Problem 1 (11 points)

(a, 7pts) Let \( f_n \) be the Fibonacci numbers: \( f_0 = 1, f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, \ldots \) and let \( g_n = (-1)^n f_n: g_0 = 1, g_1 = -1, g_2 = 2, g_3 = -3, g_4 = 5, g_5 = -8, \ldots \)

i. Please give an order 2 linear recurrence with constant coefficients for the sequence \( g_n \).

\[
\begin{align*}
 f_{n+2} &= f_{n+1} + f_n \\
 (-1)^{n+2} f_{n+2} &= -(-1)^{n+1} f_{n+1} + (-1)^n f_n \\
 g_{n+2} &= -g_{n+1} + g_n
\end{align*}
\]

ii. Please compute a closed form solution of the form \( g_n = a\alpha^n + b\beta^n \) with \( a, b, \alpha, \beta \in \mathbb{R} \).

\[
(-1)^n f_n = (-1) \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right)
\]

\[
\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]

(b, 4pts) The path on the \( 8 \times 8 \) grid corresponds to a parentheses expression with 7 pairs of balanced parentheses, as described in class. Please draw the binary tree that corresponds to the expression.
**Problem 2** (12 points): Please consider the following digraph:

\[ D = (\{1, 2, 3, 4, 5\}, \{(1, 2), (1, 3), (1, 5), (2, 2), (2, 4), (3, 2), (3, 3), (4, 5), (5, 2), (5, 3), (5, 5)\}) \]

(a, 3pts) Please draw a picture of \( D \).

(b, 3pts) Please write down the adjacency matrix \( M \) for \( D \) under the vertex order \((1, 2, 3, 4, 5)\).

\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
\end{bmatrix}
\]

(c, 3pts) Please write down \( M^2 \).

\[
\begin{bmatrix}
0 & 3 & 2 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 2 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 3 & 2 & 1 & 1 \\
\end{bmatrix}
\]

(d, 3pts) Please write down the reachability matrix \( R \) for \( D \) under the vertex order \((1, 2, 3, 4, 5)\).

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
Problem 3 (12 points):
Consider the following digraph:

(a, 4pts) Please list the vertex sets of the strong components of the above digraph.

\[ V_1 = \{1, 6, 7, 8\} \]
\[ V_2 = \{2\} \]
\[ V_3 = \{3, 9, 10\} \]
\[ V_4 = \{4\}, \quad V_5 = \{5, 11\} \]

(b, 4pts) Please draw the digraph that is the condensation of the above digraph.

(c, 2pts) Please give the vertex basis for the condensation digraph (b) and all vertex bases for the above original digraph.

\[ B^* = \{3\} \]
\[ B_1 = \{3\}, \quad B_2 = \{9\}, \quad B_3 = \{10\} \]

(d, 2pts) Please indicated two arcs in the above digraph with the property that if one changes those arcs’ directions the new digraph becomes connected.

\[ (2, 3) \text{ and } (11, 10) \]
\[ \text{or } (2, 9) \]

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Problem 4 (12 points): Please consider the following digraph, which is a subgraph of the 4-dimensional deBruijn digraph with arcs \((i_1i_2i_3i_4, 0i_1i_2i_3)\), \((i_1i_2i_3i_4, 1i_1i_2i_3)\) for \(i_k \in \{0, 1\}\) with \(1 \leq k \leq 4\) but with the arc \((1010, 0101)\) removed.

(a, 4pts) Please explain why the above digraph with the missing \((1010, 0101)\) arc remains connected.

There is a path \(1010, 1101, 1110, 1011, 0101\) that can be substituted for the missing arc, with which the digraph becomes connected.

(b, 4pts) Please give two vertices and a shortest path from the first vertex to the second whose length is the diameter of the above digraph.

\[
\text{diam} = 5; (0100, 1010, 1101, 0110, 1011, 0101)
\]

(c, 4pts) Please draw a subgraph of the digraph shown above which is a perfect binary tree with 15 vertices where the arcs point from child to parent.

Hint: a vertex with a single incoming arc from another vertex can only be a leaf.