North Carolina State University is a land-grant university and a constituent institution of The University of North Carolina

Department of Mathematics

NC **STATE** UNIVERSITY

Your Name: ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 4 problems, which are subdivided into 14 questions, where each question counts for the explicitly given number of points, adding to a total of **48 points**. Please write your answers in the spaces indicated, or below the questions, using the **back of the sheets** for completing the answers and **for all scratch work,** if necessary. You are allowed to consult **one** 8.5in × 11in sheet with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **75 minutes** to do this test.

Good luck!

Problem 1 ⎯⎯⎯

2 ⎯⎯⎯

3 ⎯⎯⎯

4 ⎯⎯⎯

Total ⎯⎯⎯

Problem 1 (10 points)

(a, 7pts) Please consider the linearly recurrent sequence $g_{n+2} = g_{n+1} + 2g_n$ for all $n \geq 0$ with $g_0 = 1$ and $g_1 = 1$. Let $h_m = g_{-m}$ for all $m \geq -1$, that is, running the linear recurrence backwards to negative subscripts, $h_{-1} = g_1 = 1, h_0 = g_0 = 1, h_1 = g_{-1}, h_2 = g_{-2}, \ldots$ such that $g_{-n+2} = g_{-n+1} + 2g_{-n}$ remains valid for $n \geq 1$.

    i. Please derive the linear recurrence $h_{m+2} = a\,h_{m+1} + b\,h_m$ for all $m \geq -1$, where a, b are rational numbers; please also compute h_1, h_2, h_3, h_4.

    ii. Please compute a closed form solution in the form $h_m = c_1 q_1^m + c_2 q_2^m$ with $c_1, c_2, q_1, q_2 \in \mathbb{Q}$.

(b, 3pts) How many arcs does the 6-dimensional de Bruijn digraph have? Please explain.

Problem 2 (12 points): Please consider the following digraph:

$$D = (\{1,2,3,4,5,6\}, \{(1,3),(1,5),(2,2),(2,5),(3,6),(4,2),(5,4),(5,5),(6,1),(6,2),(6,4),(6,6)\})$$

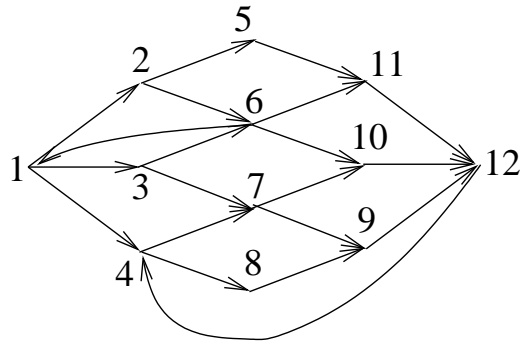(a, 3pts) Please draw a picture of D.

(b, 3pts) Please write down the adjacency matrix M for D under the vertex order $(1,2,3,4,5,6)$.

(c, 3pts) Please write down M^2.

(d, 3pts) Please write down the reachability matrix R for D under the vertex order $(1,2,3,4,5,6)$.

Problem 3 (12 points):
Consider the following
digraph:

(a, 4pts) Please list the vertex sets of the strong components of the above digraph.

(b, 4pts) Please draw the digraph that is the condensation of the above digraph.

(c, 4pts) Please give the vertex basis for the condensation digraph (b) and all vertex bases for the above original digraph.

(d, 2pts) Please indicate a single pair of vertices such that if an arc between them is added the digraph becomes strongly connected.

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Problem 4 (14 points):
Consider $K_{4,5}$ with its
vertices labeled as follows:
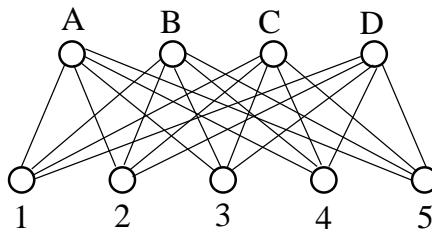
(a, 8pts) Please draw a subgraph that is homeomorphic to K_5.

(b, 3pts) What is the diameter and the chromatic number of $K_{4,5}$ drawn above? Please explain.

(c, 3pts) Consider the graph $K_5^- = (\{1,2,3,4,5\}, \{\{i,j\} \mid 1 \le i < j \le 5, \{i,j\} \ne \{1,2\}\})$, that is, K_5 with the edge between vertex 1 and vertex 2 removed. Please show that K_5^- is planar.