \exists \tilde{\mathbf{A}} \text{ with } |\tilde{\mathbf{A}} - \mathbf{A}| \leq_{\text{entry-wise}} \mathbf{E} \\ \exists \tilde{\mathbf{b}} \text{ with } |\tilde{\mathbf{b}} - \mathbf{b}| \leq_{\text{entry-wise}} \delta \end{array}\right\} \quad \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}} \iff |\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}| \leq_{\text{entry-wise}} \mathbf{E}|\tilde{\mathbf{x}}| + \delta.$$

Ill-conditioned example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{E} = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \end{bmatrix}, \delta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tilde{\mathbf{x}} = \begin{bmatrix} t+1 \\ -t \end{bmatrix}$$

$$|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}| = \begin{bmatrix} 0 \\ \varepsilon|t| \end{bmatrix} \leq \begin{bmatrix} 0 \\ \varepsilon|t| \end{bmatrix} = \mathbf{E}\tilde{\mathbf{x}} + \delta,$$

Any t yields an admissible solution.

Gastinel's nearest singular matrix estimate:

$$\left\| \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1-\varepsilon \end{bmatrix}}_{\mathbf{A}}^{-1} \right\|_{\infty,\infty} = \left\| -\frac{1}{\varepsilon} \begin{bmatrix} 1-\varepsilon & -1 \\ -1 & 1 \end{bmatrix} \right\|_{\infty,\infty} = \frac{2}{\varepsilon},$$

$$\mathbf{A} + \begin{bmatrix} 0 & -\varepsilon/2 \\ 0 & \varepsilon/2 \end{bmatrix} \text{ is singular } (\varepsilon > 0).$$

Sensitivity analysis: component-wise nearest singular matrix

Given are $2n^2$ rational numbers $\underline{a}_{i,j}, \overline{a}_{i,j}$.
Let $\mathcal{A}$ be the *interval* matrix

$$\mathcal{A} = \left\{ \begin{bmatrix} a_{1,1} & \dots & a_{n,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \mid \underline{a}_{i,j} \leq a_{i,j} \leq \overline{a}_{i,j} \text{ for all } 1 \leq i,j \leq n \right\}.$$

Does $\mathcal{A}$ contain a singular matrix?
This problem is *NP-complete* (Poljak & Rohn 1990).

Nearest approximate GCD in the Euclidean norm
(Karmarkar and Lakshman ISSAC'96)

Let $f, g \in \mathbb{C}[z]$, both monic, $\deg(f) = m$ and $\deg(g) = n$.
Assuming that $\mathrm{GCD}(f, g) = 1$,
find $\tilde{f}, \tilde{g} \in \mathbb{C}[z]$ monic of degrees m and n, such that

$\mathrm{GCD}(\tilde{f}, \tilde{g})$ is non-trivial and
$\mathcal{N} = \|f - \tilde{f}\|^2 + \|g - \tilde{g}\|^2$ is minimized.

$\|f\|$ denotes a norm of the coefficient vector of f.

Equivalent formulation:
Compute the nearest singular *Sylvester matrix* to the Sylvester matrix

$$
\begin{bmatrix}
a_m & a_{m-1} & \dots\dots & a_0 & & & \\
 & a_m & \dots\dots & a_1 & a_0 & & 0 \\
 & & \ddots & & \ddots & \ddots & \\
 & 0 & & a_m & \dots\dots\dots & & a_0 \\
b_n & b_{n-1} & \dots\dots & b_0 & & & \\
 & b_n & \dots\dots & b_1 & b_0 & & 0 \\
 & & \ddots & & \ddots & \ddots & \\
 & 0 & & b_n & \dots\dots\dots & & b_0
\end{bmatrix}
$$

The *symbolic* minimum of $\mathcal{N}$ with respect to a common root $\alpha \in \mathbb{C}$ can be obtained in closed-form:

$$\mathcal{N}_{min} = \frac{\overline{f(\alpha)}\,f(\alpha)}{\sum_{k=0}^{m-1}(\overline{\alpha}\alpha)^k} + \frac{\overline{g(\alpha)}\,g(\alpha)}{\sum_{k=0}^{n-1}(\overline{\alpha}\alpha)^k}$$

The individual perturbations of the coefficients of f and g are

$$f_i - \tilde{f}_i = \frac{(\overline{\alpha})^i f(\alpha)}{\sum_{k=0}^{m-1}(\overline{\alpha}\alpha)^k} \quad \text{and} \quad g_j - \tilde{g}_j = \frac{(\overline{\alpha})^j g(\alpha)}{\sum_{k=0}^{n-1}(\overline{\alpha}\alpha)^k}$$

($\overline{\alpha}$ is the complex conjugate).

Reduced Problem: Given $f \in \mathbb{C}[z]$ and $\alpha \in \mathbb{C}$.
Find $\tilde{f} \in \mathbb{C}[z]$, such that

$$\tilde{f}(\alpha) = 0, \quad \text{and} \quad \|f - \tilde{f}\| = \min .$$

$$\begin{aligned} \text{Let } f(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\ \tilde{f}(z) &= (z - \alpha) \sum_{k=0}^{n-1} u_k z^k \\ &= u_{n-1} z^n + (u_{n-2} - \alpha) z^{n-1} + (u_{n-3} - \alpha u_{n-2}) z^{n-2} + \\ &\quad \cdots \quad + (u_0 - \alpha u_1) z - \alpha u_0 \end{aligned}$$

In terms of linear algebra:

$$\|f - \tilde{f}\| = \min_{\mathbf{u} \in \mathbb{C}^n} \left\| \underbrace{\begin{bmatrix} -\alpha & & & 0 \\ 1 & -\alpha & & \\ & \ddots & \ddots & \\ 0 & & 1 & -\alpha \\ & & & 1 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}}_{\mathbf{u}} - \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}}_{\mathbf{b}} \right\| \quad (1)$$

(1) is an over-determined linear system of equations

LinProgr problem, if $\|\cdot\|$ is the $\begin{cases} l^\infty & \text{norm, or} \\ l^1 & \text{norm} \end{cases}$

LeastSqu problem, if $\|\cdot\|$ is the l^2 (Euclidean) norm.

Solutions for the l^2-norm in closed form:

$$\mathcal{N}_{min}(\alpha) = \|f - \tilde{f}\|^2 = \frac{\overline{f(\alpha)}\, f(\alpha)}{\sum_{k=0}^{n} (\overline{\alpha}\alpha)^k}, \qquad f_j - \tilde{f}_j = \frac{(\overline{\alpha})^j f(\alpha)}{\sum_{k=0}^{n} (\overline{\alpha}\alpha)^k}$$

(also derived in Corless et al. [ISSAC'95] via SVD)

An l^∞ example: x^2+1

$$\min_{\substack{\tilde{a}_2,\tilde{a}_1,\tilde{a}_0\in\mathbb{R}\text{ such that}\\ \exists\alpha\in\mathbb{R}\colon\ \tilde{a}_2\alpha^2+\tilde{a}_1\alpha+\tilde{a}_0=0}} \Big(\max\{|1-\tilde{a}_2|,|0-\tilde{a}_1|,|1-\tilde{a}_0|\Big)$$

$$=?$$

Sensitivity analysis: Kharitonov [1978] theorem

Given are $2n$ rational numbers $\underline{a}_i, \overline{a}_i$.
Let P be the *interval* polynomial

$$P = \{x^n + a_{n-1}x^{n-1} + \cdots + a_0 \mid \underline{a}_i \leq a_i \leq \overline{a}_i \text{ for all } 0 \leq i < n\}.$$

Then every polynomial in P is *Hurwitz* (all roots have negative real parts), if and only if the four "corner" polynomials

$$g_k(x) + h_l(x) \in P, \quad \text{where } k = 1,2 \text{ and } l = 1,2,$$

with

$$g_1(x) = \underline{a}_0 + \overline{a}_2 x^2 + \underline{a}_4 x^4 + \cdots, \quad h_1(x) = \underline{a}_1 + \overline{a}_3 x^3 + \underline{a}_5 x^5 + \cdots,$$
$$g_2(x) = \overline{a}_0 + \underline{a}_2 x^2 + \overline{a}_4 x^4 + \cdots, \quad h_2(x) = \overline{a}_1 + \underline{a}_3 x^3 + \overline{a}_5 x^5 + \cdots$$

are Hurwitz.

Constraining a Root Locus to a Curve

Let Γ be a piecewise smooth curve with finitely many segments, each having a parametrization $\gamma_k(t)$ in a single real parameter t.

For a given polynomial $f \in \mathbb{C}[z]$, we want to find a minimally perturbed polynomial $\tilde{f} \in \mathbb{C}[z]$ that has (at least) one root on Γ.

Parametric Minimization

We substitute the parametrization $\gamma_k(t)$ for the indeterminate α in $\mathcal{N}_{min}(\alpha)$. The resulting expression is a function in $t \in \mathbb{R}$.

It attains its minima at its *stationary* points. We have to compute the *real* roots of the derivative.

The derivative of the norm-expression is determined *symbolically*, the roots can be computed numerically.

Algorithm C

Input: $f \in \mathbb{C}[z]$, and a curve Γ.

Output: $\tilde{f} \in \mathbb{C}[z]$, and $\tau \in \mathbb{R}$, s.t. $\tilde{f}(\gamma_k(\tau)) = 0$ for some segment of Γ, and $\|f - \tilde{f}\|_2 = \min$.

(**C**$_1$) For each segement of Γ:

- (**C**$_{1.1}$) Substitute $\gamma_k(t)$ for α in the symbolic minimum $\mathcal{N}_{min}(\alpha) \mapsto N(t)$.
- (**C**$_{1.2}$) Symbolically determine the derivative $N'(t)$.
- (**C**$_{1.3}$) Compute the *real* roots (of the numerator) of $N'(t)$. Select the one that minimizes $N(t)$.

(**C**$_2$) From all $N(\tau_k)$ of step (**C**$_{1.3}$) determine the minimum $N(\tau)$.

(**C**$_3$) Compute the perturbations δ_j. Return $\tilde{f}$, k, and τ.

Computing the Radius of Stability in the l^2-Norm

Definition: Let $\mathcal{D} \subset \mathbb{C}$ be an open, and convex domain of the complex plane. The polynomial $f \in \mathbb{C}[z]$ is called $\mathcal{D}$-stable, if all its roots are located within $\mathcal{D}$.

Special cases: – the left half-plane: *Hurwitz* stability
– the open unit-disc: *Schur* stability

Given a $\mathcal{D}$-stable polynomial f, how much can we perturb its coefficients such that the perturbed polynomial is still $\mathcal{D}$-stable?

If we have a (piecewise) real parametrization of the boundary $\partial\mathcal{D}$ then we can apply our algorithm to find a *nearest unstable* polynomial.

Theorem: Let $f \in \mathbb{C}[z]$ be $\mathcal{D}$-stable, and

let $\hat{f} \in \mathbb{C}[z]$ be an unstable polynomial,
such that $\|f - \hat{f}\| = \varepsilon$, where $\varepsilon \in \mathbb{R}, \varepsilon > 0$.

Then, there exist $\tilde{f} \in \mathbb{C}[z]$ and $\zeta \in \partial\mathcal{D}$ such that

$\|f - \tilde{f}\| \leq \varepsilon$ and $\tilde{f}(\zeta) = 0$.

Example (of a monic polynomial)

$$\begin{aligned} f(z) \; &= \; z^3 + (2.41 - 3.50\,\mathbf{i})z^2 + (2.76 - 5.84\,\mathbf{i})z \\ &\quad -1.02 - 9.25\,\mathbf{i} \end{aligned}$$

is Hurwitz.

Root locations: $-1.04 + 3.10\,\mathbf{i}$, $-.99 - 1.30\,\mathbf{i}$, $-.37 + 1.70\,\mathbf{i}$

Nearest unstable polynomial:

$$\begin{aligned} \tilde{f}(z) \; &= \; z^3 + (2.7037 - 3.1492\,\mathbf{i})z^2 + (2.5740 - 5.6842\,\mathbf{i})z \\ &\quad -1.1026 - 9.3486\,\mathbf{i}\,. \end{aligned}$$

Radius of stability in the l^2-norm: 0.533567.

Special case to Tchebycheff approx: Stiefel's 1959 theorem
Let $\mathbf{A}$ be a matrix

$$\mathbf{A} = \begin{bmatrix} a_{0,0} & \cdots & a_{0,n-1} \\ \vdots & & \vdots \\ a_{n,0} & \cdots & a_{n,n-1} \end{bmatrix} \in \mathbb{R}^{(n+1)\times n}$$

of rank n such that no row of $\mathbf{A}$ is the zero vector, and
let $\mathbf{b} = [b_0, \ldots, b_n] \in \mathbb{R}^{n+1}$ such that $\mathbf{Ax} \neq \mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then

$$\delta = \min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{Ax} - \mathbf{b}\|_\infty = \left| \frac{\sum_{i=0}^{n} \lambda_i b_i}{\sum_{i=0}^{n} |\lambda_i|} \right|,$$

where $\Lambda = [\lambda_0, \ldots, \lambda_n]^{tr} \neq 0$ is a linear dependency among the rows of $\mathbf{A}$, i.e., $\Lambda^{tr}\mathbf{A} = 0$.

Special case: nearest polynomial with root α:

$$\delta(\alpha) = \min_{\mathbf{u}\in\mathbb{R}^n} \|\mathbf{P}\mathbf{u} - \mathbf{b}\|_\infty = \left|\frac{\sum_{i=0}^n \lambda_i a_i}{\sum_{i=0}^n |\lambda_i|}\right| = \left|\frac{f(\alpha)}{\sum_{i=0}^n |\alpha^i|}\right|. \qquad (2)$$

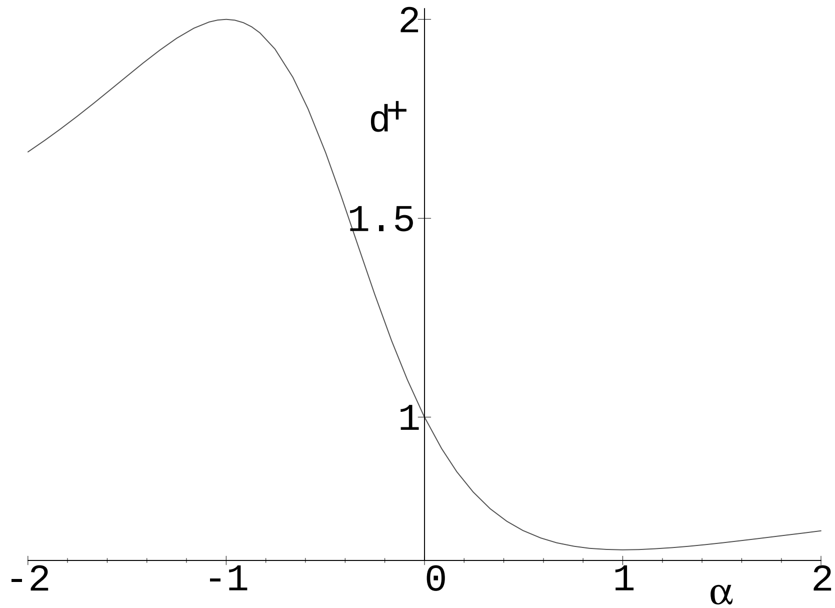
(also derived by Manocha & Demmel [1995])

Stiefel's theorem also gives algorithm for finding $\mathbf{u}$.

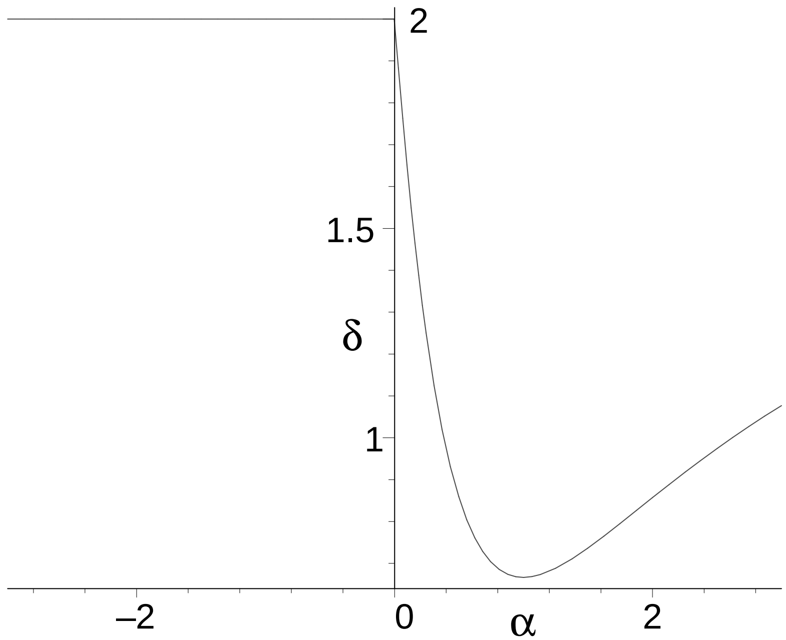
Parametric α: must minimize rational function (2).

Generalization to l^p-norm, where $1 \le p \le \infty$ (Hitz 1999):

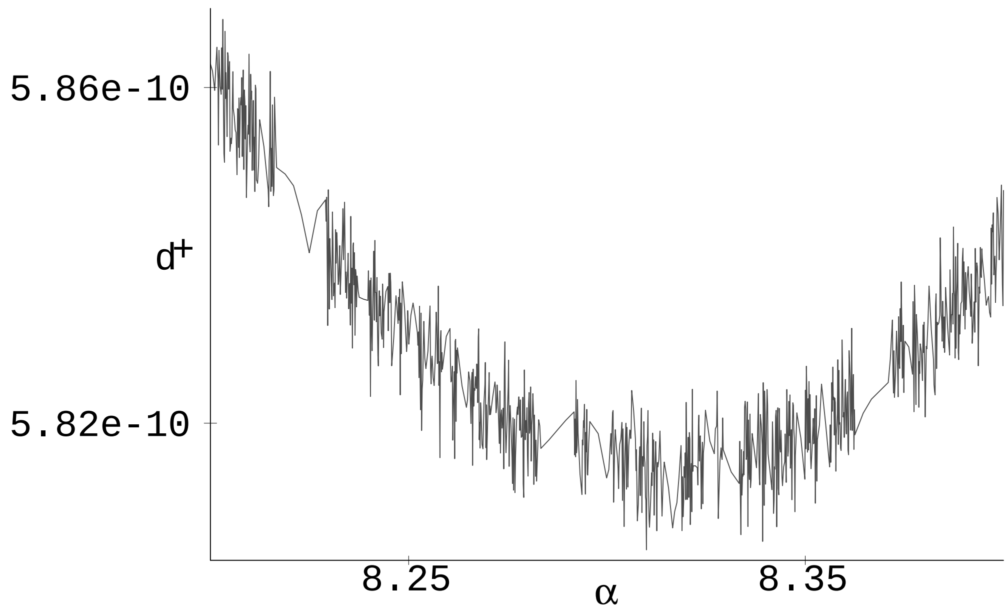
$$\delta(\alpha) = \frac{|f(\alpha)|}{\left(\sum_{k=0}^n |\alpha^k|^q\right)^{1/q}}, \quad \frac{1}{q} + \frac{1}{p} = 1, \quad \text{and} \quad \frac{1}{\infty} = 0$$



$$f(x) = x^2 + 1,\ \tilde{f}(x) = \frac{1}{3}x^2 - \frac{2}{3}x + \frac{1}{3} = \frac{1}{3}(x-1)^2,\ \delta = \frac{2}{3}.$$



$$f(x) = 2x^2 - 2x + 2,\ \tilde{f}(x) = \frac{4}{3}(x^2 - 2x + 1),\ \delta = \frac{2}{3}.$$



$$f(x) = \prod_{k=1}^{10}(x-k-\mathbf{i})(x-k+\mathbf{i}),\ \delta \leq 5.8210^{-10}.$$

Nearest matrix with a given eigenvalue (Eckart and Young 1936):
Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\mu \in \mathbb{C}$:

$$\delta_{\mathbf{A}}(\mu) = \min_{\tilde{\mathbf{A}}:\ \mu \text{ is an eigenvalue of } \tilde{\mathbf{A}}} \|\mathbf{A} - \tilde{\mathbf{A}}\| = \frac{1}{\|(\mu\mathbf{I} - \mathbf{A})^{-1}\|}$$

where $\|\cdot\| = \|\cdot\|_{p,p}$ is an **induced matrix norm**.

For

$$\|\mathbf{B}\|_{\infty,\infty} = \max_i \sum_j |b_{i,j}|, \quad \|\mathbf{B}\|_{1,1} = \max_j \sum_i |b_{i,j}|,$$

we can solve optimization problem for **real** entries and parameter μ in **polynomial-time**.

Homework: Given $f = \sum f_{i,j} x^i y^j \in \mathbb{C}[x,y]$ **absolute irreducible**, find $\tilde{f} = (c_0 + c_1 x + c_2 y) u(x,y) \in \mathbb{C}[x,y]$, $\deg(\tilde{f}) \leq \deg(f)$, such that

$$\|f - \tilde{f}\|_2 \text{ is minimal}$$

("nearest polynomial with a linear factor").

Hint: minimize parametric least square solution in the real and imaginary parts of the c_i.