We provide a general design framework for dynamic mechanisms under complex environments, coined \textit{Lossless History Compression mechanisms}. Lossless history compression mechanisms compress the history into a \textit{state} carrying the least historical information without losing any generality in terms of either revenue or welfare. In particular, the characterization works for almost arbitrary constraints on the outcomes, and any objective function defined on the historical reports, allocations, and the cumulative payments. We then apply our framework to design a non-clairvoyant dynamic mechanism under budget and ex-post individual rationality constraints that is dynamic incentive-compatible and achieves non-trivial revenue performance, even without any knowledge about the future. In particular, our dynamic mechanism obtains a constant approximation to the optimal dynamic mechanism having access to all information in advance. To the best of our knowledge, this is the first dynamic mechanism that achieves a constant approximation and strictly respects dynamic incentive-compatibility and budget constraints without relying on any forecasts of the future.

1 INTRODUCTION

As a fundamental problem in market design, dynamic mechanism design has been extensively studied in the community of economics and computation over the past decades [3, 7, 12, 13, 15, 21, 22, 29, 31]. This is partly inspired by the popularity of selling ads on online platforms via auctions, an industry totalling hundreds of billions of dollars annually. Compared to the classic static auctions, dynamic auctions open up the possibility of evolving the auctions across time to boost the revenue or welfare. Recent works have been focusing on the dynamic environment with repeated sales of heterogeneous items to one or more agents [1, 2, 5, 6, 10, 14, 23, 25–27, 30, 33]. Despite of the elegant results therein, the environments in practice could be much more complicated than the theoretical models with quasi-linear utility agents and ex-post individual rationality (ex-post IR) constraints only. Taking the online ad auction industry as an illustrative example, there could be budget constraints or max cost-per-conversion constraints. In addition, the agents’ bidding can be algorithm-defined bidding-proxies optimizing for advertiser specified CPA (\textit{Cost Per Acquisition}) or ROI (\textit{Return on Investment}) targets [11, 19]. Thus, a nature question arises:

\textit{Can we design a dynamic mechanism in a more general environment? and under what conditions the optimal mechanisms are still interpretable and preserve simple structures?}

In this paper, we study dynamic mechanism design in general environments with complicated constraints. We consider an environment where there is a fixed set of agents throughout the time horizon and the items arrive in an online manner. For simplicity, we assume that for each stage, there is one newly arrived item and the item must be sold once it arrives.

We focus on the design of \textit{dynamic incentive-compatible} (DIC) and \textit{ex-post individually rational} (ex-post IR) mechanisms. Dynamic incentive compatibility means that it is of the agent’s best interest to always report truthfully, while ex-post individual rationality requires that truthful reporting guarantees the agent non-negative cumulative utility on the ex-post level. We adopt the novel concept of \textit{non-clairvoyance} introduced by Mirrokni et al. [26]. A dynamic mechanism is \textit{non-clairvoyant}, if no prior knowledge about the future is available. Otherwise, it is \textit{clairvoyant}, if all priors are given since the start. As the seller in practice can hardly be clairvoyant, we are interested in designing non-clairvoyant dynamic mechanisms that approximates the best clairvoyant ones.
1.1 A General Dynamic Mechanism Design Framework

We begin with providing a general framework for designing clairvoyant and non-clairvoyant dynamic mechanisms in general environments. The framework helps the design of approximately optimal non-clairvoyant dynamic mechanisms under budget constraints from two aspects: i) the framework reduces without loss of generality the design space to a class of mechanisms with simple structures that we call generalized bank account mechanisms; ii) the framework provides an upper bound of the revenue from any clairvoyant dynamic mechanisms. Independent of being a tool for mechanism design and analysis, the framework also provide further insights into the implementability of non-clairvoyant dynamic mechanisms.

General Dynamic Environments. To highlight the generality of the environments that our framework can accommodate, we emphasize several key features rarely captured by existing works all at once. We allow almost arbitrary constraints on the mechanism outcomes (see Section 2.2.2). This enables us to model various supply constraints, ex-ante or ex-post individual rationality constraints, monetary transfer limits, barter exchange markets, fairness constraints, double auctions or general markets, etc. Moreover, we allow the agents having arbitrary valuation functions in each stage, and an arbitrary designer objective defined on the final allocation and total payment (see Section 2.2.3), which enables the model to capture production and social costs, revenue- or welfare-maximizing goals, etc. Our framework can be further generalized to an environment with public valuation correlation (see Appendix A).

Lossless History Compression (LHC) Mechanisms. We formalize a design framework, called Lossless History Compression (LHC) mechanisms, that works for the general dynamic environment (see Definition 3.3). In general, a dynamic mechanism at a stage may depend on the agents’ historical reports. Instead of tracking the reports, an LHC mechanism utilizes its states to carry the information that is sufficient to determine the current margins for the constraints in each stage as well as the agents’ promised expected utilities in the future. LHC mechanisms generalize the idea of the bank account mechanisms (proposed by Mirrokni et al. [24, 26] exclusively for ex-post IR). Despite of its limited structure, LHC mechanisms are rich enough to capture optimal dynamic mechanisms in general environments. We prove by construction that any direct dynamic mechanisms can be converted to an LHC mechanism without any loss on its objective.

The main challenge behind our design is to understand which information is required to carry across stages in order to preserve the optimality of the mechanism and respect all the constraints at the same time. In an environment with budget constraints only, we show that it suffices for an LHC mechanism to maintain the agents’ cumulative utilities, remaining budgets, and the agents’ promised utility in the future. In particular, the cumulative utilities keep track on the ex-post IR constraints and the remaining budgets keep track on the budget constraints. In general environments, our reduction from optimal clairvoyant mechanisms to LHC mechanisms demonstrates that once the information is enough to keep track on all the constraints, then it is immediately sufficient for achieving the optimality of the mechanism and maintaining dynamic incentive compatibility with additional records on agents’ promised future utilities (Theorem 3.5).

LHC Mechanisms and Non-clairvoyant Mechanisms. However, maintaining a record about the future, the agents’ promised future utilities, limits the application of LHC mechanisms in non-clairvoyant environments, in which the future is unpredictable. To overcome this obstacle, we aim to eliminate the dependence about future information. We prove that a non-clairvoyant mechanism satisfies dynamic incentive compatibility if and only if it is (i) stage-wise incentive compatible (STAGE-IC), and (ii) guarantees a constant expected utility in each stage for every agent independent
of the state (State-UI) (see Theorem 3.8). For Stage-IC and State-UI LHC mechanisms, additional records on agents’ promised future utilities, which are constant, become redundant.

Can we always convert an LHC mechanism to a Stage-IC and State-UI mechanism? With a mild assumption (Assumption 3.6), we show that such an conversion is possible if and only if an operation called Payment Realignment can be done (Theorem 3.9). Intuitively, Payment Realignment keeps the allocation rules the same and rearranges the payment across stages carefully. We further provide sufficient conditions under which Payment Realignment is possible (Lemma 3.11). The conditions are able to capture many practical constraints, such as ex-post IR constraints, budget constraints, and ROI constraints. Figure 1 summarizes our characterizations of LHC mechanisms.

![Fig. 1. An illustration of our characterizations of LHC mechanisms](image)

1.2 Non-clairvoyant Mechanism Design with Budget Constraints

We then apply the framework on designing non-clairvoyant mechanism with public budget constraints. Our LHC mechanism enables us to consider without loss of generality a class of dynamic mechanisms, called Generalized Bank Account Mechanisms (GBAM). A GBAM utilizes two bank accounts to maintain two balances: the cumulative utilities and the remaining budgets.

Preserving the non-clairvoyance with budget constraints introduces new challenges. Without budget, repeatedly offering a single-stage mechanism that is IC and ex-post IR per stage respects DIC, e.g., the repeated second price auction. However, in presence of budget constraints, even the repeated second price auction is not DIC. Extensive studies in budget management strategy suggest that bid-shading is a better strategy for the agent than truthful reporting in this case [4, 6, 16, 17].

The second and more challenging obstacle related to revenue performance is that even ignoring the dynamic components, repeated offering Myerson’s auction [28] is no longer an optimal strategy for every stage in isolation: because it is clearly better to exclude the buyers whose budgets have been exhausted. However, such an exclusion might encourage bid-shading. Thus, naïve adoptions of the non-clairvoyant mechanisms designed for the environments without budget constraints would break the incentive property or result in a revenue far from the optimum.

Under an assumption that holds generally in online advertising markets (Assumption 2.5), we manage to circumvent all these difficulties and develop a non-clairvoyant dynamic mechanism achieving at least a constant fraction of the optimal revenue from any dynamic mechanisms with budget constraints, including those having access to future information (Theorem 5.4). Achieving a constant approximation independent of the number of agents introduces further challenges into both the construction and analysis. In Section 5, we discuss the dilemmas with further details.

1.3 Warm-up Example

To see how an additional constraint such as the budget constraint may affect the previously known results in dynamic environments, consider the following commonly used example [22, 26, 30].

---

1We consider hard budget constraints so that the agent’s utility is negative infinity once the budget constraint is violated.
Basic Setting. A seller sells two items to one buyer in two periods. The buyer has additive valuation and quasi-linear utility. The buyer’s private values \( v_1, v_2 \sim F \) are drawn independently at the beginning of each stage, respectively. \( F \) is an equal-revenue distribution with largest value \( \bar{v} \):

\[
\Pr[v_1 \leq v] = \Pr[v_2 \leq v] = F(v) = \begin{cases} 0, & v \in [0, 1] \\ 1 - 1/v, & v \in (1, \bar{v}) \\ 1, & v \geq \bar{v} \end{cases}.
\]

The revenue optimal static mechanism, with two items sold independently, yields revenue 2 by setting a price 1 for each. However, the following revenue optimal dynamic mechanism yields expected revenue \( 2 + \ln \ln \bar{v} \), while incentivizing the buyer to always participate and report truthfully:

1. In the first stage, the buyer reports her private type as \( \hat{v}_1 \). If \( \hat{v}_1 \geq 1 \), the item is sold at price \( r_1 = \min\{\hat{v}_1, 1 + \ln \bar{v}\} \); otherwise, the item is not sold and we set \( r_1 = 1 \).

2. In the second stage, the buyer reports her private type as \( \hat{v}_2 \) and the seller sells the item via a posted-price auction with price \( r_2 = \bar{v}/e^{r_1-1} \).

The dynamic mechanism above is ex-post individually rational, because in neither stage the payment is larger than the reported value. Truthful-reporting is the best strategy, because: i) for given \( r_1 \), the auction in the second stage is a posted-price auction; ii) given best-response (truthful reporting) in the second stage, the expected utility at the second stage for any \( 1 \leq r_2 \leq \bar{v} \) is

\[
\mathbb{E}_{v_2 \sim F}[(v_2 - r_2)^+] = \int_{r_2}^{\bar{v}} (v_2 - r_2) dF(v_2) + \Pr[v_2 = \bar{v}] \cdot (\bar{v} - r_2) = \ln \bar{v} - \ln r_2.
\]

Therefore, the cumulative expected utility of reporting \( \hat{v}_1 \geq 1 \) is

\[
1\{\hat{v}_1 \geq 1\} \cdot (v_1 - r_1) + \mathbb{E}_{v_2 \sim F}[(v_2 - r_2)^+] = v_1 - r_1 + \ln \bar{v} - \ln(\bar{v}/e^{r_1-1}) = v_1 - 1,
\]

where \((x)^+ = \max(x, 0)\), while the expected utility of reporting \( \hat{v}_1 < 1 \) is zero. The expected revenue is \( \mathbb{E}[1\{v_1 \geq 1\} \cdot r_1 + 1\{v_2 \geq r_2\} \cdot r_2] = 2 + \ln \ln \bar{v} \gg 2 \).

Bank Account Mechanism. The above can be reformulated as a bank account mechanism [26]:

1. In the first stage, the buyer reports her value as \( \hat{v}_1 \) and the item is sold at a posted-price 1.
2. Let the bank account balance be \( \alpha = \min\{\ln \bar{v}, (\hat{v}_1 - 1)^+\} \).
3. In the second stage, before \( v_2 \) is drawn, the buyer pays \( \alpha \); then the buyer realizes \( v_2 \) and reports \( \hat{v}_2 \); finally, the item is sold at a posted-price \( r_2 = \bar{v}/e^{\alpha} \).

The only difference between the two mechanisms is that we postpone a payment \( \alpha = r_1 - 1 \) from the first stage to the beginning of the second stage in the bank account mechanism. In fact, the expected utility from the second stage is always zero: \( \mathbb{E}[(v_2 - r_2)^+] - \alpha = \ln \bar{v} - \ln(\bar{v}/e^{\alpha}) - \alpha \equiv 0 \). Such a property is later formalized as BI or STATE-UI. Moreover, the buyer is incentivized to report truthfully in the posted-price auction of the first stage even if the second stage may not exist.

With Budget Setting. Consider the case in which the buyer is subject to a budget constraint \( B \in (2, \bar{v}) \). Then the buyer cannot afford the second item when \( v_1 < \ln \bar{v} - \ln B + 1 \) so that \( r_2 > B \). In this case, the previous mechanism is no longer truthful for a buyer with \( 1 \leq v_1 < \ln \bar{v} - \ln B + 1 \). Note that truthful reporting yields expected utility \(-\infty\), since \( B < r_2 \leq \bar{v} \) and it is possible that \( v_2 \geq r_2 \). On the other hand, misreporting \( \hat{v}_1 = 1 \) yields utility \( v_1 - \hat{v}_1 + 0 = v_1 - 1 \geq 0 \). Fortunately, we can fix the mechanism by replacing \( r_1 \) and \( r_2 \) with \( r_1' = \min\{\hat{v}_1, 1 + \ln(B - 1)\} \) and \( r_2' = (B - 1)/e^{r_1'-1} \), respectively. One can verify that \( r_1' + r_2' \) never exceeds \( B \) and hence the budget constraint and truthfulness are respected. In this case, the expected revenue drops to \( 2 + \ln \ln(B - 1) \).

However, this may not be the optimal dynamic mechanism anymore. To see this, let \( B < 2 + \ln \bar{v} \) and consider the variation with \( r_1 \) replaced by \( r_1'' = \min\{\hat{v}_1, B\} \) and the auction in the second stage changed to allocate the item for free with probability \((r_1'' - 1)/(1 + \ln \bar{v}) \) (well-defined when
$B < 2 + \ln \overline{v})$. We omit the analysis here but conclude that this new mechanism is dynamic incentive-compatible and yields expected revenue $1 + \ln B \gg 2 + \ln \ln (B - 1)$, when $B$ is sufficiently large.

Finally, we highlight that the problem of designing (approximately) revenue-optimal mechanism can be very different under the settings with and without budgets. The budget constraint brings up many new challenges. Our example demonstrates that a direct generalization of the revenue-optimal mechanism under the without-budget setting could yield a revenue $O(\ln B)$, which could be much less than the revenue $O(\ln B)$ of the mechanism properly designed for budget constraints.

1.4 Related Work

Dynamic Mechanism Design. There is a large body of literature on dynamic mechanism design. We briefly discuss those closely related to ours, while refer readers to Bergemann and Välimäki [9] for a comprehensive survey. Bergemann and Välimäki [8] propose a welfare-maximizing dynamic pivot mechanism that is a generalization of the VCG mechanism to the dynamic environment where the buyers receive their private information over time. A team mechanism achieving efficient and budget-balanced outcomes is proposed by Athey and Segal [3]. Another related line of research is on revenue-maximization dynamic mechanism design: Pavan et al. [31] extend the Myersonian approach [28] and characterize dynamic incentive-compatibility, and Papadimitriou et al. [30] shows an arbitrarily large revenue gap between static and dynamic mechanisms.

Dynamic Mechanism Design Framework. The promised utility framework was proposed by Thomas and Worrall [34]. Ashlagi et al. [2] design a revenue-utility tradeoff framework. Our work generalizes the framework by Mirrokni et al. [24, 26]. They provide the bank account mechanism framework to design the dynamic mechanism with simple structures. Based on their framework, Mirrokni et al. [26] design a non-clairvoyant mechanism which, surprisingly, achieves a constant fraction of the revenue of the optimal clairvoyant mechanism. Mirrokni et al. [25] proposes practical bank account mechanisms and huge revenue lifts have been found from experiments.

1.5 Organization

In Section 2, we introduce a general environment for dynamic mechanism design, and we present our LHC framework in Section 3. We define the LHC mechanism with budgets, coined generalized bank account mechanisms, in Section 4, and derive an upper-bound on the revenue of any clairvoyant mechanism with budgets. We present our non-clairvoyant mechanism with budgets in Section 5.

2 A GENERAL DYNAMIC ENVIRONMENT

2.1 Dynamic Environments

We consider the problem of a single designer (he) running a dynamic mechanism among the same set of self-interested agents, i.e., $\{1, \ldots, n\}$, for $T > 0$ stages. For the sake of clarity, we focus on the single agent (she) case and all our results for LHC mechanisms (Section 3) can be generalized to multi-agent settings without changes in the proofs. In each stage $t$, there is a newly arrived item and the designer must determine the allocation of the item once it arrives.

The agent realizes her private type $\theta_t \in \Theta_t$ after she observes the item, and under allocation $x_t \in \mathcal{X}_t$, she receives value $v_t(\theta_t, x_t)$, where $v_t : \mathcal{X}_t \times \Theta_t \to \mathbb{R}$. The private type at each stage $t$ is independently drawn according to the prior distribution $F_t \in \mathcal{F}_t$. However, the distributions are not necessarily identical across stages. In the main content of this paper, we focus on the setting with independent valuations where $F_t$ is invariant of the history. In Appendix A, we argue that our framework can be extended to a more general setting with public valuation correlations. There, we allow $F_t$ to vary with any publicly observable information from the past of the mechanism, such as
We first introduce the notational basis with the mechanisms in which the prior knowledge is available to the designer from the beginning. Such mechanisms are termed clairvoyant dynamic mechanisms. Later in Section 2.3, we define the non-clairvoyant dynamic mechanisms, in which the prior knowledge of each stage is revealed to the designer at the beginning of that stage.

In a clairvoyant dynamic mechanism, the designer knows the number of stages $T$ and the sequence of priors $F_1, \ldots, F_T$ in advance. Hence the mechanism enjoys the ability of forecasting the future.

We focus on direct mechanisms throughout this paper (both in clairvoyant and non-clairvoyant settings), where the agent repeatedly reports her private types in each stage as $\hat{\theta}_t \in \Theta_t$. In each stage of a clairvoyant mechanism, upon receiving the report $\hat{\theta}_t$, the designer decides and implements the stage outcome based on the history $h_{t-1}$ and the priors, $F_{(1,T)} = (F_1, \ldots, F_T)$. Throughout the paper, we will use the notation $a(u,v')$ to represent the sequence of $a$ between stage $t'$ and $t''$.

Formally, we define a clairvoyant mechanism via the allocation and payment rules based on $h_{t-1}$.

**Definition 2.2 (Clairvoyant Dynamic Mechanism).** A clairvoyant dynamic mechanism is defined by a pair of allocation and payment rules $(\langle x_{(1,T)}, p_{(1,T)} \rangle)$:

- Allocation rule $x_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)}), x_t : \mathcal{H}_{t-1} \times \Theta_t \times (F_1 \times \cdots \times F_T) \rightarrow X_t$;
- Payment rule $p_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)}), p_t : \mathcal{H}_{t-1} \times \Theta_t \times (F_1 \times \cdots \times F_T) \rightarrow \mathbb{R}$.

Therefore, the allocation and payment rules at stage $t$ of a clairvoyant mechanism can depend on the past history, the agent’s reported type at stage $t$, and the prior knowledge of all stages:

$$\mathcal{M}^C = \langle x_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)}), p_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)}) \rangle_{t=1}^T.$$  (Clairvoyant Mechanism)

To summarize, in each stage of a clairvoyant dynamic mechanism, the following steps happen:

1. the agent observes the item and realizes her private type $\theta_t \sim F_t$;
2. the agent reports type $\hat{\theta}_t$ to the designer;
3. the designer implements the allocation $x_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)})$ and the payment $p_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)})$;
4. the agent accrues stage utility $u_t = v_t(\theta_t, x_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)})) - p_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)})$.

In the rest of the paper, we hide the prior knowledge $F_{(1,T)}$ from the inputs to clairvoyant mechanisms when there is no ambiguity, e.g., we write $u_t(h_{t-1}, \hat{\theta}_t) = v_t(\theta_t, x_t(h_{t-1}, \hat{\theta}_t)) - p_t(h_{t-1}, \hat{\theta}_t)$. In addition to the definitions above, a valid clairvoyant mechanism must meet certain incentive guarantees (see Section 2.2.1) and outcome restrictions (see Section 2.2.2).

### 2.2.1 Dynamic Incentive Compatibility

The agent’s best response in a dynamic mechanism depends on her strategy in the future stages. The classic notion of dynamic incentive-compatibility requires that for all stages, reporting truthfully is the optimal strategy, assuming that she reports truthfully in the future [26]. Notice that reporting truthfully is also the buyer’s optimal strategy in the future,

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2Interested readers can refer to [26] for discussions on the choice of DIC notions.
assuming the mechanism is dynamic incentive-compatible in the future. For a history \( h_t \), let the 
continuation utility be 
\[
U(t,h_t) = \mathbb{E} \left[ \sum_{\tau = t+1}^{T} u_{\tau}(h_{\tau-1}, \theta_{\tau}; \theta_{\tau}) \right],
\]
the expected utility from the remaining stages via truthful reporting, where \( h_{\tau} \) is the history at stage \( \tau > t \) resulted from truthful reporting from stage \( t + 1 \) to \( \tau \) after \( h_t \). The expectation here is taken over the random events in the remaining stage and the private type draws \( \theta_{\tau} \sim F_{\tau} \).

To highlight the effects from the agent’s report \( \hat{\theta}_t \) at stage \( t \), let \( h_t(t) \) be the history obtained from \( h_{t-1} \) where the agent reports \( \hat{\theta}_t \) at stage \( t \), and denote by \( h_t(t) \) a history at stage \( \tau > t \) resulted from reporting truthfully from stage \( t + 1 \) to \( \tau \) after \( h_t(t) \); and thus, the continuation utility is given by:
\[
\overline{U}_t(h_t(t)) = \mathbb{E} \left[ \sum_{\tau = t+1}^{T} u_{\tau}(h_{\tau-1}, \theta_{\tau}; \theta_{\tau}) \right].
\]
Dynamic incentive compatibility can then be defined via backward induction: in the last stage \( T \), it is an optimal strategy for the agent to report truthfully no matter what the history \( h_{T-1} \) is:
\[
\theta_T \in \arg\max_{\theta_T} u_T(h_{T-1}, \hat{\theta}_T; \theta_T),
\]
for all private type \( \theta_T \) and history \( h_{T-1} \). For the induction step, in stage \( t \), conditioned on the agent reporting truthfully for all the remaining stages, the agent should be incentivized to report truthfully no matter what the history \( h_{t-1} \) is:
\[
\forall t \in [T], \theta_t \in \Theta_t, h_{t-1} \in \mathcal{H}_{t-1}, \quad \theta_t \in \arg\max_{\hat{\theta}_t} u_t(h_{t-1}, \hat{\theta}_t; \theta_t) + \overline{U}_t(h_t(t)). \quad \text{(DIC)}
\]

2.2.2 Mechanism Constraints. As the second novelty of our general design framework, we introduce the general constraints on the mechanism outcomes. The practical constraints beyond the canonical ex-post individually rational constraints then can directly fit into our framework.

In a seminal work, Mirrokni et al. [26] introduces a design framework called bank account mechanisms to simplify the design of dynamic mechanisms with respect to the ex-post individual rationality. This paper focuses on empowering the general LHC framework to incorporate practical constraints, such as budget constraints, return on investment (ROI) constraints, etc.

More specifically, we allow most constraints on the mechanism outcomes as long as they are independent of the agent private types.\(^3\) The constraints may involve any public information like priors and the reported types. Examples include the ex-post or ex-ante individually rational constraint, the budget constraint (agents could even have separate budgets for items of different categories), ROI constraints, supply or demand constraints, etc. Nevertheless, the DIC constraint does not belong to this category, as it depends on the private types.

Without loss of generality, we classify the constraints into two categories: allocation constraints and payment constraints, denoted by \( \mathcal{X} \) and \( \mathcal{P} \), respectively (font-variants of \( x \) and \( p \)).

Definition 2.3 (Allocation Constraints). The allocation constraints \( \mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_T) \). Each \( \mathcal{X}_t \) maps the history \( h_{t-1} \), the agent’s reported type \( \hat{\theta}_t \), and priors \( F_{(1,T)} \) to a subset of \( \mathcal{X}_t \):\(^4\)
\[
x_t \in \mathcal{X}_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)}) \subseteq \mathcal{X}_t. \quad \text{(Allocation)}
\]
In this paper, we will simply write it as \( x_t \in \mathcal{X}_t \) when there is no ambiguity.

In addition to the above definition, we further assume the allocation constraints \( \mathcal{X} \) is monotone: \( \forall x'_t \in \mathcal{X}_t, \hat{\theta}_{t+1} \in \Theta_{t+1}, \mathcal{X}_{t+1}(h'_t, \hat{\theta}_{t+1}, F_{(1,T)}) \neq \emptyset \), where \( h'_t \) is the history when the allocation in stage \( t \) is set to be \( x'_t \). The assumption essentially requires the constraints to early rule out the allocations that can lead to no feasible outcomes later.

\(^3\)The feasible set of the constraints must be a properly measurable set such that all the expectation operators are well-defined.
\(^4\)As the history \( h_{t-1} \) includes all the historical information, the definition is almost universal. We won’t get too much into the mathematical details, while note that both \( \mathcal{X}_t \) and \( \mathcal{P}_t \) are properly defined measurable functions.
Definition 2.4 (Payment Constraints). The payment constraints \( P = \{ P_1, \ldots, P_T \} \). Each \( P_t \) maps the history \( h_{t-1} \), the allocation of the current stage \( x_t \), past payments \( p_{(1:t-1)} \), the agent's reported type \( \hat{\theta}_t \), and priors \( F_{(1:T)} \) to a subset of \( \mathbb{R} \):

\[
p_t \in P_t(h_{t-1}, x_t, p_{(1:t-1)}, \hat{\theta}_t, F_{(1:T)}) \subseteq \mathbb{R}.
\]

(Payment)

Similarly, we write it as \( p_t \in P_t \) and assume the payment constraints \( P \) to be monotone.

As a quick example, the objective can be to maximize revenue by setting

\[
\text{Obj}(h_T, p) : \mathcal{H}_T \times \mathbb{R} \rightarrow \mathbb{R},
\]

where \( p = \sum_{t \in [T]} p_t \) is the total payment. We summarize the design problem as a program

maximize \( \mathbb{E}[\text{Obj}(h_T, p)] \) \hspace{1cm} (Design Problem)

subject to \( \text{DIC} \) and \( x_t \in \mathcal{X}_t, p_t \in P_t, \forall t \in [T] \).

As a quick example, the objective can be to maximize revenue by setting \( \text{Obj}(h_T, p) = p \).

2.3 Non-clairvoyant Dynamic Mechanism

The non-clairvoyance notion is first introduced by Mirrokni et al. [26] to remove the unrealistic designer power of forecasting the distributional information and arriving order of future items. A designer is non-clairvoyant, if he does not know the value of \( T \) and has no access to \( F_t \) until the beginning of stage \( t \). A dynamic mechanism is non-clairvoyant, if its allocation and payment rules are independent of the prior knowledge of future stages, \( F_{(t+1:T)} \), and the total number of stages, \( T \):

\[
\mathcal{M}^{\text{NC}} = \{ x_t(h_{t-1}, \hat{\theta}_t, F_{(1:t)}), p_t(h_{t-1}, \hat{\theta}_t, F_{(1:t)}) \}_{t=1}^{\infty}.
\]

(NON-CLAIRVOYANT MECHANISM)

A mechanism is dynamic incentive compatible in the non-clairvoyant sense, if it respects (DIC) for all \( T \in \mathbb{N} \) and any priors \( F_{(t+1:T)} \) of future stages. Taking \( T = t \) (consider stage \( t \) as the last stage), we get stage-wise incentive compatible, because the corresponding continuation utility \( \bar{U}_t = 0 \):

\[
(\text{DIC}) \implies \theta_t \in \arg \max_{\theta_t} u_t(h_{t-1}, \hat{\theta}_t; \theta_t) + \bar{U}_t(h_t^{(t)}) = \arg \max_{\theta_t} u_t(h_{t-1}, \hat{\theta}_t; \theta_t).
\]

(STAGE-IC)

Finally, the mechanism constraints must respect the non-clairvoyance as well,

\[
x_t \in \mathcal{X}_t(h_{t-1}, \hat{\theta}_t, F_{(1:t)}) \subseteq \mathcal{X}_t, \quad p_t \in P_t(h_{t-1}, x_t, p_{(1:t-1)}, \hat{\theta}_t, F_{(1:t)}) \subseteq \mathbb{R}.
\]

2.4 Ad Auctions with Budget Constrained Buyers

As we mentioned before, we are particularly interested in the practical setting of ad auctions with budget constrained buyers. In particular, we consider an environment with allocation space \( \mathcal{X}_t = [0, 1] \), type space \( \Theta_t = [0, \bar{\theta}] \), and quasi-linear utility \( u_t = \theta_t x_t - p_t \). Moreover, the following (ex-post-IR) constraint and (BUDGET) constraint must be satisfied by the industrial standard:

\[
\mathcal{B}^{\text{EX-POST-IR}} = \{ p = \sum_{t \in [T]} p_t \leq \sum_{t \in [T]} u_t(\hat{\theta}_t, x_t) \}
\]

(Ex-post-IR)

\[
\mathcal{B}^{\text{BUDGET}} = \{ p = \sum_{t \in [T]} p_t \leq B \}
\]

(Budget)

Therefore, the mechanism constraint is \( \mathcal{P} = \mathcal{B}^{\text{EX-POST-IR}} \cap \mathcal{B}^{\text{BUDGET}} \), which must be satisfied for all \( T \in \mathbb{N} \) in the non-clairvoyant mechanisms.

The objective of the designer is to design a revenue-optimal mechanism that satisfies (DIC), (ex-post-IR), and (BUDGET) in clairvoyant and non-clairvoyant environments. The following mild assumption applies for our approximation analysis of our non-clairvoyant mechanism in the budget constrained environment, which is motivated by the online advertising market where the buyer’s budget is typically much larger than her valuation of every possible impression.
ASSUMPTION 2.5. There exists a sufficiently small $\epsilon$ such that the buyer’s budget satisfies $B \geq \bar{u}/\epsilon$.

Remark 2.6. Our characterizations in Section 3 and Section 4 do not rely on Assumption 2.5, but our non-clairvoyant mechanism in Section 5 does. We note that Assumption 2.5 rules out the budgeted scenario we discussed in Section 1.3. However, designing a non-clairvoyant mechanism under budget constraints with non-trivial revenue guarantee under Assumption 2.5 is already technically involved. We leave it as an interesting open question to design non-clairvoyant mechanisms without Assumption 2.5.

3 LOSSLESS HISTORY COMPRESSION (LHC) MECHANISM

In this section, we formally define our general design framework that we call Lossless History Compression (LHC) mechanisms, which serves as the bridge that connects clairvoyant dynamic mechanism designs and non-clairvoyant dynamic mechanism designs. We defer the proofs in this section to Appendix C. LHC mechanisms generalize the idea of the bank account mechanisms, a technique introduced by Mirrokni et al. [24, 26] for the classic ex-post individually rational constraints, respectively. The compression is defined as:

A succinct summary of the historical scenario we discussed in Section 1.3. However, designing a non-clairvoyant mechanism under budget constraints, without Assumption 2.5. Moreover, the design of optimal clairvoyant dynamic mechanisms can be restricted to LHC mechanisms, without losing any generality in terms of the designer OBJECTIVE (see Theorem 3.5).

At a high level, in an LHC mechanism (see Definition 3.3), a succinct summary of the historical information, which we call it state, is used to keep track on the least information that is essential for implementing the dynamic mechanism. The information is compressed as much as possible to simplify the structure of the dynamic mechanism, while not violating any mechanism constraints nor losing any prior knowledge information (for the general setting with public correlations). The following notion of lossless history compression then defines the least information to be kept.

Definition 3.1 (Lossless History Compression (LHC)). A lossless history compression (or simply compression) $s = (s_1, \ldots, s_T)$ is a sequence of functions — each maps the history and previous payments to a succinct summary $s_t(h_t, p_{1:t})$, i.e., $s_t : \mathcal{H}_t \times \mathbb{R}^t \rightarrow S_t$ — such that there exist accompanying mechanism constraints $\langle \mathcal{P}_t, \mathcal{F}_t \rangle$ defined on $s_t$,

$$
\mathcal{P}_t(s_{t-1}, \hat{\theta}_t, F_{1:T}) = \mathcal{P}_t(h_{t-1}, \hat{\theta}_t, F_{1:T}),
$$

and a summary update rule $\Lambda = (\Lambda_1, \ldots, \Lambda_T)$:

$$
s_t(h_t, p_{1:t}) = \Lambda_t(s_t(h_{t-1}, p_{1:t-1}), \hat{\theta}_t, x_t, p_t).
$$

Intuitively, the compression contains the sufficient information to perfectly recover the mechanism constraints as $\langle \mathcal{P}_t, \mathcal{F}_t \rangle$, and it can be tracked by the update rule $s_t = \Lambda_t(s_{t-1}, \hat{\theta}_t, x_t, p_t)$. At a glance, the abstract definition above seems complicated, yet it turns out to be quite simple for most of the common settings, where each of the constraints may end up with one real number in the summary. As a quick example, let’s consider the (ex-post-IR) and (budget) constraints, which will be further investigated in Section 4.

Example 3.2 (ex-post-IR and Budget). Let $S_t = \mathbb{R}^2$ and each summary $s_t$ be two real numbers $\langle s_{t}^{ir}, s_{t}^{b} \rangle$ for the (ex-post-IR) and (Budget) constraints, respectively. The compression is defined as:

$$
s_{t}^{ir} = \sum_{1=t}^{T} v_t(\hat{\theta}_t, x_t) - p_t, \quad \text{and} \quad s_{t}^{b} = B - \sum_{1=t}^{T} p_t.
$$

Finally, the summary is updated according to $s_{t}^{ir} = s_{t-1}^{ir} + v_t(\hat{\theta}_t, x_t) - p_t$ and $s_{t}^{b} = s_{t-1}^{b} - p_t$.

We are now ready to formally define an LHC mechanism.

Definition 3.3 (LHC Mechanism). An LHC mechanism $\overline{M} = (\overline{x}, \overline{p})$ is defined based on a lossless history compression $s$ with $\sigma_t = (s_t, \mu_t)$ as its state, where $\mu_t = \overline{U}_t(\sigma_t)$ is the promised continuation utility. In each stage $t$,
(1) the agent realizes her private type $\theta_t$, reports $\hat{\theta}_t$, and accrues stage utility $u_t(\sigma_{t-1}, \hat{\theta}_t; \theta_t) = v_t\left(\theta_t, \bar{x}_t(\sigma_{t-1}, \hat{\theta}_t)\right) - \bar{p}_t(\sigma_{t-1}, \hat{\theta}_t)$ from the stage allocation $\bar{x}_t(\sigma_{t-1}, \hat{\theta}_t)$ and payment $\bar{p}(\sigma_{t-1}, \hat{\theta}_t)$; 
(2) the next state $\sigma_t$ is determined according to the following state update rule 
$$\sigma_t = \langle s_t, \mu_t \rangle = \langle \Lambda_t(s_{t-1}, \hat{\theta}_t, \bar{x}_t, \bar{p}_t), \mu_t(\sigma_{t-1}, \hat{\theta}_t) \rangle,$$
(State-Update) 
where $\mathbb{E}_{\hat{\theta}_{t-1}}[u_t(\sigma_{t-1}, \hat{\theta}_t; \theta_t) + \mu_t(\sigma_{t-1}, \theta_t)] = \mu_t$; 
(3) the stage outcome $\langle \bar{x}_t, \bar{p}_t \rangle$ satisfies the following incentive constraint 
$$\theta_t \in \arg\max_{\hat{\theta}_t} u_t(\sigma_{t-1}, \hat{\theta}_t; \theta_t) + \mu_t(\sigma_{t-1}, \hat{\theta}_t),$$
(LHC-IC) 
and the allocation and payment constraints $\bar{x}_t \in \mathcal{X}_t$, $\bar{p}_t \in \mathcal{P}_t$.

Our key result is that without loss of generality, one can find an optimal solution to the Design Problem within LHC mechanisms. On one hand, by putting the definitions together (Definition 3.1, Definition 3.3, and (DIC), (Allocation), (Payment) constraints), we can directly conclude:

**Lemma 3.4.** Any LHC mechanism satisfies (DIC), (Allocation), and (Payment) constraints.

On the other hand, as we will prove in Section 3.3 by construction, one can reduce any dynamic mechanism to a LHC mechanism without losing the optimality.

**Theorem 3.5.** Given any dynamic mechanism satisfying (DIC), (Allocation), and (Payment) constraints, for any given lossless history compression $s$, there exists an LHC mechanism defined based on $s$ that yields at least the same designer Objective. In particular, the optimal solution to the Design Problem can be achieved by an optimal LHC mechanism.

### 3.1 Non-clairvoyance, Stage-Incentive Compatibility, and State-Utility Independence

Before we provide a proof sketch for Theorem 3.5, we first establish the connection between LHC mechanisms and non-clairvoyant dynamic mechanisms. We first introduce two important properties closely related to the non-clairvoyant notions: stage IC and state-utility independence. Firstly, under a mild assumption (Assumption 3.6) that holds for most commonly studied environments, we prove that these two properties together is an if-and-only-if characterization of non-clairvoyant mechanisms satisfying non-clairvoyant (DIC). Moreover, any dynamic mechanism can be converted to stage IC and state-utility independent mechanism without loss of optimality when the (Payment) constraints satisfy a further property (see Theorem 3.8) such that (Payment REALIGNMENT) is possible. Secondly, we show in Corollary 3.14, when these two properties are in place, we can further restrict our attention to a subset of LHC mechanisms with even simpler structures, where the state is simplified as $\sigma_t = \langle s_t, \mu_t \rangle$ rather than $\sigma_t = \langle s_t, \mu_t \rangle$ (which is a major simplification when the update rule for $\mu_t$ is not constructive), and the (LHC-IC) constraint is also simplified as (STATE-IC).

The mild assumption required is that the payment rule implementing the stage allocation $x_t$ is unique up to a constant, which is commonly satisfied in many standard environments. For example, the environments with continuous type spaces and additive valuations, in which the mild assumption is implied by the Myerson’s lemma [28] (in single-dimensional environments) or the envelope theorem [32] (in multi-dimensional environments).

---

5We use $\bar{U}_t(\sigma_t)$ to denote the continuation utility of the LHC mechanism $\bar{M}$ under state $\sigma_t$, which is an analog to the definition of $\bar{U}_t(h_t)$ with respect to the uncompressed history $h_t$. 
6The irregular cases where the assumption is violated are usually of less interests. One example is as follows. Consider a single item auction with value space $\Theta_t = \{0, 1\}$. For the allocation $x_t(0) = 0, x_t(1) = 1$, any payment function such that $p_t(0) = 0, p(1) \in (0, 1]$ implements $x_t$, while not satisfying the assumption.
Assumption 3.6 (Unique Payment). In any stage $t$, every stage allocation rule $x_t$ pins down a unique payment rule $p_t$ up to a constant $c$: For any fixed history $h_{t-1}$ and payment rule $p'_t$ such that truthful reporting is an optimal strategy, i.e., $\theta_t \in \arg\max_{\hat{\theta}_t} v_t(\theta_t, x_t(h_{t-1}, \hat{\theta}_t)) - p'_t(h_{t-1}, \hat{\theta}_t)$, we have

$$\forall \hat{\theta}_t \in \Theta_t, p'_t(h_{t-1}, \hat{\theta}_t) \equiv p_t(h_{t-1}, \hat{\theta}_t) + c,$$

for some constant $c$.

As previously mentioned in Section 2.3, the non-clairvoyant (DIC) constraint directly implies (Stage-IC). In fact, it also implies the following property under Assumption 3.6:

Definition 3.7 (State-Utility Independence). An LHC mechanism is state-utility independent, if for each stage $t$, there exists a constant $c_t$ such that

$$\mathbb{E}_{\theta_t \sim \mathcal{F}_t} [u_t(\sigma_{t-1}, \theta_t)] = c_t \text{ independent of the state } \sigma_{t-1}. \quad \text{(State-UI)}$$

The state-utility independence property states that the agent’s expected utility at stage $t$ is independent of the state $\sigma_{t-1}$. Since the state $\sigma_{t-1}$ is the only channel that carries the agent’s historical reports, state-utility independence directly implies that her historical reports have no effect on her future expected utility. Hence, a non-clairvoyant mechanism that is State-UI and Stage-IC, is DIC in the non-clairvoyant sense. When Assumption 3.6 stands, we have the following sufficient and necessary characterizations of non-clairvoyant dynamic mechanisms satisfying non-clairvoyant (DIC) and that dynamic mechanisms can be converted to a Stage-IC and State-UI mechanism without loss of optimality.

Theorem 3.8 (Non-clairvoyance Characterization). Under Assumption 3.6, a non-clairvoyant dynamic mechanism satisfies non-clairvoyant (DIC) if and only if it satisfies (Stage-IC) and (State-UI).

3.2 Payment Realignment

Can we convert a dynamic mechanism satisfying (DIC), (Allocation), and (Payment), to a mechanism that is Stage-IC and State-UI? The following theorem provides a necessary and sufficient condition, coined Payment Realignment, on when such a conversion is possible.

Theorem 3.9. A dynamic mechanism $M = (x, p)$ that satisfies (DIC), (Allocation), and (Payment) can be converted to a Stage-IC and State-UI mechanism $M' = (x, p')$ with equivalent stage allocation rules and final cumulative payment (hence the designer Objective is unaffected), if and only if there exist constants $c_1, \ldots, c_T \in \mathbb{R}$ such that $c_1 + \cdots + c_T = \mathcal{U}_0(\emptyset)$ and $\forall t \in [T], h_{t-1} \in \mathcal{H}_{t-1}, \theta_t \in \Theta_t,$

$$p'_t(h_{t-1}, \theta_t) = p_t(h_{t-1}, \theta_t) - \mathcal{U}_t(h_t) + \mathcal{U}_{t-1}(h_{t-1}) - c_t \in \mathcal{P}_t,$$

where $h_t$ is the history from reporting $\theta_t$ after $h_{t-1}$. Moreover, in the converted mechanism using $c_1, \ldots, c_T$ and Payment Realignment, the expected utility at stage $t$ is $c_t$, independent of the state.

Payment Realignment keeps the allocation rule the same but rearranges the payment across stages. Readers may take another look at Section 1.3 for a concrete example of payment realignment.

In a non-clairvoyant environment, the agent is willing to participate in stage $t$ before she learns $\theta_t$ only if her expected utility at stage $t$ is non-negative, which equals to $c_t$ in Payment Realignment by Theorem 3.9. Lemma 3.11 provides a sufficient condition under which the mechanism constraints enables Payment Realignment with $c_1, \ldots, c_T \in \mathbb{R}_+$ in the dynamic auction environment with $\Theta_t = \mathbb{R}_+$ for all $t$. Before Lemma 3.11, we introduce the concept of monotone history functions, which is a class of functions with non-decreasing function value as the dynamic mechanism proceeds.

Definition 3.10 (Monotone History Function). A function $\zeta$ with $\zeta(\emptyset) \leq 0$ is monotone in terms of history if for any $h_t \in \mathcal{H}_t$ and any $h_{t+1} \in \mathcal{H}_{t+1}$ resulted from $h_t$, $\zeta(h_t) \leq \zeta(h_{t+1}).$
Lemma 3.11. When $\Theta_t = \mathbb{R}^+$ for all $t$, Payment Realignment is possible with $c_1, \ldots, c_T \in \mathbb{R}^+$ for Payment constraints $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_T\}$ if there exist $a_1, \ldots, a_K \in \mathbb{R}^+$ and monotone history functions $\zeta_1, \ldots, \zeta_K$ such that $\mathcal{P}_t = \{p_t \in \mathbb{R} : \forall k \in [K], \sum_{r=1}^t \theta_r x_r - (1 + a_k) \cdot p_t \geq \zeta_k(h_t)\}$, where $x_t$ and $p_t$ are the allocation probability and the payment at stage $t$ corresponding to history $h_t$.

Note that the condition of Payment constraints in Lemma 3.11 is able to capture many practical constraints, such as Budget, ROI constraints, and hence (ex-post-IR). For (Budget) with a budget limit $B$, set $a_k = 0$ and $\zeta_k(h_t) = -B + \sum_{r=1}^t \theta_r x_r$. As for ROI constraints, which states that the agent’s cumulative allocated value must be at least $(1 + \beta)$ times of the agent’s cumulative payment, one can set $a_k = \beta$ and $\zeta_k(h_t) = 0$. (ex-post-IR) is a special case of ROI constraints with $\beta = 0$. One could also think of another constraint in which the agent requires that her average utility for any first $t$ stages must be at least $\gamma$. To represent such a constraint, set $a_k = 0$ and $\zeta_k(h_t) = \gamma \cdot t$.

3.3 A Constructive Proof for Theorem 3.5

We prove Theorem 3.5 by construction. The main obstacle of the proof is that for the given compression $s$ and the mechanism $M = (x, p)$, there might be two different histories $h_{t-1} \neq h'_{t-1}$ such that they are compressed into the same state, i.e., $s_t(h_{t-1}, p_{(1,t-1)}) = s_t(h'_{t-1}, p'_{(1,t-1)})$, but the successive stage allocations are different, i.e., $x_t(h_{t-1}, \hat{\theta}_t) \neq x_t(h'_{t-1}, \hat{\theta}_t)$. In this case, a direct construction of LHC mechanism will suffer a contradiction: $\tilde{x}_t(s_{t-1}(h_{t-1}, p_{(1,t-1)}), \hat{\theta}_t) = x_t(h_{t-1}, \hat{\theta}_t) \neq x_t(h'_{t-1}, \hat{\theta}_t) = \tilde{x}_t(s_{t-1}(h'_{t-1}, p'_{(1,t-1)}), \hat{\theta}_t) = \tilde{x}_t(s_{t-1}, \hat{\theta}_t)$. Therefore, a correct path is to first convert an arbitrary $M$ into a symmetric mechanism $M^{\text{symmetric}}$ (see Definition 3.12), and from $M^{\text{symmetric}}$, we can easily construct an LHC mechanism $\hat{M}$ such that $\tilde{x}_t(s_{t-1}(h_{t-1}, p_{(1,t-1)}), \hat{\theta}_t) = x_t^{\text{symmetric}}(h_{t-1}, \hat{\theta}_t)$, $\tilde{p}_t(s_{t-1}(h_{t-1}, p_{(1,t-1)}), \hat{\theta}_t) = p_t^{\text{symmetric}}(h_{t-1}, \hat{\theta}_t)$.

Definition 3.12 (Symmetric Mechanism). A mechanism $M = (x, p)$ is symmetric with respect to a compression $s$, if for any $t < t'$ and two histories $h_t, h'_t \in H_t$, when $s_t(h_t, p_{(1,t)}) = s_t(h'_t, p'_{(1,t)})$ and $\overline{U}_t(h_t) = \overline{U}_t(h'_t)$, we have for all $\theta_{(t+1,t')} \in \Theta_{t+1} \times \cdots \times \Theta_{t'}$, $x_{t'}(h_{t'-1}, \theta_{t'}) = x_{t'}(h'_{t'-1}, \theta_{t'})$ and $p_{t'}(h_{t'-1}, \theta_{t'}) = p_{t'}(h'_{t'-1}, \theta_{t'})$, (Symmetric) where $h_{t'-1}$ and $h'_{t'-1}$ are histories resulted from reporting $\theta_{(t+1,t'-1)}$ after history $h_t$ and $h'_t$, respectively.

Lemma 3.13 (Symmetrization). Any dynamic mechanism $M$ that satisfies (DIC), (Allocation), and (Payment), can be transformed into a mechanism $M^{\text{symmetric}}$ that is further symmetric according to the given lossless history compression $s$. Moreover, $M^{\text{symmetric}}$ yields at least the same designer Objective as $M$, and if $M$ is further Stage-IC and State-UI, then $M^{\text{symmetric}}$ preserves these properties.

The proof of Lemma 3.13 is done by applying symmetrization stage by stage, from $\tau = 1$ to $T$. On each stage $t$, for the histories that are mapped to the same state $\sigma_t$, we assign them an identical sub-mechanism to make the entire mechanism symmetric. The assigned sub-mechanism is the sub-mechanism induced by a selected representative history that is mapped to $\sigma_t$ with the maximum designer Objective among all the histories that are also mapped to $\sigma_t$. The complete proof is then a careful argument by induction. The following corollary would be useful to simplify the states of the LHC mechanisms that are Stage-IC and State-UI.

Corollary 3.14 (Stage-IC and State-UI LHC Mechanisms). When Payment Realignment is possible, any dynamic mechanism satisfying (DIC), (Allocation), and (Payment) can be converted to an LHC mechanism that is Stage-IC and State-UI and yields at least the same designer Objective. Moreover, the promised continuation utilities $\mu_t$ in its states $\sigma_t = \langle s_t, \mu_t \rangle$ at stage $t$ are constants, i.e., it is sufficient to keep its state simply as $\sigma_t = \langle s_t \rangle$. 

4 LHC MECHANISM WITH BUDGET CONSTRAINTS

We now apply the LHC mechanism framework to the budget constrained environment introduced in Section 2.4 to obtain our generalized bank account mechanism below. Recall that the buyer’s type \( \alpha \) is drawn from \([0, \overline{\alpha}]\) and her utility is \( u_t = v_t \cdot x_t - p_t \). The goal of the seller (designer) is to maximize the expected revenue under (\text{ex-post-IR}) and (\text{Budget}), i.e., \( \text{Obj}(h_T, p) = p \). The unique payment assumption (Assumption 3.6) is satisfied in this environment due to the Myerson’s lemma [28]. According to Lemma 3.11 and Corollary 3.14, we can focus on an LHC mechanism that is \text{Stage-IC} and \text{State-UI}, such that the history could be summarized by \( s_t = (s_t^u, s_t^p) \) (as illustrated in Example 3.2) in which \( s_t^u \) tracks the cumulative utility and \( s_t^p \) tracks the remaining budget: \( s_t^u = \sum_{t=1}^{T-1} v_t \cdot x_t - p_t \) and \( s_t^p = B - \sum_{t=1}^{T-1} p_t \). The update rule is \( s_{t+1}^u = s_t^u + v_t \cdot x_t - p_t \) and \( s_{t+1}^p = s_t^p - p_t \). We simply denote \( s_t^u \) by \( \alpha_t \) and \( s_t^p \) by \( \beta_t \). The generalized bank account mechanism with (\text{ex-post-IR}) and (\text{Budget}) is defined below.

**Definition 4.1 (Generalized Bank Account Mechanism).** A generalized bank account mechanism \( B \) is specified by \((x_{1,T}, p_{1,T})\) such that

- The stage mechanism \( x_t(\alpha, \beta, b_t), p_t(\alpha, \beta, b_t) \) is parameterized by the balances \( \alpha, \beta \in \mathbb{R}_+ \), which is incentive-compatible in the stage for every \( \alpha \) and \( \beta \):
  \[
  \forall b_t, \quad v_t \cdot x_t(\alpha, \beta, b_t) - p_t(\alpha, \beta, b_t) \geq v_t \cdot x_t(\alpha, \beta, b_t) - p_t(\alpha, \beta, b_t) \tag{\text{Stage-IC}}
  \]

- The stage mechanism ensures that the expected utility is balance independent under truthful reporting: there exists a non-negative constant \( c_t \) such that for all \( \alpha, \beta \in \mathbb{R}_+ \),
  \[
  \mathbb{E}_{v_t \sim F}[v_t \cdot x_t(\alpha, \beta, v_t) - p_t(\alpha, \beta, v_t)] = c_t \geq 0 \tag{\text{BI}}
  \]

- The balances \( \alpha_t \) and \( \beta_t \) are updated accordingly such that the balances are always non-negative,
  \[
  \forall b_t, \quad \alpha_t = \alpha_{t-1} + b_t \cdot x_t(\alpha, \beta, b_t) - p_t(\alpha, \beta, b_t) \geq 0 \quad \text{and} \quad \beta_t = \beta_{t-1} - p_t(\alpha, \beta, b_t) \geq 0. \tag{\text{BU}}
  \]

Here, (BI) corresponds to (\text{State-UI}) and (BU) corresponds to the summary update rule. Theorem 4.2 directly follows Lemma 3.11 and Corollary 3.14.

**Theorem 4.2.** A generalized bank account mechanism satisfies (\text{DIC}), (\text{ex-post-IR}), and (\text{Budget}). Given any clairvoyant dynamic mechanism that satisfies (\text{DIC}), (\text{ex-post-IR}), and (\text{Budget}), there exists a generalized bank account mechanism achieving at least the same revenue.

4.1 Local-stage Mechanism, Deposit, and Spend

We next provide an interpretation for the balance updates and the design of stage mechanisms. The interpretation will help establish the upper bound on the revenue of any clairvoyant mechanism with budget constraints (see Lemma 4.4). It is a key building block for our approximation results.

With Theorem 4.2, the revenue of any clairvoyant mechanism is upper bounded by a generalized bank account mechanism \( M = \langle x_{1,T}, p_{1,T} \rangle \) that is \text{Stage-IC} and \text{BI}. Since its stage mechanism \( \langle x_t, p_t \rangle \) is \text{Stage-IC} for every possible balances \((\alpha_{t-1}, \beta_{t-1})\), by Myerson’s lemma [28], we have

\[
\text{p}(\alpha_{t-1}, \beta_{t-1}, v_t) = \int_0^{v_t} \frac{\partial x_t(\alpha_{t-1}, \beta_{t-1}, u)}{\partial u} \cdot vdu + p_t(\alpha_{t-1}, \beta_{t-1}, 0).
\]

Decompose the payment as \( p_t(\alpha_{t-1}, \beta_{t-1}, v_t) = q_t(\alpha_{t-1}, \beta_{t-1}, v_t) + p_t(\alpha_{t-1}, \beta_{t-1}, 0) \) with the first component \( q_t(\alpha_{t-1}, \beta_{t-1}, v_t) = \int_0^{v_t} \frac{\partial x_t(\alpha_{t-1}, \beta_{t-1}, u)}{\partial v} \cdot vdu \). \( \langle x_t, v_t \rangle \) defines a \text{Stage-IC} and stage-wise \text{ex-post individually rational} (\text{Stage-IR}) mechanism by Myerson’s lemma. \text{Stage-IR} means:

\[
\forall \alpha_{t-1}, \beta_{t-1}, v_t \geq 0, \quad v_t \cdot x_t(\alpha_{t-1}, \beta_{t-1}, v_t) - q_t(\alpha_{t-1}, \beta_{t-1}, v_t) \geq 0. \tag{\text{Stage-IR}}
\]

Rewrite the update rule for state \( \alpha_t \) as \( \alpha_t = \alpha_{t-1} + v_t \cdot x_t - p_t = \alpha_{t-1} + (v_t \cdot x_t - q_t) - p_t(\alpha_{t-1}, \beta_{t-1}, 0). \) We can further decompose the update as the increase of \( \alpha \), \( s_t^{u+} = v_t \cdot x_t - q_t \), and the decrease of
We generalize the notation to accommodate the needs for the design of our mechanism for multiple likely. A properly designed policy to exclude buyers who have $\alpha_t = \alpha_{t-1} + s_t^{ir} - s_t^{ir}$. In particular, $s_t^{ir}$ is referred as deposit and $s_t^{ir}$ is referred as spend of the balance $\alpha$. We call $(x_t, q_t)$ the local-stage mechanism. The following alternative definition comes from the above payment decomposition.

**Lemma 4.3.** $(x_t, q_t, s_t^{ir}, \alpha_t, \beta_t)$ is a generalized bank account mechanism if for each stage $t$:

1. The local-stage mechanism $(x_t, q_t)$ is Stage-IC and Stage-IR;
2. $E_t[s_t^{ir}((\alpha_{t-1}, \beta_{t-1}, q_t))] - s_t^{ir}((\alpha_{t-1}, \beta_{t-1})) = c_t \geq 0$ and does not depend on $\alpha_{t-1}$ nor $\beta_{t-1}$;
3. $s_t^{ir}((\alpha_{t-1}, \beta_{t-1})) \leq \alpha_{t-1}$ and $\forall q_t, q_t((\alpha_{t-1}, \beta_{t-1}, v_t)) + s_t^{ir}((\alpha_{t-1}, \beta_{t-1})) \leq \beta_{t-1}$.

The proof is deferred to Appendix D. In fact, item 2 and item 3 in Lemma 4.3 correspond to (BI) and (BU), respectively. The alternative definition provides an intuitive way to derive an upper bound on the revenue that a dynamic mechanism can extract under budget constraint. For convenience, let $\text{MYE}(F_t)$ be the revenue of running a Myerson’s auction with a buyer’s distribution $F_t$.

**Lemma 4.4.** The revenue of any dynamic mechanism $(x_t, q_t, s_t^{ir}, \alpha_t, \beta_t)$ is bounded by

$$\text{REV} \leq \min \left( E \left[ \sum_{t \in [T]} s_t^{ir}(v_{(1,t)}) \right] + \sum_{t \in [T]} \text{MYE}(F_t), B \right).$$

**Proof.** Clearly, the revenue cannot exceed the buyer’s budget: $\text{REV} \leq B$. In the meantime, by $p_t = q_t + s_t^{ir}$, we have that $\text{REV} \leq E\left[ \sum_{t \in [T]} s_t^{ir}(v_{(1,t)}) \right] + E\left[ \sum_{t \in [T]} q_t(v_{(1,t)}) \right]$. Finally, $E[q_t(v_{(1,t)})] \leq \text{MYE}(F_t)$, because $\text{MYE}(F_t)$ is the optimal revenue of Stage-IC and Stage-IR mechanisms.

## 5 Non-Clairvoyant Mechanism with Budget Constraints

In this section, we design our non-clairvoyant mechanism with budget constraints for multiple buyers. Our characterization results in Section 3 (and hence Theorem 4.2) extend to multiple-buyer setting directly. However, we are facing new challenges of designing non-clairvoyant mechanisms.

At a high level, the difficulty comes from a dilemma of ensuring $(\alpha_t, \beta_t)$ and $(\alpha_{t-1}, \beta_{t-1})$.

- To achieve good revenue guarantees, if a buyer has already exhausted her budget, she must be excluded from the auction in some way;
- Naïve exclusion policies may break (BI): the excluded buyers must receive the same expected utility as what they would receive under the histories where their budgets are not exhausted.

A properly designed policy to exclude buyers who have likely exhausted budgets becomes the key.

### 5.1 Multiple-buyer Dynamic Mechanism Design with Budget Constraints

We generalize the notation to accommodate the needs for the design of our mechanism for multiple-buyer with budget constraints. We consider an environment with $n$ buyers from set $N$ and at stage $t$, buyer $i$’s valuation $v_i^t$ is drawn independently from a distribution $F_i$ over $V = [0, 3]$. We will use the vectorized symbol without superscript $\vec{\nu} = (v_1^t, \ldots, v_n^t)$ to represent the valuation profile at stage $t$, and similarly for other notations. As usual, we refer $\vec{\nu}_{-i}$ as the vector of valuations of all buyers other than buyer $i$. A clairvoyant dynamic mechanism is defined by a pair $(\vec{x}, \vec{p})$ such that $x_i^t : V^t \times (\Delta V)^{nT} \rightarrow [0, 1]$ specifies the allocation rule of buyer $i$ at stage $t$ and $p_i^t : V^t \times (\Delta V)^{nT} \rightarrow R$ specifies the payment rule of buyer $i$ at stage $t$. For each stage, the total allocation is no more than $\sum_i^n x_i^t(\hat{b}_{(1,t-1)}, \vec{\nu}_t) = 1$.

Similar to the single-buyer setting, we define the continuation utility of buyer $i$ to be his expected future utility assuming all the buyers report truthfully in the future. Formally,

$$U_t^i(\hat{b}_{(1,t)}) = \sum_{t'=t+1}^T E[\vec{\nu}_{(t+1,t)}] \left[ u_i^t(\hat{b}_{(1,t)}, \vec{\nu}_{(t+1,t)}; v_i^t) \right].$$
We are now ready to define the notion of dynamic Bayesian incentive-compatibility (DBIC)\(^7\) as follows: it is an optimal strategy for buyer \(i\) to report truthfully at stage \(t\), conditioned on the assumption that the other buyers report truthfully at stage \(t\) and all the buyers report truthfully in the future; formally, for any \(\tilde{b}_{(1,t-1)}\) and \(v_t^i\), we have
\[
v_t^i \in \arg\max_{b_t^i} \mathbb{E}_{\bar{v}_t^i} \left[ u_t^i \left( \tilde{b}_{(1,t-1)}(t), (b_t^i, \bar{v}_t^i) \right); v_t^i \right] + \bar{u}_t^i \left( \tilde{b}_{(1,t-1)}(t), (b_t^i, \bar{v}_t^i) \right). \tag{DBIC}\]

The participation requirements can be directly generalized in a natural way such that (EX-POST-IR) and (BUDGET) must be satisfied for all buyers. We denote buyer \(i\)’s public budget by \(B_t^i\). The requirement of non-clairvoyant mechanisms is similar in a way that neither the allocation rule nor the payment rule can rely on the future distributional information. Moreover, the non-clairvoyant mechanism must be DBIC for any possible future distributional information. Our design of the mechanisms is based on the following multi-buyer version of Assumption 2.5.

**ASSUMPTION 5.1.** There exists a sufficiently small \(\epsilon\) such that \(B_t^i \geq \bar{\gamma}_t^i/\epsilon\) for all buyer \(i\).

### 5.1.1 Multiple-buyer Generalized Bank Account Mechanism

We say the local-stage mechanism \(\langle \bar{x}_t, \bar{q}_t \rangle\) is Bayesian incentive-compatible if for any buyer \(i\) and any \(v_t^i\), we have
\[
v_t^i \in \arg\max_{b_t^i} \mathbb{E}_{\bar{v}_t^i} \left[ x_t^i(b_t^i, \bar{v}_t^i) \cdot v_t^i - q_t^i(b_t^i, \bar{v}_t^i) \right] \tag{STAGE-BIC}\]
and we say the stage mechanism is ex-post individual-rational if for any buyer \(i\) and \(\bar{v}_t\), we have
\[
x_t^i(\bar{v}_t) \cdot v_t^i - q_t^i(\bar{v}_t) \geq 0. \tag{STAGE-EXPOST-IR}\]

We extend Lemma 4.3 to define generalized bank account mechanisms in a multi-buyer environment:

**LEMMA 5.2.** \(\langle \bar{x}, \bar{q}, \bar{s}^{ir}, \bar{\alpha}, \bar{\beta} \rangle\) is a generalized bank account mechanism if for each stage \(t\),

- the local-stage mechanism \(\langle \bar{x}_t, \bar{q}_t \rangle\) is STAGE-BIC and STAGE-EXPOST-IR;
- for each buyer \(i\),
  \[ - \mathbb{E}_{\bar{v}_t^i} \left[ x_t^{ir,i}(\bar{a}_t-1, \bar{\beta}_t-1, \bar{\gamma}_t) \right] - s_t^{ir,i}(\bar{a}_t-1, \bar{\beta}_t-1) = c_t \geq 0 \text{ and does not depend on } \bar{a}_t-1 \text{ nor } \bar{\beta}_t-1; \]
  \[ - s_t^{ir,i}(\bar{a}_t-1, \bar{\beta}_t-1) \leq \alpha_t^i \text{ and for all } \bar{v}_t, q_t^i(\bar{a}_t-1, \bar{\beta}_t-1, \bar{\gamma}_t) + s_t^{ir,i}(\bar{a}_t-1, \bar{\beta}_t-1) \leq \beta_t^i; \]
- \(\bar{a}_t\) and \(\bar{\beta}_t\) compute the cumulative utility and the remaining budget for \(n\) buyers, respectively. Here, \(s_t^{ir,i}(\bar{a}_t-1, \bar{\beta}_t-1, \bar{\gamma}_t) = x_t^i(\bar{a}_t-1, \bar{\beta}_t-1, \bar{\gamma}_t) \cdot v_t^i - q_t^i(\bar{a}_t-1, \bar{\beta}_t-1, \bar{\gamma}_t)\) is buyer \(i\)’s utility from \(\langle \bar{x}_t, \bar{q}_t \rangle\).

### 5.2 The NonClairvoyantMulti Mechanism

In this section, we construct our NONCLAIRVOYANTMULTI mechanism for the multiple-buyer setting.

Building on top of the non-clairvoyant mechanism introduced by Mirrokni et al. [26] for (EX-POST-IR) exclusively, to handle the case with (BUDGET), we introduce an auxiliary parameter \(\bar{y}_t = (y_t^1, \cdots, y_t^n)\) such that \(y_t^i\) keeps track of the fraction of histories in which buyer \(i\)’s budget has been almost exhausted, i.e., \(\beta_t^i(\bar{v}_t) < 2\bar{\bar{\gamma}}\) for a history of valuations \(\bar{v}_t\). More formally, given a generalized bank account mechanism \(\langle \bar{x}, \bar{q}, \bar{s}^{ir}, \bar{\alpha}, \bar{\beta} \rangle\), we define \(y_t^i = \Pr_{\bar{v}_t}(\beta_t^i(\bar{v}_t) < 2\bar{\bar{\gamma}})\).

Notice that \(y_t^i\) is computed in an ex-ante manner, which is independent of the buyers’ realized valuations and bids, as well as any future prior information. Let \(\kappa\) be a constant and \(0 < \kappa < 1\). A buyer is considered as projected budget exhausted at stage \(t\) if \(y_t^{i-1} \geq \kappa\). In summary, that a buyer’s budget has been almost exhausted is a property defined on the ex-post level, while that a buyer is considered as projected budget exhausted is a property defined on the ex-ante level.

\(^7\)We note that while DIC for the single-buyer setting can be justified by an analogue of revelation principle in dynamic mechanism designs, the similar requirement for DBIC cannot be obtained. We refer the readers to Athey and Segal [3] and Pavan et al. [31] for a discussion on the relation between incentive compatibility and the revelation principle in the dynamic mechanism design. In this paper, we restrict our attention to design DBIC dynamic mechanisms.
For buyer $i$, we say she is \textit{active} at stage $t$ if $y_{t-1}^i < \kappa$. If buyer $i$ is no longer active, and hence projected budget exhausted, she will be excluded from all the future auctions; for stage $t$, let $N_t = \{ i \in N \mid y_{t-1}^i < \kappa \}$ be the set of active buyers. The next lemma provides a revenue upper bound based on $N_T$, the set of active buyers at the last stage. For convenience, let $\text{MYE}(\tilde{F}_t, N')$ be the revenue of the Myerson’s auction for the buyers in set $N'$ at stage $t$.

Lemma 5.3. The revenue of any dynamic mechanism $\langle \tilde{x}, \tilde{q}, \tilde{s}^{ir-1}, \tilde{a}, \tilde{\beta} \rangle$ can be bounded by

$$\text{REV} \leq \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i \in N_r} s_t^{ir-1}(\tilde{u}(1,t)) \right] + \sum_{t=1}^{T} \text{MYE}(\tilde{F}_t, N_T) + \sum_{i \in N \setminus N_T} B^i.$$

Intuitively, the revenue contribution from spend $s_t^{ir-1}$ is a direct extension from Lemma 4.4 for $N_T$. As for the contribution from the local-stage mechanisms, we simply bound inactive buyers’ revenue contributions by their budgets, and for the active buyers at the stage $T$, they can at most contribute $\text{MYE}(\tilde{F}_T, N_T)$ to the revenue in the local-stage mechanism at stage $t$.

We proceed to construct our non-clairvoyant mechanism via approximating each component in the upper bound stated in Lemma 5.3. We will provide a high level description of our mechanism, while the full description is deferred to Appendix E.1. For convenience, let $\langle x_t^{*,N'}, p_t^{*,N'}, \gamma \rangle$ and $\langle x_t^{S,N'}, p_t^{S,N'}, \gamma \rangle$ be the allocation rule and the payment rule for buyer $i$ in the Myerson’s auction and the second price auction for the buyers in set $N'$ at stage $t$, respectively.

5.2.1 Approximate $\sum_{t=1}^{T} \text{MYE}(\tilde{F}_t, N_T)$. At stage $t$, we will simply offer Myerson’s auction to $N_t$, the set of active buyers at stage $t$, with probability $1/5$. More precisely, given a bidding profile $\tilde{b}_t$, for a buyer $i \in N_t$ whose budget is not almost exhausted, her allocation probability is $x_t^{s,N_t}(\tilde{b}_t)$ and payment is $p_t^{s,N_t}(\tilde{b}_t)$. As for a buyer $i \in N_t$ whose budget is almost exhausted, for any bidding profile, we will always compensate the buyer via a negative payment, which equals to her expected utility in Myerson’s auction, i.e., $\mathbb{E}_{\tilde{q}_t}[x_t^{s,N_t}(\tilde{v}_t) \cdot \tilde{v}_t - p_t^{s,N_t}(\tilde{v}_t)]$, without allocating anything. As a result, for a buyer $i \in N_t$, her expected utility from the Myerson’s auction is always $\mathbb{E}_{\tilde{q}_t}[x_t^{s,N_t}(\tilde{v}_t) \cdot \tilde{v}_t - p_t^{s,N_t}(\tilde{v}_t)]$, no matter whether her budget has been almost exhausted or not.

5.2.2 Approximate $\mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i \in N_r} s_t^{ir-1}(\tilde{u}(1,t)) \right]$. To further upper bound $\sum_{i \in N_r} s_t^{ir-1}(\tilde{u}(1,t))$, we refer to Lemma 5.2, which provides two upper bounds for $\sum_{i \in N_r} s_t^{ir-1}(\tilde{u}(1,t))$:

$$\sum_{i \in N_T} s_t^{ir-1}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}) \leq \sum_{i \in N_T} \alpha_{t-1}^{i} \quad \text{and} \quad \sum_{i \in N_T} s_t^{ir-1}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}) \leq \sum_{i \in N_T} \beta_{t-1} \mathbb{E}[s_t^{ir-1}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{v}_t)],$$

where $\alpha_{t-1}^{i}$ is buyer $i$’s cumulative utility from first $t - 1$ stages, and $s_t^{ir-1}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{v}_t)$ is buyer $i$’s utility from a \textit{stage-BIC} and \textit{stage-EXPOST-IR} local-stage mechanism $\langle \tilde{x}_t, \tilde{q}_t \rangle$. Henceforth, we aim to maximize $\sum_{i \in N_T} s_t^{ir-1}(\tilde{u}(1,t))$ via maximizing $\sum_{i \in N_T} \alpha_{t-1}^{i}$ and $\sum_{i \in N_T} \beta_{t-1} \mathbb{E}[s_t^{ir-1}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{v}_t)]$.

Maximizing $\sum_{i \in N_T} \alpha_{t-1}^{i}$. Our ultimate goal is to maximize revenue, while $\sum_{i \in N_T} \alpha_{t-1}^{i}$ is the summation of cumulative utilities of all buyers. We instead aim to maximize the sum of revenue and utility, i.e., the welfare. To maximize welfare, one can simply implement the second price auction per stage. We offer second price auction to $N_t$ with probability $2/5$. Similar to the Myerson’s auction, given a bidding profile $\tilde{b}_t$, for buyer $i \in N_t$ whose budget is not almost exhausted, her allocation probability is $x_t^{S,N_t}(\tilde{b}_t)$ and payment is $p_t^{S,N_t}(\tilde{b}_t)$. For buyer $i \in N_t$ whose budget is almost exhausted, for any bidding profile, we always compensate the buyer via a negative payment, which equals to her expected utility in the second price auction, without allocating anything.

Maximizing $\sum_{i \in N_T} \mathbb{E}_{\tilde{q}_t}[s_t^{ir-1}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{v}_t)]$. To maximize the total surplus of an auction, with probability $2/5$, we offer the money burning mechanism [20] and burn the surplus. Formally, let
\[ N_t^+ = \{ i \in N_t \mid \beta_t^i \geq 2\tau \} \] be the set of active buyers whose budgets are not almost exhausted. We compute a stage-BIC and stage-expost-IR mechanism \((x_t^{\text{BURN}}, q_t^{\text{BURN}})\) that maximizes the total surplus for buyers in \(N_t^+\), such that for each buyer \(i \in N_t^+\), her expected utility is at most \(\frac{\delta}{2} \alpha_{t-1}^i\):

\[
\max \sum_{i \in N_t^+} \mathbb{E}_{\tilde{v}_t} \left[ x_t^{\text{BURN},i}(\tilde{v}_t) \cdot v_t^i - q_t^{\text{BURN},i}(\tilde{v}_t) \right] \\
\text{s.t.} \quad \mathbb{E}_{\tilde{v}_t} \left[ x_t^{\text{BURN},i}(\tilde{v}_t) \cdot v_t^i - q_t^{\text{BURN},i}(\tilde{v}_t) \right] \leq \frac{\delta}{2} \alpha_{t-1}^i \quad \forall i \in N_t^+ \\
\text{(Money Burning)}
\]

For \(i \in N_t^+\), we allocate according to \(x_t^{\text{BURN},i}\) and charge an extra payment \(\mathbb{E}_{\tilde{v}_t} \left[ x_t^{\text{BURN},i}(\tilde{v}_t) \cdot v_t^i - q_t^{\text{BURN},i}(\tilde{v}_t) \right] \) in addition to \(q_t^{\text{BURN},i}\). In such a mechanism, each buyer’s expected utility is exactly 0.

### 5.2.3 Revenue Performance

In summary, we will offer a mixture of the Myerson’s auction, the second price auction, and the money burning mechanism per stage.

**Theorem 5.4.** The revenue of the NonClairvoyantMulti mechanism is at least \(\left(\frac{1}{\delta + 2\sqrt{\delta}} - \frac{2\epsilon}{1 - 2\epsilon}\right)\) of the revenue of the optimal clairvoyant dynamic mechanism when \(\kappa = \frac{1}{\sqrt{\delta+1}}\).

We provide a high-level sketch and defer the detailed proof to Appendix E.3. First, it is straightforward that the revenue of NonClairvoyantMulti is at least \(\sum_{i \in N_t^+} (B^i - 2\tau)\) by the definition of inactive buyers. For \(\sum_{t=1}^{T} \text{MYE}(\tilde{F}_t, N_T)\), notice that at stage \(t\), each active buyer \(i \in N_t\) will pay in the Myerson’s auction for \(N_t\) with probability at least \((1 - \kappa)\). Therefore, the revenue obtained from the Myerson’s auction at stage \(t\) is at least \((1 - \kappa)\text{MYE}(\tilde{F}_t, N_T) \geq (1 - \kappa)\text{MYE}(\tilde{F}_t, N_T)\) as \(N_T \subseteq N_t\) for any \(t\). For the spend \(\mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i \in N_t} s_t^{ir^{-1}}(\tilde{a}_{1,t}) \right]\), we argued that the second price auction is welfare-maximizing. Hence, it maximizes the sum of spend (the second highest bid) and deposit (the difference between the first and the second highest bid). Moreover, the money burning mechanism maximizes the spend since \(s_t^{ir^{-1}}(\tilde{a}_{1,t-1}, \tilde{\beta}_{t-1}, \tilde{\alpha}_t) \leq \min \{ \mathbb{E}_{\tilde{v}_t} \left[ s_t^{ir^{-1}}(\tilde{a}_{1,t-1}, \tilde{\beta}_{t-1}, \tilde{\alpha}_t) \right], \alpha_{t-1}^i \} \) due to Lemma 5.2. Finally, if a buyer is excluded from the second price auction or the money burning mechanism, her contribution to the spend is almost maximized since we cannot charge her more than her budget.

### 6 Conclusion

In this paper, we provide a general framework, the LHC mechanism, for dynamic mechanism design in complex environments. The LHC mechanism enables a connection between clairvoyant mechanisms and non-clairvoyant mechanisms, and furthermore, provides an upper bound of the revenue from any clairvoyant dynamic mechanisms. We apply the LHC mechanism to the settings with budgets to obtain generalized bank account mechanisms, and design a non-clairvoyant mechanism that achieves non-trivial revenue performance against the optimal clairvoyant mechanism.

Future research can consider designing non-clairvoyant mechanisms under other practical constraints, such as ROI constraints. We also leave it as an interesting open problem to design non-clairvoyant mechanisms under budgets without Assumption 2.5, i.e., the buyers’ budgets may be small compared to their maximum valuations per stage. Moreover, it is interesting to consider the setting with private constraints, e.g., private budget constraints. However, one may need to establish a new framework beyond LHC mechanisms to tackle the incentives with private constraints.

### REFERENCES


A GENERAL ENVIRONMENT WITH PUBLIC CORRELATIONS

In this section, we extend the framework introduced in the main content for the independent valuation setting to a general with public correlation setting by highlighting the key differences.

A.1 Public Correlation

Under this general setting, we consider the correlations between private values and the historical mechanism outcome, i.e., the allocations from previous stages may influence the prior distribution for the future items. Instead, the prior distributions conditioned on the historical allocations will be independent of the buyers’ reports and private types in previous stages. In other words, the correlation is between the values of future items and the historical allocations, which are public information. In this way, we can extend our framework and results without introducing too much complications (such as belief systems).

Formally, the private type at each stage $t$ is independently drawn according to the prior distribution $F_t$, while we allow $F_t$ to vary with publicly observable information such as the historical allocations, which we call the environment history.

**Definition A.1 (Environment History).** An environment history $h^E_t \in \mathcal{H}^E_t$ refers to the sequence of mechanism outcomes until the end of stage $t$, including the historical allocation $\langle x_1, \ldots, x_t \rangle$ and even the implementation of randomized outcomes (such as lottery allocations).\(^8\)

For the multiple agent case, the environment history $h^E_t$ will be a vector. The prior distribution over $\Theta_t$ may depend on the environment history $h^E_t$. In this case, the prior $F_t \in \mathcal{F}_t$ is a function mapping an environment history to a distribution over the private types, i.e., $F_t : \mathcal{H}^E_{t-1} \rightarrow \Delta \Theta_t$, where $\Delta S$ is the set of all distributions over the ground set $S$. Formally as the following assumption.

**Assumption A.2 (Public Correlation Prior Knowledge).** In each stage $t$, the prior $F_t$ is common knowledge among the designer and the agent(s). Conditioned on the environment history $h^E_{t-1}$, the agent’s private types are drawn as $\theta_t \sim F_t(h^E_{t-1})$, independently from previous stages.

We highlight that the prior knowledge structure by Assumption A.2 based on environment histories enables us to capture many important valuation models without introducing belief systems. For example, a $k$-unit demand agent can be modeled by setting $F_t(h^E_{t-1})$ to the zero-valuation distribution (i.e., with probability 1 having zero value for upcoming items) after she receiving $k$ units from previous stages. As the items are allowed to be heterogeneous, we can easily model other complicated combinatorial demands such as substitutes, complements, decreasing/increasing marginal valuations, or even hierarchical combinations of them. However, we also notice that without belief systems, our model is not able to capture all general combinatorial valuations. As an extreme example, if the total valuation of the received items is the maximum of the values of each individual items, the marginal value from the upcoming item is then correlated with the valuation of the previous allocations rather than the previous allocation only.

With the environment history define above, the history of a general dynamic mechanism is then decomposed into the environment history and the historical reports of the agents.

**Definition A.3 (History (General setting)).** A mechanism history $h^M_t \in \mathcal{H}^M_t = \Theta_1 \times \cdots \times \Theta_t$ consists of the historical reports of the agent private types, i.e., $h^M_t = \hat{\theta}_{1:t}$. A history $h_t \in \mathcal{H}_t = \mathcal{H}^E_t \times \mathcal{H}^M_t$ is the combination of the environment history and the mechanism history, i.e., $h_t = \langle h^E_t, h^M_t \rangle \in \mathcal{H}_t$.

\(^8\)As an example of randomized outcomes, one could define the allocation space $X_t$ to be the distribution over $\{0, 1\}$ for the indivisible single item setting. Then the environment history will include the allocation as the distribution $x_t$ and the actual implementation variable $X_t \sim x_t$. 
A.2 Lossless History Compression with Public Correlation

We next extend the definitions of Lossless History Compression (LHC) as well as the LHC mechanisms to the with public correlation setting. Naturally, with the additional correlation, a lossless history compression must contain certain information to encode the correlation. In other words, one must be able to recover the correct prior distribution from the compression.

**Definition A.4 (Lossless History Compression (LHC) with Public Correlation).** A lossless history compression (or simply compression) $s = (s_1, \ldots, s_T)$ is a sequence of functions — each maps the history and previous payments to a succinct summary $s_t(h_t, p_{(1,t)})$, i.e., $s_t: \mathcal{H}_t \times \mathbb{R} \rightarrow \mathcal{S}_t$ — such that there exist accompanying mechanism constraints $\langle \mathcal{F}_t, \mathcal{P}_t \rangle$ and prior knowledge $\hat{\mathcal{F}}_t$ defined on $s_t$, $\hat{\mathcal{F}}_t(s_{t-1}) = F_t(h_{t-1}^F)$, $\mathcal{F}_t(s_{t-1}, \hat{\theta}_t, \hat{F}_{(1,T)}) = \mathcal{F}_t(h_{t-1}, \hat{\theta}_t, F_{(1,T)})$, $\mathcal{P}_t(s_{t-1}, x_t, \hat{\theta}_t, \hat{F}_{(1,T)}) = \mathcal{P}_t(h_{t-1}, x_t, p_{(1,t-1)}, \hat{\theta}_t, F_{(1,T)})$, and a corresponding summary update rule $\Lambda = \langle \Lambda_1, \ldots, \Lambda_T \rangle$: $s_t(h_t, p_{(1,t)}) = \Lambda_t(s_{t-1}(h_{t-1}, p_{(1,t-1)}), \hat{\theta}_t, x_t, p_t)$.

Compared with Definition 3.1, the only difference is on the prior knowledge recovery, $\hat{\mathcal{F}}_t(s_{t-1}) = F_t(h_{t-1}^F)$. Such a recovery is not required in the independent setting because the priors are invariant with the environment history. Below, we restate the definition of LHC mechanisms with public correlation. The only difference, compared with Definition 3.3, is that the expectation below (State-Update) is taken over $\hat{\mathcal{F}}(s_{t-1})$ rather than $\mathcal{F}_t$.

**Definition A.5 (LHC Mechanism).** An LHC mechanism $\vec{M} = (\vec{x}, \vec{p})$ is defined based on a lossless history compression $s$ with $\sigma_t = (s_t, \mu_t)$ as its state, where $\mu_t = \vec{U}_t(\sigma_t)$ is the promised continuation utility.\(^9\) In each stage $t$,

1. the agent realizes her private type $\theta_t$, reports $\hat{\theta}_t$, and accrues stage utility $u_t(\sigma_{t-1}, \hat{\theta}_t; \theta_t) = v_t(\hat{\theta}_t, x_t(\sigma_{t-1}, \hat{\theta}_t)) - \hat{p}_t(\sigma_{t-1}, \hat{\theta}_t)$ from the stage allocation $x_t(\sigma_{t-1}, \hat{\theta}_t)$ and payment $\hat{p}_t(\sigma_{t-1}, \hat{\theta}_t)$;
2. the next state $\sigma_t$ is determined according to the following state update rule $\sigma_t = \langle s_t, \mu_t \rangle = \langle \Lambda_t(s_{t-1}, \hat{\theta}_t, x_t, \hat{p}_t), \mu_t(s_{t-1}, \hat{\theta}_t) \rangle$, (State Update)

   where $\mathbb{E}_{\hat{\theta}_t \sim \hat{F}_t(s_{t-1})} [u_t(\sigma_{t-1}, \hat{\theta}_t; \theta_t) + \mu_t(\sigma_{t-1}, \hat{\theta}_t)] = \mu_t$;
3. the stage outcome $\langle x_t, \hat{p}_t \rangle$ satisfies the following incentive constraint $\theta_t \in \arg\max_{\hat{\theta}_t} u_t(\sigma_{t-1}, \hat{\theta}_t; \theta_t) + \mu_t(\sigma_{t-1}, \hat{\theta}_t)$, (LHC-IC)

and the allocation and payment constraints $x_t \in \mathcal{X}_t, \hat{p}_t \in \mathcal{P}_t$.

A.3 All Results Apply for the General Setting

All our previous results about the framework directly extend to the with public correlation setting, namely, Theorem 3.5, Theorem 3.8, Theorem 3.9, and Lemma 3.11.

\(^9\)We use $\vec{U}_t(\sigma_t)$ to denote the continuation utility of the LHC mechanism $\vec{M}$ under state $\sigma_t$, which is an analog to the definition of $U_t(h_t)$ with respect to the uncompressed history $h_t$. Note that the expectations in the definition of $\vec{U}_t(\sigma_t)$ is taken with respect to the prior knowledge $\hat{\mathcal{F}}(s_{t-1}), \ldots, \hat{\mathcal{F}}(s_{t-1})$, which can be evaluated based on the compression $s_t$ rather than the entire history $h_t^F$. 
B ECONOMICS MOTIVATIONS OF THE GENERAL MODEL

In this section, we briefly discuss the economics motivations behind our modeling choice and the limitations of our current setup.

Not involving belief systems. The key limitation comes from the fact that we did not introduce the belief system in our general environment. The goal of this paper is not to study a universal design paradigm for arbitrary dynamic environments, but to highlight a special set of general environments for which we can restrict our attention to the simple LHC mechanisms while not losing any optimality in terms of the design objective. The belief systems, however, are usually quite complicated and hardly be summarized by a simple state. Whereas in our current setup, for most practical settings, the state could be simply several numbers. The implication of not introducing belief systems is that the agents cannot have hidden private states evolving over time. Otherwise, to guarantee the dynamic incentive compatibility, the designer must model the agents’ belief systems or otherwise can only use state-wise independent mechanisms.

Therefore, we are able to model cross-stage correlations based on publicly observable information, but our model does not allow any cross-stage correlations based on privately observable information. Such a limitation further restricts us to valuation models that are additive across different stages. Because the marginal valuation gain from the current stage must be independent with the historical private types. In other words, the partial derivative of the marginal valuation gain in this stage with respect to any historical private types must be zero, implying an additive formulation of the overall valuation. As a result, we define the overall utility to be

$$u = \sum_{t \in [T]} u_t = \sum_{t \in [T]} v_t(\theta_t, x_t) - p_t.$$  

Monetary transfer. In our model, we force the payment at each stage to be equivalent in the agent utility and designer objective, i.e., both are defined on the total payment $p = \sum_{t \in [T]} p_t$ rather than each single payment $p_t$. However, we would emphasize that such an enforcement does not limit our model. In fact, as our definition for item valuations are general enough such that even the monetary transfers could be modeled as the allocation of a special item, whose allocation could be any real number. Even the current explicitly defined payments could be disabled by setting the payment constraints to $P_t = \{0\}$.

One should interpret our enforcement on the current payments only as a notational convenience for the monetary transfers that are value persistent over time. For any other type of monetary transfers that are not value persistent, one can still model it in our model as special items. Finally, as a side note, our framework can also model agents with general financial constraints beyond budget constraints, i.e., the marginal cost of the payments is increasing, by using the trick that transfers the additional cost of agents to an increasing tax inside the designer’s objective (Goel et al. [18]). By financial constraints, we refer to the settings where the buyer, for example, can borrow some money and spend more in the mechanism, while paying interests as additional cost. In general, the cost of the buyer can be an increasing and continuous function of her total payment. Moreover, we briefly describe the trick here: one can formalize the program by using allocation and cost as the variables, rather than allocation and payment. So the constraints remain the same as the basic setting. Instead, the revenue in the objective will be the sum over all the payments which can be formulated as functions of the costs by each buyer. Note that the payment-to-cost functions must be invertible so that we can get well-defined cost-to-payment functions. As a side note, the trick won’t apply to the budget constraint because it cannot be modeled via a continuous and invertible payment-to-cost function.
C   MISSING PROOFS IN SECTION 3

C.1   Proof of Theorem 3.8

Proof. The direction that if a dynamic mechanism \( M \) satisfies (State-UI) and (Stage-IC), then it satisfies non-clairvoyant (DIC) is easy to see: the two terms inside the argmax operator of (DIC) are \( u_t \) and \( \bar{U}_t \). By removing the always constant term \( \bar{U}_t \), (DIC) becomes exactly equivalent to (Stage-IC), which is satisfied by \( M \) according to the hypothesis.

We then prove the direction that dynamic mechanism \( M \) satisfies non-clairvoyant (DIC) and then it also satisfies (Stage-IC) and (State-UI).

We now proceed to prove the hypothesis that \( M \) satisfies (State-UI) up to stage \( T \) by induction from \( \tau = 1 \) to infinity. In particular, we are going to show that for all \( t < T \) and \( h_t \)

\[
\bar{U}_{t,T}(h_t) = \mathbb{E}\left[ \sum_{t=t+1}^{T} u_t(h_{t-1}, \theta_t; \theta_t) \right] = g_{t,T}(F_{1,T}),
\]

where \( h_t \) is the history resulted from reporting truthfully from stage \( t + 1 \) to \( \tau \) after \( h_t \). In other words, \( \bar{U}_{t,T}(h_t) \) is a constant depending on the prior \( F_{1,T} \) only. In particular, State-UI is equivalent to that the above condition holds for all \( T = t + 1 \).

When \( t = 0 \), it is obviously true as there is only one possible history \( h_0 = \emptyset \). Suppose the hypothesis is true up to stage \( t - 1 \) and we then prove it for stage \( t \). Consider any state (or history) \( h_{t-1} \), according to the hypothesis, we know that \( \forall T \geq t \)

\[
\bar{U}_{t-1,T}(h_{t-1}) = g_{t-1,T}(F_{1,T}).
\]

where \( h_{t-1}^{(t)} \) is the history obtained from \( h_{t-1} \) where the agent reports \( \theta_t \) at stage \( t \). Moreover, by definition, we have

\[
\bar{U}_{t-1,T}(h_{t-1}) = \mathbb{E}_{\theta_t}\left[ u_t(h_{t-1}, \theta_t; \theta_t) + \bar{U}_{t,T}(h_{t-1}^{(t)}) \right] = \bar{U}_{t-1,T}(h_{t-1}) + \mathbb{E}\left[ \bar{U}_{t,T}(h_{t-1}^{(t)}) \right].
\]

As a result, we have that

\[
\mathbb{E}\left[ \bar{U}_{t,T}(h_{t-1}^{(t)}) \right] = \bar{U}_{t-1,T}(h_{t-1}) - \bar{U}_{t-1,T}(h_{t-1}) = g_{t-1,T}(F_{1,T}) - g_{t-1,T}(F_{1,T})
\]

is a constant independent of \( h_{t-1} \). All that remains to show is that \( \bar{U}_{t,T}(h_{t-1}^{(t)}) \) is a constant independent of \( \theta_t \).

As we previously argued in Section 2.3, \( M \) must satisfy (Stage-IC). Putting (DIC) and (Stage-IC) together, we have that for each stage \( t \), both \( \langle x_t, p_t \rangle \) and \( \langle x_t, p_t \rangle \) are stage-wise IC mechanisms with the same allocation rule, where

\[
p_t^{(t)}(h_{t-1}, \theta_t) = p_t(h_{t-1}, \theta_t) - \bar{U}_{t,T}(h_{t-1}^{(t)}).
\]

According to Assumption 3.6, for any given history \( h_{t-1} \), the difference between \( p_t \) and \( p_t^{(t)} \), i.e., \( \bar{U}_{t,T}(h_{t-1}^{(t)}) \), must be a constant of \( \theta_t \), i.e., \( \forall \theta_t \in \Theta_t, \bar{U}_{t,T}(h_{t-1}^{(t)}) \) remains the same, which finishes the proof of the induction, and therefore, \( M \) satisfies (State-UI).

\[\Box\]

C.2   Proof of Theorem 3.9

Proof. First of all, note that the (Payment Realignment) does not change the total payment of the mechanism on any path \( h_T \in \mathcal{H}_T \):

\[
\sum_{t \in [T]} p_t' = \sum_{t \in [T]} p_t - \bar{U}_t + \bar{U}_{t-1} - c_t = \bar{U}_0 - (c_1 + \cdots + c_T) - \bar{U}_T + \sum_{t \in [T]} p_t = \sum_{t \in [T]} p_t.
\]
where we omit the dependence on \( h_t \) for \( \bar{U}_t \) for convenience. For the above equation, \( \bar{U}_T = 0 \) because there is no future stage after stage \( T \), and \( \bar{U}_0 - (c_1 + \cdots + c_T) = 0 \) by the hypothesis of (Payment Realignment).

We start with the direction that \( M \) is converted into \( M' \) under a valid (Payment Realignment) \( p' \), then \( M' \) is Stage-IC and State-UI. By substituting \( p'_t = p_t - \bar{U}_t + \bar{U}_{t-1} - c_t \) into the (DIC) constraint of \( M \), we have

\[
\theta_t \in \argmax u_t + \bar{U}_t = \argmax u_t + p_t - (p'_t + \bar{U}_t - \bar{U}_{t-1} + c_t) + \bar{U}_t
\]

\[
= \argmax u'_t + \bar{U}_{t-1} - c_t = \argmax u'_t.
\]

The last equality is because \( \bar{U}_{t-1} - c_t \) is constant of the report \( \hat{\theta}_t \). Therefore \( M' \) is Stage-IC.

On the other hand, because the expected stage utility

\[
\mathbb{E}[u'_t] = \mathbb{E}[u_t + \bar{U}_t - \bar{U}_{t-1} + c_t] = \mathbb{E}[u_t + \bar{U}_t] - \bar{U}_{t-1} + c_t = c_t
\]

is a constant, \( M' \) is also State-UI, where \( \bar{U}_{t-1} = \mathbb{E}[u_t + \bar{U}_t] \) by definition. Finally, \( M' \) also respect all the (Allocation) and (Payment) constraints, because the allocation \( x \) remains the same as in \( M \), and the (Payment) constraints are respected according to (Payment Realignment).

We now prove the other direction that if \( M \) can be transformed into a Stage-IC and State-UI mechanism \( M' = \langle x, p' \rangle \), then there exist constants \( c_1 + \cdots + c_T = \bar{U}_0 \) such that \( p'_t = p_t - \bar{U}_t + \bar{U}_{t-1} - c_t \in \mathcal{P}_t \). First of all, because \( M' \) is a valid dynamic mechanism, it must satisfy the (Payment) constraints and hence \( p'_t \in \mathcal{P}_t \).

Similar to the argument in the proof of Theorem 3.8, since both \( p'_t \) and \( p_t - \bar{U}_t \) implement the allocation \( x_t(h_{t-1}, \hat{\theta}_t) \), by Lemma 3.6, their difference \( p_t - \bar{U}_t - p'_t \) must be a constant in terms of \( h_{t-1} \). Moreover, \( \mathbb{E}[u'_t(h_{t-1}, \theta_t; \theta_t)] \) is a constant of \( h_{t-1} \) by (State-UI), where

\[
\mathbb{E}[u'_t] = \mathbb{E}[u_t + p_t - p'_t] = \mathbb{E}[u_t + \bar{U}_t] + \mathbb{E}[p_t - \bar{U}_t - p'_t] = \bar{U}_{t-1} + p_t - \bar{U}_t - p'_t.
\]

Here \( \mathbb{E}[p_t - \bar{U}_t - p'_t] = p_t - \bar{U}_t - p'_t \) because it is a constant in terms of \( \hat{\theta}_t \). Therefore, we conclude that \( c_t := p_t - \bar{U}_t + \bar{U}_{t-1} - p'_t = \mathbb{E}[u'_t] \) is a constant.

\[\square\]

C.3 Proof of Lemma 3.11

Proof. Let \( a_0 = 0 \) and \( \zeta_0 \) be a function such that for any history \( h \), \( \zeta_0(h) = 0 \). For convenience, let \( \mathcal{K} = \{0, \cdots, K\} \). Moreover, let

\[
t_t = \min_{h_t \in \mathcal{H}_t} \left[ \min_{k \in \mathcal{K}} \left\{ \sum_{r=1}^{t} \frac{\theta_r x_r}{1 + \alpha_k} - p_r - \frac{\zeta_k(h_r)}{1 + \alpha_k} + \bar{U}_t(h_t) \right\} \right],
\]

so that \( t_0 = \bar{U}_0(\emptyset) \). Moreover, let \( c_t = -t_t + t_{t-1} \) for \( t < T \) and \( c_T = t_{T-1} \) and clearly, we have \( \sum_{t \in [T]} c_t = t_0 = \bar{U}_0(\emptyset) \). We first claim that \( c_t \geq 0 \) for all \( t \). To prove such a claim, let

\[
\hat{h}_t \in \argmin_{h_t \in \mathcal{H}_t} \left[ \min_{k \in \mathcal{K}} \left\{ \sum_{r=1}^{t} \frac{\theta_r x_r}{1 + \alpha_k} - p_r - \frac{\zeta_k(h_r)}{1 + \alpha_k} + \bar{U}_t(h_t) \right\} \right].
\]
In addition, let \( \hat{h}_{t|t'} \) be the history of \( \hat{h}_t \) restricted to the first \( t' \) stages, and \( x_{t|t'} \) and \( p_{t|t'} \) be the allocation probability and the payment at stage \( t' \) corresponding to history \( \hat{h}_t \). Then, for \( t < T \),
\[
\hat{t}_{t-1} = \min_{k \in K} \left\{ \sum_{r=1}^{t-1} \frac{\theta_r x_{t-1|r}}{1 + \alpha_k} - p_{t-1|r} - \frac{\zeta_k(\hat{h}_{t-1})}{1 + \alpha_k} \right\} + \hat{U}_t(\hat{h}_{t-1})
\]
\[
= \min_{k \in K} \left\{ \sum_{r=1}^{t-1} \frac{\theta_r x_{t-1|r}}{1 + \alpha_k} - p_{t-1|r} - \frac{\zeta_k(\hat{h}_{t-1})}{1 + \alpha_k} \right\} + \mathbb{E}_{\theta_t} \left[ u_t(\hat{h}_{t-1}, \theta_t; \theta_t) + \hat{U}_t(\hat{h}_{t|t}) \right]
\]
\[
\geq \min_{k \in K} \left\{ \sum_{r=1}^{t-1} \frac{\theta_r x_{t-1|r}}{1 + \alpha_k} - p_{t-1|r} - \frac{\zeta_k(\hat{h}_{t-1})}{1 + \alpha_k} \right\} + \min_{\theta_t} \left[ u_t(\hat{h}_{t-1}, \theta_t; \theta_t) + \hat{U}_t(\hat{h}_{t|t}) \right]
\]
where \( \hat{h}_{t|t'} \) is the history resulted from reporting \( \theta_t \) after \( \hat{h}_t \), and the first equation follows the definition of continuation utility and the inequality follows a basic algebraic relationship between minimum and expectation. Moreover, we have
\[
\hat{t}_{t-1} \geq \min_{k \in K} \left\{ \sum_{r=1}^{t-1} \frac{\theta_r x_{t-1|r}}{1 + \alpha_k} - p_{t-1|r} - \frac{\zeta_k(\hat{h}_{t-1})}{1 + \alpha_k} \right\} + \min_{\theta_t} \left[ u_t(\hat{h}_{t-1}, \theta_t; \theta_t) + \hat{U}_t(\hat{h}_{t|t}) \right]
\]
\[
\geq \min_{k \in K, h_t} \left\{ \sum_{r=1}^{t-1} \frac{\theta_r x_{t-1|r}}{1 + \alpha_k} - p_{t-1|r} - \frac{\zeta_k(\hat{h}_{t-1})}{1 + \alpha_k} \right\} + \min_{\theta_t} \left[ u_t(\hat{h}_{t-1}, \theta_t; \theta_t) + \hat{U}_t(\hat{h}_{t|t}) \right]
\]
\[
\geq \min_{k \in K, h_t} \left\{ \sum_{r=1}^{t-1} \frac{\theta_r x_{t-1|r}}{1 + \alpha_k} - p_{t-1|r} - \frac{\zeta_k(\hat{h}_{t|t})}{1 + \alpha_k} \right\} + \min_{\theta_t} \left[ u_t(\hat{h}_{t-1}, \theta_t; \theta_t) + \hat{U}_t(\hat{h}_{t|t}) \right]
\]
where the first inequality follows that a summation of minimums is at least the minimum of a summation, and the second inequality follows \( \zeta_k \) is a monotone history function and \( \alpha_k \in \mathbb{R}_+ \). As for \( t = T \), we clearly have \( c_T = t_{T-1} \geq t_T \geq 0 \) since the original mechanism satisfies (PAYMENT) and \( \hat{U}_T(h_T) = 0 \) for \( h_T \in \mathcal{H}_T \).

We finish the proof by showing \( \forall t \in [T], h_{t-1} \in \mathcal{H}_{t-1}, \theta_t \in \Theta_t, \)
\[
p'_t = p_t - \hat{U}_t(h_t) + \hat{U}_{t-1}(h_{t-1}) - c_t \in \mathcal{P}_t.
\]
Notice that for each \( k \in \mathcal{K} \), we have
\[
\sum_{r=1}^{t} \frac{\theta_r x_r}{1 + \alpha_k} - p'_t - \frac{\zeta_k(h_t)}{1 + \alpha_k}
\]
\[
= \sum_{r=1}^{t} \frac{\theta_r x_r}{1 + \alpha_k} - \left( p_r - \hat{U}_t(h_t) + \hat{U}_{t-1}(h_{t-1}) - c_t \right) - \frac{\zeta_k(h_t)}{1 + \alpha_k}
\]
\[
= \left[ \sum_{r=1}^{t} \frac{\theta_r x_r}{1 + \alpha_k} - p_r - \frac{\zeta_k(h_t)}{1 + \alpha_k} \right] - \left( \hat{U}_0(\emptyset) - \hat{U}_t(h_t) - t_0 + t_t \right)
\]
\[
= \sum_{r=1}^{t} \frac{\theta_r x_r}{1 + \alpha_k} - p_r - \frac{\zeta_k(h_t)}{1 + \alpha_k} + \hat{U}_t(h_t) \geq 0,
\]
where the second equality is due to \( \hat{U}_0(\emptyset) = t_0 \) and the last equality follows the definition of \( t_t \). \( \Box \)
C.4 Proof of Lemma 3.13

Proof. We will proceed by induction on stage \( \tau \) from 0 to \( T \) to show that the mechanism can be turned to a mechanism that (i) satisfies (DIC), (ALLOCATION), and (PAYMENT) constraints, and (ii) is symmetric with respect to the given compression \( s \) for the first \( \tau \) stages, i.e., for any \( t \leq \tau \) and \( t' > t \), when \( s_t(h_t, p_{(1,t)}) = s_t(h'_t, p'_{(1,t)}) \) and \( \mathcal{U}_t(h_t) = \mathcal{U}_t(h'_t) \), we have for all \( \theta_{(t+1,t')} \),

\[
\begin{align*}
x_t'(h_{t'-1}, \theta_{t'}) &= x_t(h^*_{t'-1}, \theta_{t'}) \quad \text{and} \quad p_t'(h_{t'-1}, \theta_{t'}) = p_t(h^*_{t'-1}, \theta_{t'}).
\end{align*}
\]

It is obvious that the base \( \tau = 0 \) is true. For the inductive step, let \( \mathcal{M} = \langle x, p \rangle \) be the mechanism that is symmetric for the first \( \tau - 1 \) stages. We partition the set of all possible histories \( \mathcal{H}_t \) from the first \( \tau \) stages into classes with the same pair of compressed history \( \tilde{s} \) and continuation utility \( \tilde{u} \), i.e., for \( \langle \tilde{s}, \tilde{u} \rangle \), define

\[
S_t(\tilde{s}, \tilde{u}) = \{ h_t \mid s_t(h_t, (p_{(1,t)}) = \tilde{s}, \mathcal{U}_t(h_t) = \tilde{u} \}.
\]

Next, we choose \( \hat{h}_t(\tilde{s}, \tilde{u}) \) that maximizes the expected designer objective \( ^{10} \)

\[
\hat{h}_t(\tilde{s}, \tilde{u}) \in \arg\max_{h_t \in S_t(\tilde{s}, \tilde{u})} \mathbb{E}[\text{OBJ}(h_T, p) \mid h_t],
\]

where \( h_T \) is the history resulted from reporting truthfully for stages between \( \tau + 1 \) and \( T \) after \( h_\tau \), and the expectation is taken over the agent’s types for stages between \( \tau + 1 \) and \( T \). We then define the mechanism \( \langle x', p' \rangle \) such that \( x_t' = x_t \) and \( p_t' = p_t \) for \( t \leq \tau \). For \( t > \tau \), we have

\[
x_t'(h_{t-1}, \hat{\theta}_t) = x_t(h^*_{t-1}, \hat{\theta}_t) \quad \text{and} \quad p_t'(h_{t-1}, \hat{\theta}_t) = p_t(h^*_{t-1}, \hat{\theta}_t),
\]

where \( h_t' = \hat{h}_t(s_t(h_\tau, (p_{(1,t)}), \mathcal{U}_t(h_\tau)) \) and \( h^*_{t-1} \) is the history resulted from reporting the same types between stages \( \tau + 1 \) and \( t - 1 \) as \( h_{t-1} \) after \( h_t' \).

We argue that \( \mathcal{M}' = \langle x', p' \rangle \) satisfies the desired properties.

The mechanism satisfies (ALLOCATION) and (PAYMENT) constraints. This is guaranteed by Definition 3.1 of lossless history compression \( s \). For \( t \leq \tau \), the allocations and payments are not changed; for \( t = \tau + 1 \), the constraints \( \mathcal{F}_{\tau + 1} \) and \( \mathcal{F}_{\tau + 1} \) do not change according to Definition 3.1 as \( s_t \) remains the same, and both the original and new allocations and payments satisfy the same constraints; for \( t > \tau + 1 \), the allocations and payments constraints are satisfied by the summary update rule.

The mechanism is DIC. For \( t \geq \tau + 1 \), \( \mathcal{M}' \) is DIC since \( \mathcal{M} \) is DIC. As for \( t = \tau \), we use the fact that

\[
\theta_\tau \in \arg\max_{\hat{\theta}_\tau} u_\tau(h_{\tau-1}, \hat{\theta}_\tau; \theta_\tau) + \mathcal{U}_\tau(h_\tau) = \arg\max_{\hat{\theta}_\tau} u'_\tau(h_{\tau-1}, \hat{\theta}_\tau; \theta_\tau) + \mathcal{U}'_\tau(h_\tau),
\]

where \( \mathcal{U}'_\tau(h_\tau) = \mathcal{U}_\tau(h'_\tau) = \mathcal{U}_\tau(h_\tau) \). The expression holds because (i) we didn’t change the stage mechanism at stage \( \tau \) so that \( u_\tau = u'_\tau \), and (ii) we carefully changed the mechanism while preserving the expected continuation utilities \( \mathcal{U}_\tau(h_\tau) \). For \( t < \tau \), we can apply a similar argument to show that the mechanism is DIC since the stage utility and the expected continuation utilities are preserved.

The mechanism is symmetric for \( t \leq \tau \). The mechanism is symmetric for \( t = \tau \) by the construction. As for \( t < \tau \), consider two histories \( h_t \) and \( h'_t \) falling into the same class \( S_t \). Then they must induce the same succinct summary and continuation utility in the original mechanism. Hence they also induce the same succinct summary and continuation utility in the new mechanism. By the induction hypothesis, the allocation and payment must be the same for \( h_t' \) and \( h_t^* \) for any \( \theta_{(t+1,t')} \) with \( t' > t \) in the original mechanism. Therefore, \( h_t' \) and \( h_t^* \) must share the same succinct summary and continuation utility, and thus, they are in the same class. Therefore, the mechanism is symmetric for any \( t < \tau \) by our construction.

\(^{10}\)When the \( \arg\max \) does not exist (e.g., \( S_t(\tilde{s}, \tilde{u}) \) is an open set), we can select any \( h_t \) such that the objective is no less than the average objective over the set \( S_t(\tilde{s}, \tilde{u}) \).
The expected Objective does not decrease. Since we always replace a suffix of the mechanism with the one with at least the same expected Objective, it never decreases.

C.5 Proof of Corollary 3.14

Proof. The proof directly follows from the proof of Lemma 3.13, because the symmetrization construction breaks neither (STAGE-IC) nor (STATE/UI).

D MISSING PROOFS IN SECTION 4

D.1 Proofs of Lemma 4.3

Proof. Notice that
\[ \mathbb{E}_{v_t}[x_t \cdot x_t(\alpha_{t-1}, \beta_{t-1}, v_t)] = E_{v_t} \left[ x_t \cdot x_t(\alpha_{t-1}, \beta_{t-1}, v_t) - q_t(\alpha_{t-1}, \beta_{t-1}, v_t) - s^{ir^-}_t(\alpha_{t-1}, \beta_{t-1}) \right] = E_{v_t} \left[ s^{ir^+}_t(\alpha_{t-1}, \beta_{t-1}, v_t) - s^{ir^-}_t(\alpha_{t-1}, \beta_{t-1}) \right] \]

Therefore, since the \( \langle x, p \rangle \) is BI, \( E_{v_t}[s^{ir^+}_t(\alpha_{t-1}, \beta_{t-1}, v_t)] - s^{ir^-}_t(\alpha_{t-1}, \beta_{t-1}) \) must be non-negative and be a constant independent of \( \alpha_{t-1} \) and \( \beta_{t-1} \). Relating the deposit and spend rule to \( \alpha_t \) and \( \beta_t \), we have

\[
\begin{align*}
\alpha_t &= \alpha_{t-1} + s^{ir^+}_t(\alpha_{t-1}, \beta_{t-1}, v_t) - s^{ir^-}_t(\alpha_{t-1}, \beta_{t-1}) \\
\beta_t &= \beta_{t-1} - q_t(\alpha_{t-1}, \beta_{t-1}, v_t) - s^{ir^-}_t(\alpha_{t-1}, \beta_{t-1})
\end{align*}
\]

To ensure the mechanism is feasible, since both \( s^{ir^+}_t(\alpha_{t-1}, \beta_{t-1}, v_t) \) and \( p_t(\alpha_{t-1}, \beta_{t-1}, v_t) \) are non-decreasing in \( v_t \) by Myerson’s lemma, we must have \( s^{ir^-}_t(\alpha_{t-1}, \beta_{t-1}) \leq \alpha_{t-1} \) and \( q_t(\alpha_{t-1}, \beta_{t-1}, v_t) + s^{ir^-}_t(\alpha_{t-1}, \beta_{t-1}) \leq \beta_{t-1} \).

E OMITTED MATERIALS FOR NONCLAIRVOYANTMULTI MECHANISM

E.1 Full Descriptions of NonClairvoyantMulti

Our stage mechanisms are parameterized by the cumulative utility \( \tilde{a}_{t-1} \), the remaining budget \( \tilde{\beta}_{t-1} \), and the fractions of almost budget-exhausted history \( \tilde{y}_{t-1} \). For buyer \( i \notin N_t \), we will not allocate nor charge anything; and for buyer \( i \in N_t \), if her budget is almost exhausted, we will compensate the buyer via a negative payment, which equals to her expected utility in the corresponding auction when her budget is not almost exhausted.

1. **Second price auction:**
   - If \( \beta^i_{t-1} \geq \tilde{\beta} \), implement buyer \( i \)’s outcome in the second-price auction among \( N_t \),
     \[ x_t^{S,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{y}_{t-1}, \tilde{b}_t) = x_t^{S,N_t,i}(\tilde{b}_t) \quad \text{and} \quad p_t^{S,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{y}_{t-1}, \tilde{b}_t) = p_t^{S,N_t,i}(\tilde{b}_t) \]
   - If \( \beta^i_{t-1} \leq \tilde{\beta} \), notice that \( E_{\tilde{b}_t}[x_t^{S,N_t,i}(\tilde{v}_t) \cdot y_t^i - p_t^{S,N_t,i}(\tilde{v}_t)] \) would be her expected utility in the second price auction if her budget was not almost exhausted. In this case, the item is not allocated to her and she will be paid as much as the expected utility:
     \[ x_t^{S,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{y}_{t-1}, \tilde{b}_t) = 0 \]
     \[ p_t^{S,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{y}_{t-1}, \tilde{b}_t) = -E[ x_t^{S,N_t,i}(\tilde{v}_t) \cdot y_t^i - p_t^{S,N_t,i}(\tilde{v}_t)] \]

2. **Money burning mechanism:**
\( N^+_t = \{ i \in N_t | \beta^t_{i-1} \geq 2\varepsilon \} \), the subset of active buyers whose budgets are not almost exhausted. We will compute a stage-BIC and stage-expost-IR mechanism \((x_t^{\text{Burn}}, q_t^{\text{Burn}})\) that maximizes the sum of expected utilities for buyers in \( N^+_t \), subject to the constraints that for \( i \in N^+_t \), her expected utility is at most \( \frac{1}{2} \alpha^t_{i-1} \):

\[
\max \sum_{i \in N^+_t} E_{\tilde{v}_i} \left[ x_t^{\text{Burn},i}(\tilde{v}_i) \cdot \upsilon^i_t - q_t^{\text{Burn},i}(\tilde{v}_i) \right] \\
\text{s.t.} \ E_{\tilde{v}_i} \left[ x_t^{\text{Burn},i}(\tilde{v}_i) \cdot \upsilon^i_t - q_t^{\text{Burn},i}(\tilde{v}_i) \right] \leq \frac{1}{2} \alpha^t_{i-1} \quad \forall i \in N^+_t
\]

(Money Burning) (Stage-BIC), (Stage-Expost-IR)

For each \( i \in N^+_t \), we then allocate according to \( x_t^{\text{Burn},i} \) and charge her \( q_t^{\text{Burn},i} \) plus her expected utility as the extra payment:

\[
x_t^{R,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{\gamma}_{t-1}, \tilde{b}_t) = x_t^{\text{Burn},i}(\tilde{b}_t) \\
p_t^{R,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{\gamma}_{t-1}, \tilde{b}_t) = q_t^{\text{Burn},i}(\tilde{b}_t) + E_{\tilde{v}_i} \left[ x_t^{\text{Burn},i}(\tilde{v}_i) \cdot \upsilon^i_t - q_t^{\text{Burn},i}(\tilde{v}_i) \right].
\]

- If \( \beta^t_{i-1} < 2\varepsilon \), do not allocate and charge nothing.

(3) Myerson’s auction:

- If \( \beta^t_{i-1} \geq \varepsilon \), implement the buyer \( i \)'s outcome in the Myerson’s auction among \( N_t \):

\[
x_t^{M,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{\gamma}_{t-1}, \tilde{b}_t) = x_t^{*,N,i}(\tilde{b}_t) \quad \text{and} \quad p_t^{M,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{\gamma}_{t-1}, \tilde{b}_t) = p_t^{*,N,i}(\tilde{b}_t);
\]

- If \( \beta^t_{i-1} < \varepsilon \), notice that \( E_{\tilde{v}_i} [x_t^{*,N,i}(\tilde{v}_i) \cdot \upsilon^i_t - p_t^{*,N,i}(\tilde{v}_i)] \) was her expected utility in the Myerson’s auction when her budget was not almost exhausted. Allocate nothing to the buyer and pay her expected utility, no matter what her bid is:

\[
x_t^{M,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{\gamma}_{t-1}, \tilde{b}_t) = 0 \\
p_t^{M,i}(\tilde{a}_{t-1}, \tilde{\beta}_{t-1}, \tilde{\gamma}_{t-1}, \tilde{b}_t) = -E_{\tilde{v}_i} [x_t^{*,N,i}(\tilde{v}_i) \cdot \upsilon^i_t - p_t^{*,N,i}(\tilde{v}_i)].
\]

Our mechanism for active buyers is a combination of the above three mechanisms:

\[
x_t^i = \frac{5}{2} x_t^i + \frac{3}{2} x_t^{R,i} + \frac{1}{2} x_t^{M,i} \\
p_t^i = \frac{5}{2} p_t^i + \frac{3}{2} p_t^{R,i} + \frac{1}{2} p_t^{M,i}
\]

The parameters \( \tilde{a}_t, \tilde{\beta}_t, \tilde{\gamma}_t \) are updated accordingly. We emphasize that there is a subtle difference between the budget criterion used in the money burning mechanism and the other two mechanisms, such that we exclude a buyer from the money burning mechanism when \( \beta^t_{i-1} < 2\varepsilon \), but for the other two mechanisms, we pay a buyer back when \( \beta^t_{i-1} < \varepsilon \). Notice that we will only compensate the buyer by a negative payment in the second-price auction and the Myerson’s auction when \( \beta^t_{i-1} < \varepsilon \) and the compensation can be at most \( \varepsilon \) per stage. Therefore, once \( \beta^t_{i-1} < 2\varepsilon \) for some \( t \), then for all \( t' > t, \beta^t_{i-1} < 2\varepsilon \). As a result, we exclude a buyer from the money burning mechanism from the first time when her budget becomes almost exhausted. However, a buyer can participate in the second-price auction and Myerson’s auction again once her budget becomes larger than \( \varepsilon \).

It is straightforward to verify that \text{NonClairvoyantMulti} is stage-BIC and BU. As for BI, first of all, note that the definition of \( N_t \) is independent of the history. Moreover, if \( i \notin N_t \), then her expected utility at stage \( t \) is definitely 0; otherwise, for an active buyer \( i \in N_t \), her expected utility is always \( E_{\tilde{v}_i} [x_t^{*,N,i}(\tilde{v}_i) \cdot \upsilon^i_t - p_t^{*,N,i}(\tilde{v}_i)] \) from the second price auction, 0 from money-burning mechanism, and \( E_{\tilde{v}_i} [x_t^{*,N,i}(\tilde{v}_i) \cdot \upsilon^i_t - p_t^{*,N,i}(\tilde{v}_i)] \) from the Myerson’s auction.
\section*{E.2 Valid Non-clairvoyant Mechanism}

We first verify that the \textsc{NonClairvoyantMulti} mechanism is a non-clairvoyant generalized bank account mechanism. The non-clairvoyance and Stage-IC are clear from the construction of the mechanism. As for BI, our construction ensures that for stage \( t \), an inactive buyer always earns 0 utility. For an active buyer \( i \), her expected utility is always \( \mathbb{E}_{\tilde{v}_i} \left[ x^*_{N_l,i}(\tilde{v}_i) \cdot v^i_t - \tilde{p}^{S,N_l,i}_t(\tilde{v}_i) \right] \geq 0 \) from the second price auction, 0 from the money-burning mechanism, and \( \mathbb{E}_{\tilde{v}_i} \left[ x^*_{N,l,i}(\tilde{v}_i) \cdot v^i_t - \tilde{p}^{S,N_l,i}_t(\tilde{v}_i) \right] \geq 0 \) from the Myerson’s auction. Moreover, since whether a buyer is active or not is defined by \( y^i_{t-1} \), which only depends on the past distributional information, we conclude that the expected utility of any buyer is independent of the historical bids \( \tilde{b}_{(1,T-1)} \). Hence BI is guaranteed.

The mechanism is \textsc{ex-post-IR} since buyer \( i \)'s utilities are non-negative for second price auction and the Myerson’s auction, while the utility decreases by at most \( a^i_{t-1} \) in the money burning mechanism. Lastly, for the \textsc{Budget} constraint, notice that if the remaining budget \( \tilde{p}^B_{t-1,i} \) is less than \( \tilde{\nu} \), the payment is non-positive, and otherwise, the payment collected from all mechanisms is at most \( \tilde{\nu} \) except the money burning mechanism and the payment in the money burning mechanism is at most \( 2\tilde{\nu} \).

\section*{E.3 Revenue Performance}

\textbf{Proof of Lemma 5.3.} First, the revenue that can be extracted from the inactive buyers at stage \( T \) is at most the sum of their budget \( \sum_{i \in N_T} B^i \). As for the active buyers at stage \( T \), since \( \tilde{p}_t = \tilde{q}_t + \tilde{s}^{ir} \), we have that the revenue contributed by buyers in \( N_T \) is at most

\[ \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i \in N_T} s^{ir,j}(\tilde{v}_{(1,t)}) \right] + \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i \in N_T} q^i_t(\tilde{v}_{(1,t)}) \right]. \]

Finally, notice that

\[ \mathbb{E} \left[ \sum_{i \in N_T} q^i_t(\tilde{v}_{(1,t)}) \right] \leq \text{\textsc{Mye}}(\tilde{F}_T, N_T) \]

since \textsc{Mye}(\( \tilde{F}_T, N_T \)) is the revenue of the optimal \textsc{stage-BIC} and \textsc{stage-expost-IR} mechanism for the active buyers in the final stage \( T \). \hfill \Box

\textbf{Proof of Theorem 5.4.} First, by the definition of \( \tilde{y}_T \) and the set of inactive buyers \( N_T \), we have that the revenue contributed by the inactive buyers \( N_T \) is at least

\[ \text{Rev} \geq \sum_{i \in N_T} \kappa \left( B^i - 2\tilde{\nu} \right) \] (3)

Therefore, it suffices to show that for the optimal dynamic mechanism \( \langle \tilde{x}^*, \tilde{q}_*^*, \tilde{s}^{ir,*}, \tilde{\nu}^*, \tilde{\beta}^* \rangle \),

\[ \text{Rev} = \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i \in N} \frac{1}{5} P^M_t(\tilde{v}_{(1,t)}) + \frac{2}{5} P^S_t(\tilde{v}_{(1,t)}) + \frac{2}{5} P^B_t(\tilde{v}_{(1,t)}) \right] \]

\[ \geq \frac{1}{5} \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i \in N_T} s^{ir,j}(\tilde{v}_{(1,t)}) \right] + \frac{1}{5} \sum_{t=1}^{T} \text{\textsc{Mye}}(\tilde{F}_t, N_T). \]

For any sequence of valuations \( \tilde{v}_{(1,t)} \), let \( e_i(\tilde{v}_{(1,t)}) \) be the last stage before stage \( t \) (including stage \( t \)) in which the remaining budget of buyer \( i \) before the beginning of the stage is at least \( 2\tilde{\nu} \) and buyer \( i \) is active, i.e., \( e_i(\tilde{v}_{(1,t)}) = \max \{ t' \leq t \mid \tilde{p}^B_{t-1,i}(\tilde{v}_{(1,t-1)}) \geq 2\tilde{\nu} \text{ and } i \in N_T \} \). Notice that for a fixed \( \tilde{v}_{(1,t)} \), then \( e_i(\tilde{v}_{(1,t)}) = t \) if and only if \( e_i(\tilde{v}_{(1,t)}) = t \).
We now proceed to bound $E \geq t$ equality follows the definition of $\gamma$. Therefore, for all $t \geq e_l(\tilde{v}(1,T))$, we have $\beta^i_l(\tilde{v}(1,T)) \leq 2\tilde{v}$. As a result, for any fixed $\tilde{v}(1,T)$, we have

$$\sum_{t=e_l(\tilde{v}(1,T))}^{T} p^i_l(\tilde{v}(1,T)) \geq -2\tilde{v}. \quad (4)$$

Therefore,

$$\text{REV} = \mathbb{E} \left[ \sum_{t \in N} \sum_{i=1}^{T} \frac{1}{5} p^M_i(\tilde{v}(1,t)) + \frac{2}{5} p^S_i(\tilde{v}(1,t)) + \frac{2}{5} p^B_i(\tilde{v}(1,t)) \right] = \mathbb{E} \left[ \sum_{i \in N} \sum_{t=1}^{T} \frac{1}{5} p^M_i(\tilde{v}(1,t)) + \frac{2}{5} p^S_i(\tilde{v}(1,t)) + \frac{2}{5} p^B_i(\tilde{v}(1,t)) \right] + \mathbb{E} \left[ \sum_{i \in N} \sum_{t=e_l(\tilde{v}(1,T))}^{T} p^i_l(\tilde{v}(1,t)) \right] \quad (5)$$

We now proceed to bound $\mathbb{E} \left[ \sum_{i \in N} \sum_{t=1}^{T} \frac{1}{5} p^M_i(\tilde{v}(1,t)) + \frac{2}{5} p^S_i(\tilde{v}(1,t)) + \frac{2}{5} p^B_i(\tilde{v}(1,t)) \right]$. 

Myerson’s Revenue. Recall that once $y'_i \geq \kappa$ for some $t$, then for all $t' > t$, $y'_i = y'_i$ since buyer $i$ is excluded from any future auctions. Henceforth, we have $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_T$. Moreover, it is clear that $\text{Mye}(F_i, N_T) \geq \text{Mye}(F_i, N_T)$ since a Myerson’s auction with more buyers can extract more revenue. Moreover, for $\tilde{v}(1,T)$ and each round $t$, if buyer $i \in N_t$ and $t \leq e_l(\tilde{v}(1,T))$, then it will pay $p^*_{N_i,l}(\tilde{v}_i)$. Therefore, we have

$$\mathbb{E} \left[ \sum_{i \in N} \sum_{t=1}^{T} p^M_i(\tilde{v}(1,t)) \right] = \mathbb{E} \left[ \sum_{i \in N_t} \sum_{t=1}^{T} p^*_{N_i,l}(\tilde{v}_i) \cdot 1 \{ \beta^i_l(\tilde{v}(1,t-1)) \geq 2\tilde{v} \} \right]$$

$$= \sum_{t=1}^{T} \sum_{i \in N_t} \mathbb{E} \left[ p^*_{N_i,l}(\tilde{v}_i) \right] \cdot \mathbb{E} \left[ 1 \{ \beta^i_l(\tilde{v}(1,t-1)) \geq 2\tilde{v} \} \right]$$

$$= \sum_{t=1}^{T} \sum_{i \in N_t} \mathbb{E} \left[ p^*_{N_i,l}(\tilde{v}_i) \right] \cdot (1 - y'_i)$$

$$\geq (1 - \kappa) \sum_{t=1}^{T} \sum_{i \in N_t} \mathbb{E} \left[ p^*_{N_i,l}(\tilde{v}_i) \right],$$

where the second equality is because that $\tilde{v}_i$ is drawn independently of $\tilde{v}(1,t-1)$, and the third equality follows the definition of $y'_i$. Moreover, as we mentioned before that $N_t \supseteq N_T$, we have

$$\mathbb{E} \left[ \sum_{i \in N} \sum_{t=1}^{T} p^M_i(\tilde{v}(1,t)) \right] \geq (1 - \kappa) \sum_{t=1}^{T} \text{Mye}(F_i, N_t) \geq (1 - \kappa) \sum_{t=1}^{T} \text{Mye}(F_i, N_T). \quad (6)$$

Revenue from Spend. By the update rule of $\alpha$, we can express the spend $s^r_{l-t}$ of our NONCLAIRVOYANTMULTI mechanism as

$$s^r_{l-t}(\tilde{v}(1,t)) = \alpha^i_{l-1}(\tilde{v}(1,t-1)) - \alpha^i_l(\tilde{v}(1,t)) + z^i_l(\tilde{v}(1,t))$$

where $z^i_l(\tilde{v}(1,t)) = x^i_l(\tilde{v}(1,t)-\alpha^i_l(\tilde{v}(1,t))$ denotes the buyer $i$’s utility from the local stage mechanism $\langle \tilde{x}_i, \tilde{q}_i \rangle$. Moreover, notice that when the buyer $i$ has the lowest valuation 0, both $p^M_i$ and $p^S_i$ are 0.
As a result, by construction, the spend of our NonClairvoyantMulti mechanism only comes from the money burning mechanism. Therefore, we have

\[
\mathbb{E} \left[ \sum_{t \in N} \sum_{t=1}^{S_t} \frac{2}{5} p_t^i (\bar{v}_t, \bar{u}_i) + \frac{2}{5} p_t^{B_i} (\bar{v}_t, \bar{u}_i) \right] \geq \mathbb{E} \left[ \sum_{t \in N} \sum_{t=1}^{S_t} \frac{2}{5} p_t^i (\bar{v}_t, \bar{u}_i) + s_{t}^{ir-i} (\bar{v}_t, \bar{u}_i) \right] .
\]

We will proceed to show that

\[
\mathbb{E} \left[ \sum_{t \in N} \sum_{t=1}^{S_t} \frac{2}{5} p_t^i (\bar{v}_t, \bar{u}_i) + s_{t}^{ir-i} (\bar{v}_t, \bar{u}_i) \right] \geq 1 - \frac{\kappa}{5} \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{i \in N_t} s_{t}^{ir-i} (\bar{v}_t, \bar{u}_i) \right] .
\]

(7)

**Lower bound the sum of cumulative utility and spend.** Fix \( \bar{u}_{(1,t)} \). By induction, we would like to show that for any buyer \( i \) and any historical valuation profiles \( \bar{v}_{(1,t)} \) with \( t \leq e_i(\bar{v}_{(1,t)}) \), we have

\[
\alpha_t^i + \sum_{r=1}^{t} s_{r}^{ir-i} \geq \frac{2}{5} \left( \alpha_{t+1}^i + \sum_{r=1}^{t} s_{r}^{ir,i} - \sum_{r=1}^{t} \chi_{r}^i \cdot b_{r}^{\max,2,N_t} - z_{t+1}^i \right)
\]

where \( b_{r}^{\max,2,N_t} \) is the second highest bid in the valuation profile \( \bar{v}_r \) among buyers in \( N_t \) at stage \( r \). Moreover, we omit the dependence on \( \bar{v}_{(1,r)} \) for simplicity.

The base is clearly true since \( \alpha_0^i = 0 \). Assume the induction hypothesis is true for stage \( t < e_i(\bar{v}_{(1,t)}) \), and we substitute \( \alpha_t^i \) by \( \alpha_t^i - z_{t+1}^i + s_{t}^{ir-i} \) for both our NonClairvoyantMulti mechanism and the optimal dynamic mechanism:

\[
\alpha_{t+1}^i + \sum_{r=1}^{t+1} s_{r}^{ir-i} - z_{t+1}^i \geq \frac{2}{5} \left( \alpha_{t+1}^i + \sum_{r=1}^{t+1} s_{r}^{ir,i} - \sum_{r=1}^{t+1} \chi_{r}^i \cdot b_{r}^{\max,2,N_t} - z_{t+1}^i \right)
\]

(9)

Notice that \( z_{t+1}^i \leq x_{t+1}^i \cdot \bar{v}_{t+1}^i \leq \bar{v}_{t+1}^i \), since \( z_{t+1}^i \) is buyer \( i \)'s utility from the local-stage mechanism \( \langle \bar{v}_{t+1}^i, \bar{q}_{t+1}^i \rangle \). If \( b_{t+1}^{\max,2,N_t} \geq \bar{v}_{t+1}^i \), then we can finish the induction step using the fact that \( z_{t+1}^i \geq 0 \) and \( z_{t+1}^i \leq x_{t+1}^i \cdot \bar{v}_{t+1}^i \leq b_{t+1}^{\max,2,N_t} \). If not, notice that \( z_{t+1}^i \) can be decomposed into

\[
z_{t+1}^i = \frac{1}{5} \cdot z_{t+1}^{M_i} + \frac{2}{5} \cdot z_{t+1}^{S_i} + \frac{2}{5} \cdot z_{t+1}^{B_i} \geq \frac{2}{5} \cdot z_{t+1}^{S_i}
\]

where \( z_{t+1}^{M_i}, z_{t+1}^{S_i}, \) and \( z_{t+1}^{B_i} \) are buyer \( i \)'s utilities from the local-stage mechanisms induced by the Myerson’s auction, the second price auction, and the money burning mechanism, respectively. Therefore,

\[
z_{t+1}^i \geq \frac{2}{5} \cdot z_{t+1}^{S_i} = \frac{2}{5} \left( \bar{v}_{t+1}^i - b_{t+1}^{\max,2,N_t} \right) \geq \frac{2}{5} \left( \bar{v}_{t+1}^i - b_{t+1}^{\max,2,N_t} \right) x_{t+1}^i \geq \frac{2}{5} \left( z_{t+1}^{ir} - x_{t+1}^{ir} \cdot b_{t+1}^{\max,2,N_t} \right)
\]

(10)

We can now finish the induction step to show that (8) holds for \( (t+1) \) by plugging (10) into (9).

**A charging scheme for the spend.** By Lemma 5.2, the optimal dynamic mechanism satisfies that \( s_{t}^{ir-i} \leq E_{\bar{v}_{t}^{\text{BURN},t}} \left[ x_{t}^{ir-i} (\bar{v}_{t}) \cdot \bar{v}_{t}^{i} - q_{t}^{ir-i} (\bar{v}_{t}) \right] \), and thus, there exists a solution to the money burning mechanism with constraints:\n
\[
E_{\bar{v}_{t}^{\text{BURN},t}} \left[ x_{t}^{\text{BURN},t} (\bar{v}_{t}) \cdot \bar{v}_{t}^{i} - q_{t}^{\text{BURN},t} (\bar{v}_{t}) \right] = \min \left( max(0, s_{t}^{ir-i}), 0, \frac{5}{2} \alpha_{t-1}^i \right).
\]

\footnote{Note that the optimal dynamic mechanism itself is a feasible solution to the money burning problem (Money Burning), except that when \( s_{t}^{ir-i} < 0 \). In case of \( s_{t}^{ir-i} < 0 \), we can still construct another feasible solution by simply letting the allocation and payment of buyer \( i \) to be zero. Similarly, to achieve an expected local stage utility no more than \( \frac{5}{2} \alpha_{t-1}^i \), we can scale the allocation and payment of buyer \( i \) by a factor if needed.}
Let $\mu'_t = \min\left(\max\left(\frac{2}{5}s_r^{ir.i}, 0\right), \alpha'_{t-1}\right)$ be a charging scheme that we would like the spend of our \textsc{NonClairvoyantMulti} mechanism to compete against. If $\alpha'_{t-1} \leq \frac{2}{5} \max(s_r^{ir.i}, 0)$, by (8), we have for $t \leq e_i(\tilde{v}_{(1,T)})$

$$\mu'_t = \alpha'_{t-1} \geq \frac{2}{5}\left(\alpha'_{t-1} + \sum_{r=1}^{t-1} \max(s_r^{ir.i}, 0) - \sum_{r=1}^{t-1} x_r^{e_i} \cdot b_r^{max,2,N_r}\right) - \sum_{r=1}^{t-1} s_r^{ir.i}. $$

Rearranging the above expression, we have

$$\mu'_t + \sum_{r=1}^{t-1} s_r^{ir.i} + \frac{2}{5} \sum_{r=1}^{t-1} x_r^{e_i} \cdot b_r^{max,2,N_r} \geq \frac{2}{5} \alpha'_{t-1} + \sum_{r=1}^{t-1} \max(s_r^{ir.i}, 0) \geq \frac{2}{5} \sum_{r=1}^{t} s_r^{ir.i},$$

(11)

where the last inequality is due to $\alpha'_{t-1} \geq \max(s_r^{ir.i}, 0)$ by Lemma 5.2.

On the other hand, if $\alpha'_{t-1} > \frac{2}{5} \max(s_r^{ir.i}, 0)$, let $t'$ be the last stage before $t$ in which $\alpha'_{t-1} \leq \frac{2}{5} \max(s_r^{ir.i}, 0)$ holds. As a result, (11) holds for $t'$ and for stages between $(t'+1)$ and $t$, we can simply use the inequality that $\sum_{t=1}^{t-1} \mu_r^{i} \geq \sum_{t=1}^{t} \max(s_r^{ir.i}, 0)$:

$$\sum_{t'=t}^{t} \mu_r^{i} + \sum_{r=1}^{t'-1} s_r^{ir.i} + \frac{2}{5} \sum_{r=1}^{t'-1} x_r^{e_i} \cdot b_r^{max,2,N_r} \geq \frac{2}{5} \sum_{r=1}^{t} \max(s_r^{ir.i}, 0).$$

(12)

Since all of $\mu_r^{i}, x_r^{e_i}, \text{ and } b_r^{max,2,N_r}$ are non-negative, from either (11) or (12), we can have

$$\sum_{t=1}^{t} \mu_r^{i} + \sum_{r=1}^{t-1} s_r^{ir.i} + \frac{2}{5} \sum_{r=1}^{t-1} x_r^{e_i} \cdot b_r^{max,2,N_r} \geq \frac{2}{5} \sum_{r=1}^{t} \max(s_r^{ir.i}, 0).$$

(13)

**Approximation guarantee on the spend.** Let $N'(\tilde{v}_{(1,T)}) = \{i \in N_T | e_i(\tilde{v}_{(1,T)}) < T\}$ and $N''(\tilde{v}_{(1,T)}) = \{i \in N_T | e_i(\tilde{v}_{(1,T)}) = T\}$, and therefore, $N'(\tilde{v}_{(1,T)}) \cup N''(\tilde{v}_{(1,T)}) = N_T$. For $i \in N'$, the buyer’s budget is almost exhausted at the end of $T$ and it is clear that the total spend from buyer $i$ cannot exceed his budget, and thus, we have

$$\sum_{i \in N'} \sum_{r=1}^{T} s_r^{ir.i}(\tilde{v}_{(1,T)}) \leq \sum_{i \in N'} B^{i} \leq \sum_{i \in N'} \left[ e_i(\tilde{v}_{(1,T)}) \mu_r^{i}(\tilde{v}_{(1,T)}) + 2\tilde{\sigma} \right].$$

Taking the expectation over $\tilde{v}_{(1,T)}$, we have

$$\mathbb{E}_{\tilde{v}_{(1,T)}} \left[ \sum_{i \in N'} \sum_{r=1}^{T} s_r^{ir.i}(\tilde{v}_{(1,T)}) \right] \leq \mathbb{E}_{\tilde{v}_{(1,T)}} \left[ \sum_{i \in N'} \sum_{t=1}^{\infty} e_i(\tilde{v}_{(1,T)}) p_r^{i}(\tilde{v}_{(1,T)}) + 2\tilde{\sigma} \right]$$

$$\leq \text{REV} + \mathbb{E}_{\tilde{v}_{(1,T)}} \left[ \sum_{i \in N'} \sum_{t=1}^{\infty} e_i(\tilde{v}_{(1,T)}) \sum_{i=1}^{2\tilde{\sigma}} \right].$$

(14)

Summing (13) across all the buyers, we have

$$\sum_{i \in N} \sum_{r=1}^{T} \mu_r^{i} + \sum_{i \in N} \sum_{r=1}^{T} s_r^{ir.i} + \frac{2}{5} \sum_{i \in N} \sum_{r=1}^{T} x_r^{e_i} \cdot b_r^{max,2,N_r} \geq \frac{2}{5} \sum_{i \in N} \sum_{r=1}^{T} \max(s_r^{ir.i}, 0).$$

Notice that we have $\sum_{i \in N} \sum_{r=1}^{T} \mu_r^{i} \leq \sum_{i \in N} \sum_{r=1}^{T} s_r^{ir.i}$, since for every stage $r$, $\sum_{i \in N} e_i(\tilde{v}_{(1,T)}) \geq r s_r^{ir.i}$ is the optimal solution of the money burning program (\textsc{Money Burning}) including the buyers.
whose budget have not been exhausted, and there exists a money burning mechanism achieving total utility \( \sum_{i \in T} e_i(\tilde{\nu}_{(1,T)}) \mu^i_1 \) due to the construction of \( \mu_1^i \). Therefore, we have

\[
2 \sum_{i \in N} \sum_{r=1}^{\nu} e_i(\tilde{\nu}_{(1,T)}) \left( s_{ir}^i + \frac{1}{5} x_{ir}^i \cdot b_{ir}^{\text{max},i} \right) \geq \frac{2}{5} \sum_{i \in N} \sum_{r=1}^{\nu} e_i(\tilde{\nu}_{(1,T)}) \max(s_{ir}^i, 0) \geq \frac{2}{5} \sum_{i \in N} \sum_{r=1}^{\nu} \sum_{t=1}^{T} s_{ir}^i,
\]

where the last inequality follows from that \( e_i(\tilde{\nu}_{(1,T)}) = T \) for \( i \in N''(\tilde{\nu}_{(1,T)}) \). By an argument similar to the revenue contribution of the Myerson’s auction, we can show that

\[
\mathbb{E}_{\tilde{\nu}_{(1,T)}} \left[ \sum_{i \in N} \sum_{t=1}^{\nu} e_i(\tilde{\nu}_{(1,T)}) \frac{1}{5} \cdot p_i^j(s_{ir}^i + \tilde{s}_{ir}^i - \tilde{\nu}_{(1,T)}) \right] \geq \mathbb{E}_{\tilde{\nu}_{(1,T)}} \left[ \frac{1 - \kappa}{5} \sum_{i \in N''(\tilde{\nu}_{(1,T)})} \sum_{r=1}^{\nu} s_{ir}^i \right].
\]

Plugging \((16)\) into \((15)\) and taking the expectation over \( \tilde{\nu}_{(1,T)} \), we have

\[
\mathbb{E}_{\tilde{\nu}_{(1,T)}} \left[ \sum_{i \in N} \sum_{t=1}^{\nu} e_i(\tilde{\nu}_{(1,T)}) \frac{1}{5} \cdot p_i^j(s_{ir}^i + \tilde{s}_{ir}^i - \tilde{\nu}_{(1,T)}) \right] \geq \mathbb{E}_{\tilde{\nu}_{(1,T)}} \left[ \frac{1 - \kappa}{5} \sum_{i \in N''(\tilde{\nu}_{(1,T)})} \sum_{r=1}^{\nu} s_{ir}^i \right].
\]

Putting Everything Together. Let \( w_i = \mathbb{E}_{\tilde{\nu}_{(1,T)}} [e_i(\tilde{\nu}_{(1,T)}) < T] \) be the probability the buyer \( i \)'s budget is almost exhausted before the end of the horizon. Let \( \text{Rev}^* \) be the revenue of the clairvoyant dynamic mechanism. Notice that we have

\[
\text{Rev}^* \geq \text{Rev} \geq \sum_{i} w_i (B^i - 2\bar{\nu}) \geq \sum_{i} w_i \cdot \bar{\nu} \left( \frac{1}{\epsilon} - 2 \right) \Rightarrow \sum_{i} w_i \cdot 2\bar{\nu} \leq \frac{2\epsilon}{1 - 2\epsilon} \cdot \text{Rev}^*.
\]

Summing \((6)\) + \((17)\), we have

\[
\text{Rev} \geq \mathbb{E}_{\tilde{\nu}_{(1,T)}} \left[ \sum_{i \in N} \sum_{t=1}^{\nu} \frac{1}{5} \cdot p_i^j(s_{ir}^i + \tilde{s}_{ir}^i - \tilde{\nu}_{(1,T)}) \right] \geq \frac{1 - \kappa}{5} \sum_{t=1}^{\nu} \text{MYE}(\tilde{F}_t, N_T) + \mathbb{E}_{\tilde{\nu}_{(1,T)}} \left[ \sum_{i \in N''(\tilde{\nu}_{(1,T)})} s_{ir}^i \right].
\]

Summing \([\frac{5}{1 - \kappa} \cdot \text{MYE} + \frac{1}{\kappa}] \cdot \text{Rev} + \sum_{i \in N} w_i \cdot 2\bar{\nu} \geq \mathbb{E}_{\tilde{\nu}_{(1,T)}} \left[ \sum_{t=1}^{\nu} \sum_{i \in N_T} s_{ir}^i - \tilde{\nu}_{(1,T)} \right] + \text{MYE}(\tilde{F}_t, N_T) + \sum_{i \in N_T} B^i \geq \text{Rev}^*.\]

Let \( \kappa = \frac{1}{1 + \sqrt{5}} \) and we have

\[
\text{Rev} \geq \frac{1}{7 + 2 \sqrt{5}} \cdot \text{Rev}^* - \sum_{i} w_i \cdot 2\bar{\nu} \geq \frac{1}{7 + 2 \sqrt{5}} - \frac{2\epsilon}{1 - 2\epsilon} \text{Rev}^*.
\]

\(\square\)