Multi-unit Supply-monotone Auctions with Bayesian Valuations

Yuan Deng* Debmalya Panigrahi†

Abstract

We design multi-unit auctions for budget-constrained bidders in the Bayesian setting. Our auctions are supply-monotone, which allows the auction to be run online without knowing the number of items in advance, and achieve asymptotic revenue optimality. We also give an efficient algorithm for implementing our auction by using a succinct and efficiently implementable characterization of supply-monotonicity in the Bayesian setting.

1 Introduction

Dobzinski et al. [17] initiated the study of multi-unit auctions with budget-constrained bidders, where multiple indivisible units of an item are sold to a set of bidders with private valuations whose spending is constrained by a budget limit. They showed that if the budgets are public information, then an adaptation of Ausubel's clinching auction [2] is incentive-compatible and achieves Pareto-optimal outcomes. In many real world settings, such as in Internet advertising, multi-unit auctions need to be run *online*, where in each step, a new unit of the item has to be sold without knowing the total number of units available. To address this situation, Goel et al. [18] showed that the adaptive clinching auction is, in fact, a *supply-monotone* auction, which implies that the allocation to a bidder is monotonically non-decreasing as the supply increases. Both these results apply to the worst case setting, where the valuations of individual bidders can be arbitrary. A more commonly studied setting is that of Bayesian valuations, i.e., where bidders' valuations are drawn identically and independently from a fixed distribution. In this paper, we study the problem of designing a supply-monotone, multi-unit auction for Bayesian bidders.

The first hurdle is to define supply-monotonicity in the Bayesian setting, which requires that the *ex-post mechanism* is monotonically non-decreasing as the supply increases for any possible valuation profile. However, an ex-post mechanism requires exponentially many values to be specified. For single unit auctions, this problem is circumvented by considering the *interim mechanism*, which specifies the expected allocation for a bidder given her valuation. In a seminal work, Border [6] obtained a succinct mathematical characterization of *feasible* interim mechanisms, i.e., which can be generated by an ex-post mechanism, for independent and identical bidders. This was later extended to independent (but possibly non-identical) bidders in Border [7] (see also Che et al. [14] for a network flow formulation).

From a computational perspective, Cai et al. [9] developed an efficient algorithm to check feasibility of an interim mechanism, and additionally demonstrated that any feasible mechanism can be represented by a distribution over polynomially many *hierarchical* mechanisms. They also designed an efficient algorithm to implement a feasible mechanism by efficiently sampling a hierarchical mechanism.

Our first contribution is to extend the above literature from single-unit auctions to multi-unit, supply-monotone auctions. The latter is complicated by the fact that there are non-trivial correlations between allocation rules for individual units. We provide:

> A succinct characterization of feasible interim supply-monotone mechanisms, and efficient algorithms to verify and implement such mechanisms.

This characterization allows us to circumvent the exponentially large valuation space, and specify our ex-post mechanism succinctly and implement the mechanism efficiently.

Next, we turn to the question of maximizing *revenue* for a supply-monotone, multi-unit auction with Bayesian valuations. For *offline* auctions, i.e., if the supply were known in advance and we do not require supply-monotonicity, Laffont and Robert [19] and Pai and Vohra [24] characterized revenue-maximizing multi-unit auctions in the Bayesian setting for *symmetric* budget-constrained bidders.[1] (A set of bidders is symmetric if their valuations are i.i.d. and budgets are

*Department of Computer Science, Duke University. Email: ericdy@cs.duke.edu. This work was supported in part by a Google Ph.D. Fellowship.

†Department of Computer Science, Duke University. Email: debmalya@cs.duke.edu. This work was supported in part by NSF grants CCF-1527084, CCF-1535972, and an NSF CAREER Award CCF-1750140.

[1] These papers only consider a single-unit auction for budget-constrained bidders. However, when the supply is known, one can

identical.) However, if the supply is unknown, applying the optimal offline auction to every supply violates supply monotonicity. In this case, we say that a supply-monotone auction has *competitive ratio* τ if it always generates at least τ fraction of the revenue of a revenue-maximizing offline auction. Our second contribution is to provide a supply-monotone auction that achieves a tight competitive ratio:

> An efficient algorithm for computing a Bayesian incentive-compatible, supply-monotone auction for symmetric budget-constrained bidders that achieves the best possible competitive ratio.

Interestingly, we show that the offline optimal auction satisfies the supply-monotonicity property asymptotically as the number of bidders becomes large, which implies that the competitive ratio of our supply-monotone auction goes to 1 asymptotically.

1.1 Techniques. In the case of a single unit, feasible interim mechanisms are characterized by a set of inequalities defined on subsets of valuations. The main complication of the multi-unit setting over a single unit is the correlation between the allocation rules for individual units. Thus, applying the characterization of feasibility from the single unit case to each individual unit in the multi-unit setting is not sufficient. Instead, we use a hierarchy of subsets of valuations, in which the subset of valuations of the current supply is a superset of the subset of valuations of the next supply, and show that a set of inequalities over all possible hierarchies can characterize feasible supply-monotone interim allocations. To establish necessity of these conditions, we generalize the probabilistic argument over subsets of valuations to the hierarchies. As for sufficiency, we provide a generalization of the network flow approach from Che et al. [14] that is carefully tailored to handle the interdependencies between units, and show that feasibility can be characterized by a duality condition on this network. However, our flow network being exponential in size, it does not permit efficient verification or implementation of the auction. To overcome this hurdle, we show that any interim supply-monotone allocation can be represented by a combination of interim allocation rules for individual units of the item. This allows us to reduce the verification and implementation of feasible supply-monotone allocation rules to the corresponding problems for individual units of the item, which can be solved in polynomial time.

In order to obtain revenue-maximizing supply-monotone auctions, we first acquire a better understanding of the structure of optimal offline auctions. Typically, such auctions are described as optimal solutions to linear programming formulations. Instead, we give a more transparent view of these auctions, where we provide a novel characterization of the gradient of the allocation rules in optimal offline auctions with the change in supply. In particular, we show that there are only two categories of gradients, which leads to only polynomially many different gradients in terms of the number of valuations. This characterization allows us to enumerate over these possibilities to design a supply-monotone auction with a tight competitive ratio.

1.2 Related Work. The problem of selling items online to maximize revenue has a long and rich history, particularly in the context of selling advertising slots or impressions to prospective advertisers in Internet search engines. The traditional approach in this literature has been to optimize revenue in the presence of budget constraints but without strategic considerations, pioneered by the AdWords problem due to Mehta et al. [22] which has led to a large volume of subsequent research (see the survey by Mehta [21]). In contrast, the traditional setting in auction theory captures strategic behavior but does not consider budget constraints. Recent literature in Ad auctions seeks to overcome these limitations by including budget constraints as an essential feature, and investigate its impact on auction design [1, 8, 18].

When auctions need to be run online, the overall supply is not known in advance. This requires that the auction satisfies the supply-monotonicity property, i.e., increasing the overall supply does not decrease the allocation to any bidder. The study of auctions with supply monotonicity was initiated by Mahdian and Saberi [20]. They provide a constant competitive auction with the optimal offline single-price revenue. Babaioff et al. [3] study online supply-monotone auctions for unit-demand bidders with the objective of maximizing social welfare. They show that all truthful mechanisms achieve a diminishing fraction of the optimal social welfare in the adversarial setting, where the supply might be arbitrary, and however, they present a truthful mechanism that achieves a constant approximation in the stochastic model, where the distribution of the supply is known. All these results are obtained in an non-Bayesian environment. Devanur and Hartline [15] describes the Bayesian optimal mechanism for the online supply-monotone setting and extend their discussion to prior-free cases.

A different direction of research, initiated by Che and Gale [12] (see also Benoit and Krishna [4], Che and

apply their result by treating multiple units of an item as a single unit.

Gale [13]), has investigated mechanism design with budget constraints in non-Bayesian settings. In a seminal work, Dobzinski et al. [17] show that in multi-unit settings, if the budgets are private, there is no Pareto optimal and incentive compatible mechanism, and for public budgets they demonstrate that an adaptive clinching auction is the unique mechanism that is Pareto optimal and incentive compatible, which was later extended to divisible units by Bhattacharya et al. [5]. Goel et al. [18] show that the adaptive clinching auction satisfies the supply-monotonicity property.

Another consideration that has received substantial attention is revenue maximization for the seller. The revenue-maximizing mechanism for independent buyers without budget constraints is characterized by the celebrated Myerson's auction [23]. Pai and Vohra [24] characterize revenue-maximizing auctions for budget-constrained bidders. Chawla et al. [11] provide constant factor approximations for both social welfare and revenue for budget-constrained bidders. Recently, Devanur and Weinberg [16] characterize the revenue-maximizing mechanism for a single buyer with private budget in a general setting.

Our work also extends the line of work that characterizes the feasibility of a reduced form auction. In a celebrated work, Border [6] provided the first characterization of a reduced form auction for identical and independent bidders. Cai et al. [9] extended these results to obtain a characterization of feasible, Bayesian, multi-item, multi-bidder mechanisms with independent, additive bidders as distributions over hierarchical mechanisms. Later, Cai et al. [10] generalized the results to other feasibility constraints of the interim allocation rule via a reduction to virtual welfare maximization[2]. Che et al. [14] provides a network flow interpretation of feasible reduced form auction that we also use in our work.

2 Preliminaries

2.1 The environment. We denote the number of bidders by n and the total supply by S, which is unknown to the bidders and the seller. Each bidder has a private valuation v_i from $V = \{\varepsilon, 2\cdot\varepsilon, \cdots, v_{\max}\cdot\varepsilon\}$.[3] For convenience and without loss of generality, we take $\varepsilon = 1$ throughout the paper. Moreover, we consider a single-parameter setting with symmetric bidders: their valuations are drawn identically and independently from a commonly known distribution $f = f_i$ over V and their budgets, denoted B, are identical and public. In line with the literature, we require that f satisfies the monotone hazard rate condition: $\frac{1-F(v)}{f(v)} \geq \frac{1-F(v')}{f(v')}, \forall v < v'$ where $F(v) = \sum_{v'=1}^{v} f(v')$, which implies that the *virtual valuation* $\nu(v) = v - \frac{1-F(v)}{f(v)}$ is monotonically increasing.

Before the arrival of the first unit, each bidder i draws her valuation v_i from distribution f and reports a valuation $\hat{v}_i$ to the seller, which collectively form a valuation profile $\boldsymbol{v} = (\hat{v}_1, \cdots, \hat{v}_n)$. The valuation profile remains unchanged during the auction. When the s-th unit arrives, given the valuation profile $\boldsymbol{v}$, an allocation rule $x_i^s(\boldsymbol{v})$ specifies the number of units that bidder i receives from the first s units and a payment rule $\chi_i^s(\boldsymbol{v})$ gives the amount that bidder i should pay to the seller. The utility for bidder i under a reported valuation profile $\boldsymbol{v}$ is $u_i^s(\boldsymbol{v}) = v_i \cdot x_i^s(\boldsymbol{v}) - \chi_i^s(\boldsymbol{v})$ if $\forall s' \leq s, \chi_i^{s'}(\boldsymbol{v}) \leq B_i$; otherwise, $u_i^s(\boldsymbol{v}) = -\infty$. As usual, we use $-i$ to represent bidders other than i.

Supply monotonicity requires that the ex-post allocation rule must be monotonically non-decreasing for all possible valuation profiles. In other words, the seller cannot retrieve a previously allocated unit during the auction.

Definition 2.1. (Supply Monotonicity) *For all $s \in \mathbb{Z}^+$ and any bidder i,*

$$\forall \boldsymbol{v} \in V^n,\ x_i^s(\boldsymbol{v}) \leq x_i^{s+1}(\boldsymbol{v}).$$

Remark: Our definition corresponds to the *weak online supply model* [18] where the seller is allowed to charge the payments only at the end, or equivalently, reimbursements to the bidders is allowed if necessary.

2.2 Interim Allocation Rules. Note that defining the auction via allocation rule $x_i^s(\cdot)$ and payment rule $\chi_i^s(\cdot)$ requires $2S \cdot |V|^n$ values to be specified. To reduce the complexity of this description to linear in the number of bidders, we use the interim allocation rule $a_i^s(\hat{v}_i)$, which specifies the expected units of the first s units that bidder i receives if her reported valuation is $\hat{v}_i$. More precisely, $a_i^s(\hat{v}_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^s(\hat{v}_i, \boldsymbol{v}_{-i})]$. Similarly, an interim payment rule is $p_i^s(\hat{v}_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[\chi_i^s(\hat{v}_i, \boldsymbol{v}_{-i})]$.

Since bidders' valuations are identically and independently drawn, we consider symmetric ex-post allocation rules and payment rules. An ex-post allocation rule is *symmetric* if for any permutation $\pi : [n] \to [n]$, $x_{\pi(i)}(v_{\pi(1)}, v_{\pi(2)}, \cdots, v_{\pi(n)}) = x_i(v_1, \cdots, v_n)$. Therefore, the interim allocation rule and payment rule are the same for all buyers with the same valuation and we set $a(\cdot) = a_i(\cdot)$ and $p(\cdot) = p_i(\cdot)$ for all i.

[2] Due to the estimation of the virtual welfare in the reduction, directly applying their results provides an algorithm to check whether the interim allocation is approximately supply-monotone, i.e., the allocation rule only violates the supply-monotone constraints by a small constant, but it does not provide an efficient algorithm to verify supply monotonicity exactly.

[3] Such discretization is standard in the literature, e.g., in Pai and Vohra [24].

Given the number of bidders and the distribution over the valuations, Border [6, 7] characterizes the space of feasible interim allocation rules for a single unit by a set of linear inequalities.

Theorem 2.1. (Border's Theorem [6]) *An interim allocation rule $a(\cdot)$ is feasible if and only if for all $V' \subseteq V$,*

$$n \cdot \sum_{v \in V'} f(v) \cdot a(v) \leq 1 - (1 - \sum_{v \in V'} f(v))^n.$$

Intuitively, Border's theorem states that an interim allocation rule is feasible if and only if for every set of valuations V', the probability that a bidder with valuation in set V' receives the item must be less than or equal to the probability that at least one bidder has valuation in set V'. The necessity of this condition is obvious, but sufficiency is non-trivial. Although there are exponentially many linear constraints in this theorem, Border [6] points out that it suffices to check only a linear number of constraints:

Theorem 2.2. (Border [6]) *Let $E^\beta = \{v \in V \mid a(v) \geq \beta\}$ for $\beta \in [0,1]$. $a(\cdot)$ is feasible if and only if for all β,*

$$n \cdot \sum_{v \in E^\beta} f(v) \cdot a(v) \leq 1 - (1 - \sum_{v \in E^\beta} f(v))^n.$$

Note that given an allocation rule $a(\cdot)$, there are at most $|V|$ different sets E^β. Therefore, we can check the feasibility of an interim allocation rule in polynomial time.

2.3 Bayesian IC and IR. To incentivize bidders to participate in the auction and report truthfully, we require the auction satisfies Bayesian individual rationality (BIR) and Bayesian incentive compatibility (BIC) for all supplies.

Definition 2.2. (Bayesian Individual Rational) *For all $s \in \mathbb{Z}^+$,*

$$\forall v \in V, v \cdot a^s(v) - p^s(v) \geq 0 \text{ and } p^s(v) \leq B.$$

Definition 2.3. (Bayesian Incentive Compatible) *For all $s \in \mathbb{Z}^+$,*

$$\forall \hat{v} \in V, v \cdot a^s(v) - p^s(v) \geq v \cdot a^s(\hat{v}) - p^s(\hat{v}).$$

Myerson [23] provides a simple characterization of interim allocation rules and payment rules that ensure that the auction satisfies both BIC and BIR properties.

Theorem 2.3. (Myerson's Lemma [23])

- *An auction is BIC for supply s if and only if the interim allocation rule $a^s(\cdot)$ is monotonically non-decreasing;*
- *If $a^s(\cdot)$ is monotonically non-decreasing, then there is a unique payment rule $p^s(\cdot)$ such that the auction is both BIC and BIR (assuming $a^s(0) = 0$ and $p^s(0) = 0$), where $p^s(v) = v \cdot a^s(v) - \sum_{v'=1}^{v-1} a^s(v')$.*

Moreover, the expected revenue is given by $n \cdot \sum_{v \in V} f(v) \cdot \nu(v) \cdot a^S(v)$.

3 Feasible Interim Allocation Rule for Supply-Monotone Auctions

In this section, we characterize and give an efficient implementation of feasible interim allocation rules for supply-monotone, multi-unit auctions with i.i.d. bidders. Our characterization is for the general supply-monotone, multi-unit auctions, so that it does not depend on the budget constraints. The budget constraints will be incorporated in Section 4. We note that the results of this section can be extended to identical but not independent bidders. We give this extension in Appendix A. We first provide a mathematical characterization of feasibility in Section 3.1; in Section 3.2, we show how to efficiently check feasibility; and finally, we discuss how to efficiently implement a supply-monotone auction in Section 3.3.

3.1 Necessary and Sufficient Conditions for Feasibility. We say an interim allocation rule $a^s(\cdot)$ with $1 \leq s \leq S$ is *feasible* if there exists a corresponding ex-post allocation rule $x^s(\cdot)$ that satisfies supply monotonicity. The following theorem gives necessary and sufficient conditions for feasibility:

Theorem 3.1. *A monotonically non-decreasing interim allocation rule $a^s(\cdot)$ with $1 \leq s \leq S$ is feasible if and only if for all $V^S \subseteq V^{S-1} \subseteq \cdots \subseteq V^1 \subseteq V$,*

$$n \cdot \sum_{s=1}^{S} \sum_{v \in V^s} f(v) \cdot (a^s(v) - a^{s-1}(v))$$

$$\leq \sum_{s=1}^{S} (1 - (1 - \sum_{v \in V^s} f(v))^n). \tag{3.1}$$

This theorem extends Border's theorem for a single unit (Theorem 2.1) to the multi-unit setting. For a single unit, the Border's theorem characterizes a necessary and sufficient condition via a set of inequalities over subsets of valuations. In a multi-unit setting, in order to capture the relationship between different units, we introduce a hierarchy of subsets $V^S \subseteq V^{S-1} \subseteq \cdots \subseteq V^1 \subseteq V$, in which the subset of the current supply is a superset of the subset of the next supply.

To prove Theorem 3.1, we first replace Eq. (3.1) by a slightly more involved but equivalent characterization.

LEMMA 3.1. *For all $V^S \subseteq V^{S-1} \subseteq \cdots \subseteq V^1 \subseteq V$, Eq.* (3.1) *holds if and only if for all i and $V_i^S \subseteq V_i^{S-1} \subseteq \cdots \subseteq V_i^1 \subseteq V$,*

$$\sum_i \sum_{s=1}^{S} \sum_{v_i \in V_i^s} f(v_i) \cdot (a^s(v_i) - a^{s-1}(v_i))$$

$$\leq \sum_{s=1}^{S} (1 - \prod_i (1 - \sum_{v_i \in V_i^s} f(v_i))). \tag{3.2}$$

Proof. For the if direction, by setting $V_i^s = V^s$ for all i in Eq. (3.2), we have

$$\sum_{i,s,v_i \in V_i^s} f(v_i)(a^s - a^{s-1})(v_i) = n \sum_{s,v \in V^s} f(v)(a^s - a^{s-1})(v)$$

and

$$\sum_{s=1}^{S} (1 - \prod_i \sum_{v_i \notin V_i^s} f(v_i)) = \sum_{s=1}^{S} (1 - (\sum_{v \notin V^s} f(v))^n).$$

For the only-if direction, notice that from Eq. (3.1), summing over bidders, we have

$$\sum_{i,s,v_i \in V_i^s} f(v_i)(a^s - a^{s-1})(v_i)$$

$$\leq \sum_i \frac{1}{n} \sum_{s=1}^{S} (1 - (\sum_{v_i \notin V_i^s} f(v_i))^n)$$

It suffices to show that

$$\sum_{i,s} \frac{1}{n} (1 - (\sum_{v_i \notin V_i^s} f(v_i))^n) \leq \sum_{i,s} \frac{1}{n} (\sum_{v \notin V_i^s} f(v))^n$$

Due to the fact that the arithmetic mean is at least the geometric mean, we have for any $V_1^s, \cdots, V_n^s$

$$\prod_i \sum_{v_i \notin V_i^s} f(v_i) \leq \frac{1}{n} \cdot \sum_i (\sum_{v_i \notin V_i^s} f(v_i))^n$$

Therefore, we have

$$\prod_i \sum_{v_i \notin V_i^s} f(v_i) \leq \frac{1}{n} \cdot \sum_i (\sum_{v_i \notin V_i^s} f(v_i))^n$$

which implies

$$S - \sum_{s=1}^{S} \sum_i \frac{1}{n} \cdot (\sum_{v \notin V_i^s} f(v))^n \leq S - \sum_{s=1}^{S} \prod_i (\sum_{v \notin V_i^s} f(v)),$$

and therefore, we have

$$\sum_i \frac{1}{n} \cdot \sum_{s=1}^{S} (1 - (\sum_{v \notin V_i^s} f(v))^n) \leq \sum_{s=1}^{S} (1 - \prod_i (\sum_{v \notin V_i^s} f(v))).$$

We now show that a monotonically non-decreasing interim allocation rule is supply-monotone if and only if for all i and $V_i^S \subseteq V_i^{S-1} \subseteq \cdots \subseteq V_i^1 \subseteq V$, Eq. (3.2) holds.

Only-if direction. Let $x_i^s(\cdot)$ be the ex-post supply-monotone allocation rule that induces $a^s(\cdot)$. For the convenience, we assume $a^0(\cdot) = 0$ and let $V_i^{S+1} = \emptyset, V_i^0 = V$. Let $U_i^s = V_i^s \setminus V_i^{s+1}$ for all $0 \leq s \leq S$. Notice that U_i^s are disjoint sets. Also, define functions $g_i : V \to \{0, \cdots, S\}$ such that $g_i(v_i) = s$ if $v_i \in U_i^s$.

Let us randomly select a valuation profile $\boldsymbol{v}$ such that v_i is independently drawn from distribution f, and define $s_{\max} = \max_i g_i(v_i)$. Bidder i is given a lottery that provides $s_{\max}$ units with probability $x_i^{g_i(v_i)}(\boldsymbol{v})/s_{\max}$ and 0 units with probability $1 - x_i^{g_i(v_i)}(\boldsymbol{v})/s_{\max}$. Then, the expected number of allocated units for bidder i is

$$\sum_{s=1}^{S} \sum_{v_i \in U_i^s} \sum_{\boldsymbol{v}_{-i}} \Pr_{\boldsymbol{v}}[\boldsymbol{v} = (v_i, \boldsymbol{v}_{-i})] \cdot x_i^s(v_i, \boldsymbol{v}_{-i})$$

$$= \sum_{s=1}^{S} \sum_{v_i \in U_i^s} f(v_i) \sum_{\boldsymbol{v}_{-i} \in V^{n-1}} \Pr_{\boldsymbol{v}'}[\boldsymbol{v}' = \boldsymbol{v}_{-i}] \cdot x_i^s(v_i, \boldsymbol{v}_{-i})$$

$$= \sum_{s=1}^{S} \sum_{v_i \in U_i^s} f(v_i) a^s(v_i)$$

$$= \sum_{s=1}^{S} \sum_{v_i \in U_i^s} \sum_{s'=1}^{s} f(v_i)(a^{s'}(v_i) - a^{s'-1}(v_i))$$

$$= \sum_{s=1}^{S} \sum_{v_i \in V_i^s} f(v_i)(a^s(v_i) - a^{s-1}(v_i))$$

Note that the summation of the last formula over bidders is equivalent to the LHS of Eq. (3.2).

Next, consider another random process: (1) randomly select a valuation profile $\boldsymbol{v}$ such that v_i is independently drawn according to distribution f; (2) generate $s_{\max} = \max_i g_i(v_i)$ units, i.e., for each $s \in \{1, \cdots, S\}$, if there exists i such that $g_i(v_i) \geq s$, then generate 1 unit. Therefore, the expected number of generated units is

$$\sum_{s=1}^{S} \Pr[\exists v_i \in \cup_{s'=s}^{S} U_i^{s'}] = \sum_{s=1}^{S} \Pr[\exists v_i \in V_i^s]$$

$$= \sum_{s=1}^{S} (1 - \prod_i (1 - \sum_{v \in V_i^s} f(v)))$$

which is the RHS of Eq. (3.2).

Since $x(\cdot)$ is supply-monotone, we have $\sum_i x_i^{g_i(v_i)}(\boldsymbol{v}) \leq \sum_i x_i^{s_{\max}}(\boldsymbol{v}) \leq s_{\max}$ where the last inequality uses the fact that $x_i^{s_{\max}}$ is a valid ex-post allocation rule for supply $s_{\max}$. Therefore, the number of units generated by the first random process, which corresponds to the LHS of (3.2), is at most the number of units generated by the second random process. This concludes the only if direction.

If direction. The proof of this direction is based on the max-flow min-cut theorem. We form a six-layer s-t flow network (see Figure 1). The first layer has the source *source* and the last layer has the sink *sink*. In the second layer, vertex $source_s$ corresponds to the source on supply s and there exists an arc $(source, source_s)$ with capacity ∞ for each vertex $source_s$. Moreover, in the fifth layer, vertex $sink_s$ corresponds to sink on supply s and there exists an arc $(sink_s, sink)$ with capacity ∞ for each vertex $sink_s$.

In the third layer X, vertex $(s, \boldsymbol{v}) \in X$ corresponds to supply s and a possible valuation profile $\boldsymbol{v}$. For each $(s, \boldsymbol{v})$, there exists an arc $(source_s, (s, \boldsymbol{v}))$ with capacity $\prod_i f(v_i)$. We slightly abuse the notation to define $f(\boldsymbol{v}) = \prod_i f(v_i)$. Since f is a probability distribution, the total outgoing capacity from $source_s$ is 1. Moreover, for each $s < S$ and valuation profile $\boldsymbol{v}$, there exists an arc $((s, \boldsymbol{v}), (s+1, \boldsymbol{v}))$ with capacity ∞.

In the fourth layer Y, vertex $(s, i, v_i) \in Y$ corresponds to supply s, bidder i, and valuation $v_i \in V$. For each $(s, \boldsymbol{v}) \in X$, where $\boldsymbol{v} = (v_1, \cdots, v_n)$, there exists an arc $((s, \boldsymbol{v}), (s, i, v_i))$ with capacity ∞. Moreover, there exists an arc $((s, i, v_i), sink_s)$ with capacity $f(v_i) \cdot (a^s(v_i) - a^{s-1}(v_i))$. Finally, there exists a special vertex *noSell* in the fourth layer and for each vertex $(S, \boldsymbol{v})$ in the third layer, there exists an arc $((S, \boldsymbol{v}), noSell)$ with capacity ∞. Finally, we add an arc $(noSell, sink_S)$ with capacity $S - \sum_i \sum_{v_i \in V} f(v_i) a^S(v_i)$.

We show that once $a_i^s(\cdot)$ satisfies inequality (3.2), the max-flow/min-cut of the constructed network is at least S. Note that the maximum possible flow of the constructed network is S since there are totally 1 outgoing capacity from each $source_s$ and the summation of all incoming capacity for $sink_s$ is exactly S. Moreover, once the network has full flow, we can construct the ex-post allocation rule by setting $(x_i^s(\boldsymbol{v}) - x_i^{s-1}(\boldsymbol{v}))$ as the amount of flow on the arc $((s, \boldsymbol{v}), (s, i, v_i))$ divided by $f(\boldsymbol{v})$. By flow conservation property on node (s, i, v_i), we have $(a^s(v_i) - a^{s-1}(v_i)) \cdot f(v_i) = \sum_{\boldsymbol{v}:(v_i, \boldsymbol{v}_{-i})} (x_i^s(\boldsymbol{v}) - x_i^{s-1}(\boldsymbol{v})) \cdot f(\boldsymbol{v})$, i.e., $a_i^s(v_i) - a_i^{s-1}(v_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^s(\boldsymbol{v}_{-i}) - x_i^{s-1}(\boldsymbol{v}_{-i})]$.

To show the capacity of the min-cut is at least S, we argue that the capacity of any cut is at least S. Consider an arbitrary cut $(L, \overline{L})$ without any arc with ∞ capacity. Since the capacity of arcs $(source, source_s)$ and $(sink_s, sink)$ are ∞, we have $source_s \in L$ and $sink_s \in \overline{L}$ for all s. Therefore, the cut is constituted by the edges between $\{source_s\}$ and $\overline{L} \cap X$ in the third layer, and the edges between $L \cap Y$ in the fourth layer and $\{sink_s\}$. Moreover, if $noSell \in \overline{L}$, then for all $(s, \boldsymbol{v}) \in X$, $(s, \boldsymbol{v}) \in \overline{L}$. Thus, the capacity of the cut is exactly S. From now on, we consider the case when $noSell \in L$.

Let $A^s = \{\boldsymbol{v} \mid (s, \boldsymbol{v}) \in L\}$ and $B_i^s = \{v_i \mid (s, i, v_i) \in L\}$. First, note that $A^s \subseteq A^{s+1}$ for all $s < S$: if not, there exists $(s, \boldsymbol{v}) \in L$ but $(s+1, \boldsymbol{v}) \in \overline{L}$, and however, the arc connecting them has capacity ∞. By a similar argument, if $\boldsymbol{v} \in A^s$, then for all i, $v_i \in B_i^s$. To minimize the size of the cut, we assume $B_i^s = \{v_i' \mid \exists \boldsymbol{v} \in A^s, v_i = v_i'\}$. Therefore, since $A^s \subseteq A^{s+1}$, we have $B_i^s \subseteq B_i^{s+1}$ for all $s < S$. Moreover, we have $\boldsymbol{v} \in A^s$ only if for all i, $v_i \in B_i^s$. Let $V_i^s = V \setminus B_i^s$. Since $B_i^s \subseteq B_i^{s+1}$, we have $V_i^{s+1} \subseteq V_i^s$.

The total capacity between $\{source_s \mid 1 \leq s \leq S\}$ and $L \cap X$ in the third layer is $\sum_{s=1}^S \sum_{\boldsymbol{v}:\forall i, v_i \in B_i^s} \prod_i f(v_i) = \sum_{s=1}^S \prod_i \sum_{v_i \in B_i^s} f(v_i) = \sum_{s=1}^S \prod_i (1 - \sum_{v_i \in V_i^s} f(v_i))$. Henceforth, the total capacity between $\{source_s \mid 1 \leq s \leq S\}$ and $\overline{L} \cap X$ in the third layer is $S - \sum_{s=1}^S \prod_i (1 - \sum_{v_i \in V_i^s} f(v_i)) = \sum_{s=1}^S (1 - \prod_i (1 - \sum_{v_i \in V_i^s} f(v_i)))$, which is the RHS of the inequality (3.2).

The total capacity between $L \cap Y$ in the fourth layer and $\{sink_s \mid 1 \leq s \leq S\}$ is

$$
\begin{aligned}
&\sum_{s=1}^S \sum_i \sum_{v_i \in B_i^s} f(v_i) \cdot (a^s(v_i) - a^{s-1}(v_i)) \\
&\quad + \left[S - \sum_i \sum_{v_i \in V} f(v_i) a^S(v_i)\right] \\
&= S - \sum_{s=1}^S \sum_i \sum_{v_i \in V} f(v_i)(a^s(v_i) - a^{s-1}(v_i)) \\
&\quad + \sum_{s=1}^S \sum_i \sum_{v_i \in B_i^s} f(v_i) \cdot (a^s(v_i) - a^{s-1}(v_i)) \\
&= S - \sum_{s=1}^S \sum_i \sum_{v_i \in V \setminus B_i^s} f(v_i)(a^s(v_i) - a^{s-1}(v_i)) \\
&= S - \sum_{s=1}^S \sum_i \sum_{v_i \in V_i^s} f(v_i)(a^s(v_i) - a^{s-1}(v_i))
\end{aligned}
$$

which is S minus the LHS of the inequality (3.2). Since we assume that the LHS is less than or equal to the RHS, we have the total capacity of the cut is at least S.

3.2 Efficient Verification of Feasibility. In Theorem 3.1, we gave an exponential set of inequalities that

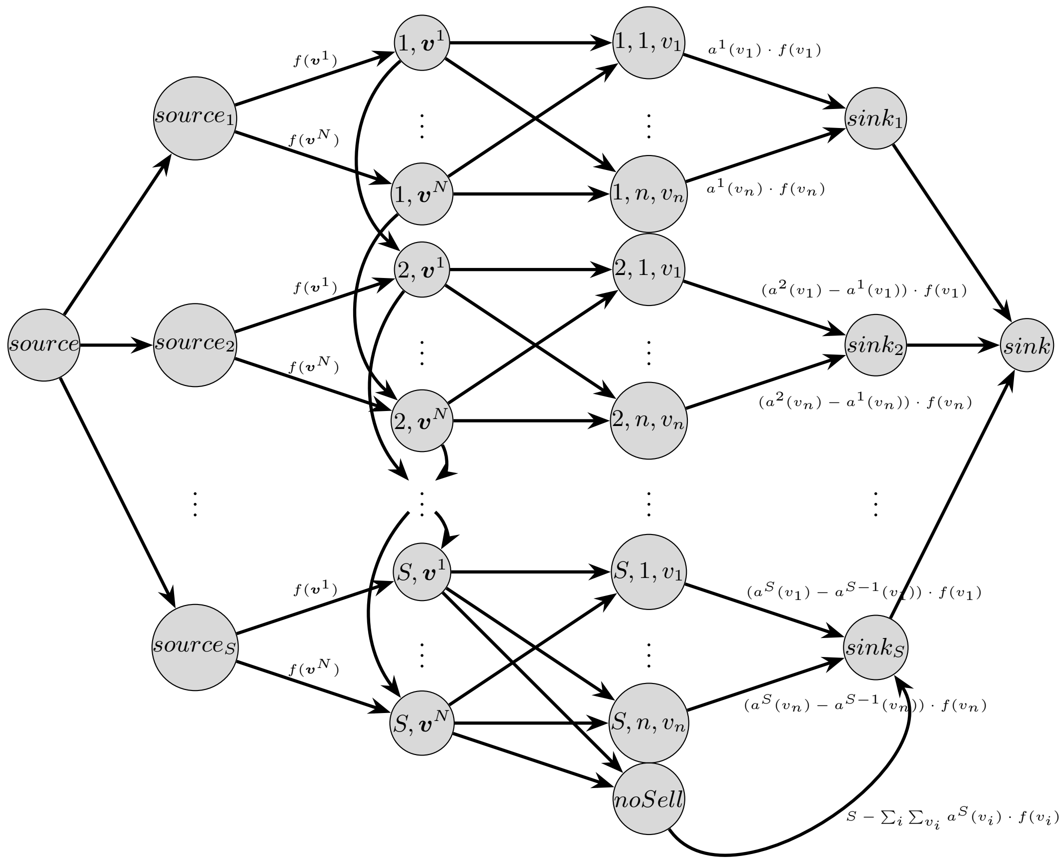

Figure 1: Network Construction: edges without labels are of capacity ∞

exactly characterizes feasibility of an interim allocation rule. In this section, we design an efficient algorithm to check these inequalities. Given a symmetric ex-post allocation rule $x_i^s(\cdot)$ satisfying supply monotonicity, we first decompose $x_i^s(\cdot)$ into $y_i^s(k, \cdot)$, denoting the probability that that bidder i receives the k-th unit when s units have arrived.

LEMMA 3.2. *For a symmetric supply-monotone ex-post allocation rule $x_i^s(\cdot)$, there exists symmetric $y_i^s(k, \cdot)$ such that for all $\boldsymbol{v}$, the following hold:*

1. $y_i^s(k, \boldsymbol{v}) = 0$ *if* $k > s$ *for all* i*;*
2. $y_i^s(k, \boldsymbol{v}) \le y_i^{s+1}(k, \boldsymbol{v})$ *for all* i*,* s*, and* k*;*
3. $\sum_i y_i^S(k, \boldsymbol{v}) \le 1$ *for all* k*;*
4. $x_i^s(\boldsymbol{v}) = \sum_{k=1}^s y_i^s(k, \boldsymbol{v})$ *for all* i *and* s*.*

Proof. Intuitively, given a valuation profile $\boldsymbol{v}$, we construct $y_i^s(k, \boldsymbol{v})$ by creating a plan to allocate the units in order to fulfill the given ex-post allocation rule $x_i^s(\boldsymbol{v})$. One possible plan is to enumerate s in the increasing order and then enumerate each buyer i for each s. For each $x_i^s(\boldsymbol{v})$, we fulfill the allocation rule with the unallocated unit that arrives earliest (see Algorithm 3.1).

Formally, we first partition the valuation profile into groups such that for valuation profile $\boldsymbol{v}$ and $\boldsymbol{v}'$, if there exists a permutation $\pi : [n] \to [n]$ such that $(v'_1, \cdots, v'_n) = (v_{\pi(1)}, \cdots v_{\pi(n)})$, then $\boldsymbol{v}$ and $\boldsymbol{v}'$ are in the same group. For each group, we select a representative valuation profile $\boldsymbol{v}$ from the group and run Algorithm 3.1 to obtain $y_i^s(k, \boldsymbol{v})$. The algorithm enumerates the interim allocation rule in an increasing order of supply and for each supply, enumerates all the buyers. Then, the algorithm allocates the remaining fraction from the earliest supply to fulfill the demand $x_i^s(\boldsymbol{v}) - x_i^{s-1}(\boldsymbol{v})$. To make sure $y_i^s(k, \cdot)$ is symmetric, for each $\boldsymbol{v}'$ such that $\boldsymbol{v}' = (v_{\pi(1)}, \cdots v_{\pi(n)})$, we set $y_{\pi(i)}^s(k, \boldsymbol{v}') = y_i^s(k, \boldsymbol{v})$.

To show that the output enjoys the properties in the statement of the lemma, we first notice that since $x_i^s(\boldsymbol{v})$ is a valid ex-post allocation rule for supply s, we have

$\sum_i x_i^s(\boldsymbol{v}) \le s$. Therefore, *currentUnit* is never larger than s. Combining with the assignment of $y_i^s(k,\boldsymbol{v})$ at line 8-10 we have $y_i^s(k,\boldsymbol{v}) = 0$ if $k > s$ for all i. Furthermore, by the assignment of $y_i^s(k,\boldsymbol{v})$ at line 8-10 and the fact that *allocate* is always positive, we have $y_i^s(k,\boldsymbol{v}) \le y_i^{s+1}(k,\boldsymbol{v})$ for all i, s, and k. Moreover, for the k-th unit of supply, when a fraction of the unit is allocated by the amount *allocate*, i.e. *remain* is deducted by *allocate*, there is exactly one buyer i such that $y_i^S(k,\boldsymbol{v})$ is added by *allocate*. Therefore, we have $\sum_i y_i^S(k,\boldsymbol{v}) \le 1$ for all k.

For the final property, we will show by induction that after fulfilling the demand of $x_i^s(\boldsymbol{v}) - x_i^{s-1}(\boldsymbol{v})$, we have for all $s' \ge s$, $\sum_{k=1}^{s'} y_i^{s'}(k,\boldsymbol{v}) = x_i^s(\boldsymbol{v})$. Assume the hypothesis is correct for s and we now consider the demand of $x_i^{s+1}(\boldsymbol{v}) - x_i^s(\boldsymbol{v})$. After fulfilling the demand, we must have $x_i^{s+1}(\boldsymbol{v}) = \sum_{k=1}^{s+1} y_i^{s+1}(k,\boldsymbol{v})$. Notice that the algorithm will increase $y_i^{s'}(k,\boldsymbol{v})$ for all $s' \ge s+1$ with the same amount. Therefore, combining with the induction hypothesis, the hypothesis holds for $s+1$.

Algorithm 3.1.

```
Let curUnit = 1 and remain = 1;
for s: 1 to S do
  for i: 1 to n do
    Let demand = x_i^s(v) − x_i^{s−1}(v);
    while demand > 0 do
      Let allocate = min{remain, demand};
      for s': s to S do
        y_i^{s'}(curUnit, v) = y_i^{s'}(curUnit, v)+allocate;
      end for
      demand = demand − allocate
      remain = remain − allocate
      if remain = 0 then
        curUnit = curUnit + 1;
        remain = 1;
      end if
    end while
  end for
end for
return y_i^s(k, v);
```

Consider the interim allocation rule $\gamma_i^s(k,v_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[y_i^s(k,(v_i,\boldsymbol{v}_{-i}))]$. Since $y_i^s(k,\cdot)$ is symmetric, $\gamma_i^s(k,v_i)$ is the same for all buyers with the same valuation and we let $\gamma^s(k,v) = \gamma_i^s(k,v)$ for simplicity. Therefore, an interim allocation $a^s(\cdot)$ can be represented by $a^s(v) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^s(v,\boldsymbol{v}_{-i})] = \sum_{k=1}^s \mathbb{E}_{\boldsymbol{v}_{-i}}[y_i^s(k,(v,\boldsymbol{v}_{-i}))] = \sum_{k=1}^s \gamma^s(k,v) \le \sum_{k=1}^s c^k(v)$, where $c^k(v) = \gamma^S(k,v)$, representing the total interim allocation from the k-th unit to a bidder with valuation v. Since $c^k(\cdot)$ is the interim allocation rule of a valid ex-post allocation rule $y_i^S(k,\cdot)$, $c^k(\cdot)$ must satisfy Border's theorem (Theorem 2.1). In other words, it must be a feasible solution to LP (Supply-monotone). In fact, we show that this is not only a necessary condition but also a sufficient one, i.e., every feasible solution to LP (Supply-monotone) yields a feasible interim allocation rule.

Theorem 3.2. *An interim allocation rule $a^s(\cdot)$ for $s = 1,\cdots,S$, is supply-monotone if and only if there exists $c^s : V \to \mathbb{R}_+$ for all i and $s = 1,\cdots,S$ that is feasible for LP* (Supply-monotone).

Proof. We have already argued the "only-if" direction, so we focus on showing the "if" direction now. If there exists $c^s : V \to \mathbb{R}$ for $s = 1,\cdots,S$ that satisfies the statement of the theorem, by Theorem 2.1, there exists an ex-post allocation rule $\alpha_i^s(\cdot)$ such that $c^s(v_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[\alpha_i^s(v_i,\boldsymbol{v}_{-i})]$. Let $\beta_i^s(\boldsymbol{v}) = \sum_{k=1}^s \alpha_i^k(\boldsymbol{v})$, then $\beta_i^s(\cdot)$ satisfies supply monotonicity. Moreover, we have

$$\begin{aligned} a^s(v) &\le \sum_{k=1}^s c^k(v) = \sum_{k=1}^s \mathbb{E}_{\boldsymbol{v}_{-i}}[\alpha_i^s(v,\boldsymbol{v}_{-i})] \\ &= \mathbb{E}_{\boldsymbol{v}_{-i}}[\beta_i^s(v,\boldsymbol{v}_{-i})]. \end{aligned}$$

We want to create a supply-monotone ex-post allocation rule $x_i^s(\cdot)$ such that $a^s(v) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^s(v,\boldsymbol{v}_{-i})]$ with additional property that $x_i^s(\cdot) \le \beta_i^s(\cdot)$.

For $s = 1$, if setting $x_i^1(\cdot) = 0$, we have

$$\mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^1(v_i,\boldsymbol{v}_{-i})] = 0 \le a^1(v_i).$$

Note that changing $x_i^1(v_i,\boldsymbol{v}_{-i})$ only affects the value of $a^1(v_i)$. Therefore, there exists $0 \le x_i^1(\cdot) \le \beta_i^1(\cdot)$ such that $a^1(v) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^1(v,\boldsymbol{v}_{-i})]$.

By induction, assume that there exists a $x_i^s(\cdot)$ satisfies supply monotonicity, $x_i^s(\cdot) \le \beta_i^s(\cdot)$, and

$$a^s(v_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^s(v_i,\boldsymbol{v}_{-i})].$$

for $s \le \overline{s}$. For $s = \overline{s}+1$, if setting $x_i^{\overline{s}+1}(\cdot) = x_i^{\overline{s}}(\cdot)$, then

$$\mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^{\overline{s}+1}(v_i,\boldsymbol{v}_{-i})] = a^{\overline{s}}(v_i) \le a^{\overline{s}+1}(v_i)$$

since $a^s(\cdot)$ is monotonically non-decreasing. Moreover, if setting $x_i^{\overline{s}+1}(\cdot) = \beta_i^{\overline{s}+1}(\cdot)$, then

$$\mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^{\overline{s}+1}(v_i,\boldsymbol{v}_{-i})] = c_i^{\overline{s}+1}(v_i,\boldsymbol{v}_{-i}) \ge a^{\overline{s}+1}(v_i).$$

Therefore, by the fact that changing $x_i^{\overline{s}+1}(v_i,\boldsymbol{v}_{-i})$ only affects the value of $a^{\overline{s}+1}(v_i)$, there exists $x_i^{\overline{s}+1}(\cdot)$ such that $x_i^{\overline{s}}(\cdot) \le x_i^{\overline{s}+1}(\cdot) \le \beta_i^{\overline{s}+1}(\cdot)$ and $a^{\overline{s}+1}(v_i) = \mathbb{E}_{\boldsymbol{v}_{-i}}[x_i^{k+1}(v_i,\boldsymbol{v}_{-i})]$.

Although there are exponentially many constraints in this linear program, by Theorem 2.2, we can construct an efficient separation oracle to identify a violating constraint. Therefore, we can determine whether a monotonically non-decreasing interim allocation rule is supply-monotone in $O(\text{poly}(|V|,S))$ time.

(Supply-monotone)

$$\begin{array}{llll} \max & 0 & & \\ s.t. & a^s(v) \le \sum_{k=1}^{s} c^k(v) & \forall v \in V, s \le S & \text{(Supply-monotone I)} \\ & n \cdot \sum_{v\in V'} f(v)c^s(v) \le 1 - (1 - \sum_{v\in V'} f(v))^n & \forall V' \subseteq V, s \le S & \text{(Supply-monotone II)} \\ & a^s(v) \le a^{s+1}(v) & \forall v \in V, s < S & \text{(Supply-monotone III)} \\ & c^s(v) \ge 0 & \forall v \in V, s \le S & \end{array}$$

3.3 Efficient Implementation for a Feasible Interim Allocation Rule. Till now, we have shown that we can efficiently verify if an interim allocation rule is supply-monotone. However, can we implement such an auction efficiently? More precisely, can we design a polynomial-time algorithm to recover the underlying ex-post allocation rule $x_i^s(\cdot)$ from a supply-monotone interim allocation rule $a^s(\cdot)$? Cai et al. [9] show that an interim allocation rule for a single unit can be represented by a convex combination of $|V|+1$ hierarchical mechanisms.

Definition 3.1. (Hierarchical Mechanisms) *A hierarchical mechanism consists of a function $H : V \to \{0, 1, \cdots, |V|\}$. On bid vector $(v_1, \cdots, v_n)$, if $H(v_i) = 0$ for all i, the mechanism throws the unit away; otherwise, the unit is awarded uniformly at random to a bidder in $\arg\max_i H(v_i)$.*

We generalize their definition to *supply-monotone hierarchical mechanisms.*

Definition 3.2. *A supply-monotone hierarchical mechanism consists of a set of functions H^s that maps V to $\{0, \cdots, |V|\}$ and G^s that maps V to $\{1, \cdots, S\}$ for the s-th unit. On bid vector $(v_1, \cdots, v_n)$, for the s-th unit, if $H^s(v_i) = 0$ for all i, the mechanism throws the unit away; otherwise, the unit is divided uniformly at random to bidders in $\arg\max_i H^s(v_i)$ and allocated to bidder i when the $G^s(v_i)$-th unit arrives.*

A supply-monotone hierarchical mechanism is a hierarchical mechanism that defers allocations based on bidders' valuations. When the s-th unit appears, the mechanism computes how to allocate the unit according to the function H^s. However, the actual allocation only happens when the $G^s(v_i)$-th unit appears in the future.

Theorem 3.3. *A supply-monotone mechanism can be represented by a convex combination of at most $|V| \cdot S^2 + 1$ supply-monotone hierarchical mechanisms.*

Proof. Similar to the ex-post allocation rule, we can decompose the interim allocation rule $a^s(\cdot)$ as $a^s(v) = \sum_{k=1}^{s} \gamma^s(k, v)$ where $\gamma^s(k, v)$ represents the expected number of units that a bidder receives from the k-th unit when the supply is s. Let $\pi^k : \{1, \cdots, |V|\} \to V$ be an ordering over V such that for all $1 \le j < |V|$, $\gamma^S(k, \pi^k(j)) \le \gamma^S(k, \pi^k(j+1))$. By Theorem 2.2, $\gamma^S(k, \cdot)$ is feasible if and only if

$$n \sum_{j' \ge j} f(\pi^k(j')) \cdot \gamma^S(k, \pi^k(j')) \le 1 - (1 - \sum_{j' \ge j} f(\pi^k(j')))^n.$$

Consider the following set of feasibility constraints:

$$\theta^s(k, \pi^k(j)) \le \theta^{s+1}(k, \pi^k(j))$$
$$\theta^S(k, \pi^k(j)) \le \theta^S(k, \pi^k(j+1))$$

and

$$n \cdot \sum_{j' \ge j} f(\pi^k(j')) \cdot \theta^S(k, \pi^k(j')) \le 1 - (1 - \sum_{j' \ge j} f(\pi^k(j')))^n$$

while $\theta^s(k, \pi^k(j)) \ge 0$ for all s, j, k. Notice that $\gamma^r(\cdot, \cdot)$ is inside this polytope. Cai et al. [9] demonstrates that for any corner of the polytope, $\theta^S(k, \cdot)$ corresponds to a hierarchical mechanism. As for $\theta^s(k, \pi^k(j))$ with $s < S$, at a corner of the polytope, there must exists a $G^k(\pi^k(j)) \ge k$ such that for all $s < G^k(\pi^k(j))$, $\theta^s(k, \pi^k(j)) = 0$ and for all $s \ge G^k(\pi^k(j))$, $\theta^s(k, \pi^k(j)) = \theta^S(k, \pi^k(j))$. Therefore, any corner of the polytope corresponds to a supply-monotone hierarchical mechanism. Finally, since there are totally $|V| \cdot S^2$ variables. By Carathéodory theorem, $\gamma^s(\cdot, \cdot)$ can be represented by $(|V| \cdot S^2 + 1)$ corners of the polytope.

An efficient algorithm to sample a supply-monotone hierarchical mechanism can be obtained by applying the algorithm developed by Cai et al. [9] for the single unit case. As for the payment, we can maintain BIC and BIR by simply charging the interim payment $p^s(v)$ for a bidder with valuation v no matter which supply-monotone hierarchical mechanism is sampled.

4 Online Auction with Budget Constrained Bidders

The previous section allows us to focus on feasible interim allocation rules when designing a supply-monotone auction. In this section, we design an algorithm to compute a revenue-maximizing supply-monotone auction for symmetric budget constrained buyers.

Laffont and Robert [19] and Pai and Vohra [24] characterized the optimal auction for symmetric budget constrained buyers with known supply. They showed that the *revenue-maximizing auction* $OPTa^s(\cdot)$ corresponding to supply s is the optimal solution of the following linear program (RevOpt). This LP encodes a revenue-maximizing feasible interim mechanism, which also satisfies incentive compatibility and budget constraints. In particular, by Lemma 2.3, the interim allocation rule must be monotone, i.e., $a^s(v) \le a^s(v+1)$ for all $v \in V$. Therefore, by Theorem 2.2, possible choice of E^β are $\{v' \in V \mid v' \ge v\}$ for all v.

We consider the setting where the total supply S is unknown to the bidders and the seller, and design a supply-monotone auction with a tight competitive ratio.

Definition 4.1. *A supply-monotone auction $a^s(\cdot)$ is τ-competitive for maximum possible supply S if for all $1 \le s \le S$,*

$$n \cdot \sum_{v \in V} f(v)\nu(v)a^s(v) \ge \tau \cdot n \cdot \sum_{v \in V} f(v)\nu(v)OPTa^s(v).$$

By Theorem 3.2 and 3.3, one can simply compute an auction with optimal competitive ratio by a linear program to optimize the competitive ratio τ. Assuming the maximum possible supply is S, we can obtain a linear program LP (Supply-monotone RevOpt) by replacing the Border's constraints in LP (RevOpt) with the supply-monotone constraints in LP (Supply-monotone) and introducing an additional constraint to track the competitive ratio.

However, the complexity of solving this LP depends on the (possibly unknown) supply S, which could be exponentially large compared to the number of valuations $|V|$. To alleviate this problem, we show there are only $O(|V|)$ *critical* supply values that must be included in the LP. Our main tool is a novel characterization of the gradient of the optimal offline allocation rule $OPTa^s(\cdot)$. For convenience, we assume $OPTa^s(0) = 0$ for all s.

4.1 Gradient Characterization.

Consider the gradient of the optimal interim allocation rule:

$$g^s(\cdot) = \lim_{\Delta \to 0^+} \frac{OPTa^{s+\Delta}(\cdot) - OPTa^s(\cdot)}{\Delta}$$

Note that if for all $s \in \mathbb{R}^+$, $g^s(\cdot)$ satisfies the Border's theorem (Theorem 2.1), then $OPTa^s(\cdot)$ is supply-monotone. Motivated by this observation, we explore how the gradients violate Theorem 2.1. Note that until the supply is large enough to make the budget constraint tight in the optimal solution, the optimal auction is essentially Myerson's auction [23] on a single unit multiplied by the supply. Therefore, at this stage, the gradient is the interim allocation rule of Myerson's auction. Let s^B be the supply when the budget constraint becomes tight for the first time.

The following lemma provides an LP that characterizes the gradients for $s \ge s^B$.

Lemma 4.1. *For $s \ge s^B$, if the budget constraint in LP (RevOpt) is tight by $OPTa^s(\cdot)$, then $g^s(\cdot)$ is the optimal solution of LP (OPTGrad) where $v \in T^s$ if and only if $n \cdot \sum_{v' \ge v} f(v) \cdot OPTa^s(v) = s \cdot (1 - (1 - \sum_{v' \ge v} f(v))^n)$.*

Proof. We first show that if $a(\cdot)$ is a feasible solution to the LP (OPTGrad), then there exists $\overline{\Delta} > 0$ such that for all $0 < \Delta < \overline{\Delta}$, $OPTa^s(\cdot) + \Delta \cdot a(\cdot)$ is a feasible solution to the LP (RevOpt) on supply $s + \Delta$. Let

$$\Delta_{IC} = \min_{a(v) > a(v+1)} \frac{OPTa^s(v+1) - OPTa^s(v)}{a(v) - a(v+1)}$$

First, since $OPTa^s(\cdot)$ satisfies the incentive compatibility, we have $OPTa^s(v+1) \ge OPTa^s(v)$. Moreover, notice that if $a(v) > a(v+1)$, we must have $OPTa^s(v+1) > OPTa^s(v)$ since $OPTa^s(v+1) = OPTa^s(v)$ implies $a(v) \le a(v+1)$ because $a(\cdot)$ is a feasible solution. Thus, by our definition, $\Delta_{IC} > 0$. Intuitively, Δ_{IC} represents the maximum Δ such that $OPTa^s(\cdot) + \Delta \cdot a(\cdot)$ is still monotonically non-decreasing in v. Let $\Delta_{Border} = \min_{v \in \bar{T}^s} \frac{(1-(1-\sum_{v' \ge v} f(v'))^n) - n \cdot \sum_{v' \ge v} f(v') \cdot OPTa^s(v')}{n \cdot \sum_{v' \ge v} f(v') \cdot a(v') - (1-(1-\sum_{v' \ge v} f(v'))^n)}$, where $v \in \overline{T}^s$ if $n \cdot \sum_{v' \ge v} f(v') \cdot a(v') > 1 - (1 - \sum_{v' \ge v} f(v'))^n$, i.e., $a(\cdot)$ violates the Border's constraint corresponding to v. Since $a(\cdot)$ is a feasible solution, $v \in \overline{T}^s$ only if $v \notin T^s$. Therefore, $\Delta_{Border} > 0$. Intuitively, Δ_{Border} represents the maximum Δ such that $OPTa^s(\cdot) + \Delta \cdot a(\cdot)$ still satisfies all Border's constraints. Finally, note that the budget constraint is always satisfied since for a feasible solution, the increment of the payment is always non-positive.

Therefore, by setting $\overline{\Delta} = \min(\Delta_{IC}, \Delta_{Border})$, we have for all $\Delta < \overline{\Delta}$, $OPTa^s(\cdot) + \Delta \cdot a(\cdot)$ is a feasible solution to the LP (RevOpt) on supply $s + \Delta$.

Next, we show that if $a'(\cdot)$ is not a feasible solution to the LP (OPTGrad), $OPTa^s(\cdot) + \Delta a'(\cdot)$ is infeasible to the LP (RevOpt) for any $\Delta > 0$:

- If $a'(v) > a'(v+1)$ for a v such that $OPTa^s(v) = OPTa^s(v+1)$, then $OPTa^s(v) + \Delta a'(v) > OPTa^s(v+1) + \Delta \cdot a'(v+1)$ for any $\Delta > 0$, violating the incentive compatibility;

(RevOpt)

$$
\begin{array}{llll}
\max & n \cdot \sum_{v \in V} f(v)\nu(v)a^s(v) & & \\
s.t. & a^s(v) \le a^s(v+1) & \forall v < v_{\max} & \text{(Incentive Compatible)} \\
& v_{\max} \cdot a^s(v_{\max}) - \sum_{v=1}^{v_{\max}-1} a^s(v) \le B & & \text{(Budget Constraint)} \\
& n \cdot \sum_{v' \ge v} f(v')a^s(v') \le s \cdot (1 - (1 - \sum_{v' \ge v} f(v'))^n) & \forall v \in V & \text{(Border's Constraint)} \\
& a^s(v) \ge 0 & \forall v \in V &
\end{array}
$$

(Supply-monotone RevOpt)

$$
\begin{array}{llll}
\max & \tau & & \\
s.t. & \sum_{v \in V} f(v)\nu(v)a^s(v) \ge \tau \cdot \sum_{v \in V} f(v)\nu(v)OPTa^s(v) & \forall s \le S & \text{(Competitive Ratio)} \\
& a^s(v) \le a^s(v+1) & \forall v \in V, s \le S & \text{(Incentive Compatible)} \\
& v_{\max} \cdot a^s(v_{\max}) - \sum_{v=1}^{v_{\max}-1} a^s(v) \le B & \forall s \le S & \text{(Budget Constraint)} \\
& a^s(v) \le \sum_{k=1}^{s} c^k(v) & \forall s \le S, v \in V & \text{(Supply-monotone I)} \\
& n \cdot \sum_{v' \in V'} f(v')c^s(v') \le 1 - (1 - \sum_{v' \in V'} f(v'))^n & \forall V' \subseteq V, s \le S & \text{(Supply-monotone II)} \\
& a^s(v) \le a^{s+1}(v) & \forall v \in V, s < S & \text{(Supply-monotone III)} \\
& a^s(v), c^s(v) \ge 0 & \forall s \le S, v \in V &
\end{array}
$$

- Recall that we assume the budget constraint is tight by $OPTa^s(\cdot)$. If $v_{\max} \cdot a'(v_{\max}) - \sum_{v'=1}^{v_{\max}-1} a'(v') > 0$, then $v_{\max} \cdot (OPTa^s(v_{\max}) + \Delta \cdot a'(v_{\max})) - \sum_{v'=1}^{v_{\max}-1}(OPTa^s(v') + \Delta \cdot a'(v')) > B$ for any $\Delta > 0$, violating the budget constraint;

- If $n \cdot \sum_{v' \ge v} f(v') \cdot a(v') > (1 - (1 - \sum_{v' \ge v} f(v'))^n)$ for a v such that $n \cdot \sum_{v' \ge v} f(v') \cdot OPTa^s(v') = s \cdot (1 - (1 - \sum_{v' \ge v} f(v'))^n)$, then $n \cdot \sum_{v' \ge v} f(v') \cdot (OPTa^s(v') + \Delta \cdot a'(v')) > (s + \Delta) \cdot (1 - (1 - \sum_{v' \ge v} f(v'))^n)$ for any $\Delta > 0$, violating the Border's constraints.

Therefore, $g^s(\cdot) = \lim_{\Delta \to 0} \frac{OPTa^{s+\Delta}(\cdot) - OPTa^s(\cdot)}{\Delta}$ is a feasible solution to the LP (OPTGrad). Moreover, since $OPTa^{s+\Delta}(\cdot)$ optimizes the revenue, $g^s(\cdot)$ must optimize the revenue among the feasible solution of the LP (OPTGrad), which concludes the proof.

Intuitively, for supply s, a constraint appears in LP (OPTGrad) if and only if such a constraint is tight for $OPTa^s(\cdot)$ in LP (RevOpt). Therefore, if $a(\cdot)$ is not a feasible solution, then for any $\Delta > 0$, we have that $OPTa^s(\cdot) + \Delta \cdot a(\cdot)$ is infeasible for LP (RevOpt) on supply $s+\Delta$. On the other hand, it turns out that there exists $\overline{\Delta} > 0$ such that for all $0 < \Delta < \overline{\Delta}$, $OPTa^s(\cdot) + \Delta \cdot a(\cdot)$ is a feasible solution for LP (RevOpt) on supply $s + \Delta$. Thus, by definition, $g^s(\cdot)$ is a feasible solution for LP (OPTGrad). Moreover, since $OPTa^{s+\Delta}(\cdot)$ maximizes revenue, $g^s(\cdot)$ also does so among feasible solutions of LP (OPTGrad).

For convenience, we say $n \cdot \sum_{v' \ge v} f(v') \cdot a(v') \le 1 - (1 - \sum_{v' \ge v} f(v'))^n$ is *the Border's constraint corresponding to valuation* v. Given supply s, we say that *valuation* v *binds the Border's constraint* if $n \cdot \sum_{v' \ge v} f(v) \cdot OPTa^s(v) = s \cdot (1 - (1 - \sum_{v' \ge v} f(v))^n)$. Similarly, we say $g^s(\cdot)$ *binds a constraint* if $g^s(\cdot)$ makes it tight in LP (OPTGrad).

We define the set of critical supplies as the ones where the gradient changes.

Definition 4.2. (Critical Supplies) *Let C_s be the set of critical supplies such that*

$$C_s = \{s \ge s^B \mid g^s(\cdot) \ne \lim_{\Delta \to 0^+} g^{s-\Delta}(\cdot)\}$$

We are interested in the size of the set C_s and also in how we can compute this set efficiently. In particular, we characterize the gradient by an induction on C_s. Our induction is based on the following condition for supply $s \in C_s$,

Condition 4.1. *We say supply $s \in C_s$ satisfies Condition 4.1 if*

- *the budget constraint in LP (RevOpt) is tight by $OPTa^s(\cdot)$;*

- *for $v \in V$, either $OPTa^s(v) = OPTa^s(v-1)$ or v binds the Border's constraint;*

- *for $v \in V$ with $\nu(v) > 0$, we have $OPTa^s(v) > 0$.*

For the base case, we note that supply s^B satisfies Condition 4.1.

Lemma 4.2. *$s = s^B$ satisfies Condition 4.1.*

$$\text{(OPTGrad)}\quad \begin{array}{lll} \max & \sum_{v\in V} f(v)\nu(v)a(v) & \\ s.t. & a(v) \le a(v+1) & \forall v: OPTa^s(v) = OPTa^s(v+1) \\ & v_{\max}\cdot a(v_{\max}) - \sum_{v=1}^{v_{\max}-1} a(v) \le 0 & \\ & n\cdot \sum_{v'\ge v} f(v')\cdot a(v') \le 1-(1-\sum_{v'\ge v} f(v'))^n & \forall v\in T^s \\ & a(0)=0 & \end{array}$$

Proof. First, by the definition of s^B, we have the budget constraint in LP (RevOpt) is tight by $OPTa^s(\cdot)$. Recall that for $s = s^B$, the optimal auction is s^B times the Myerson's auction on $s = 1$. Note that in the Myerson's auction, given a valuation profile $\boldsymbol{v} = (v_1, \cdots, v_n)$, it throws away the item if $\max_i \nu(v_i) \le 0$. Therefore, under the monotone hazard rate condition, there exists a $\underline{v} \in V$ such that $v \le \underline{v}$ if and only if $\nu(v) < 0$, which implies $OPTa^{s^B}(v) = 0$ for all $v \le \underline{v}$.

Moreover, the auction uniformly allocates the item to the bidders in $\arg\max_i \nu(v_i)$. Since $\nu(v)$ is monotonically increasing, we have $\nu(v) > \nu(v-1)$ for all v and the auction allocates the item to valuations higher than $v-1$ once it exists. In other words, let $V' = \{v' \in V \mid v' \ge v\}$, the probability that a bidder with valuations in V' gets the item equals to the probability that at least one bidder is with valuations in V'. Thus, it binds the Border's constraint corresponding to v and for all $\nu(v) > 0$, $OPTa^s(v) > 0$.

4.1.1 Properties and Implications.

Before describing the induction, we first prove several properties and implications of Condition 4.1. For convenience, we introduce a *valuation hierarchy*:

DEFINITION 4.3. (VALUATION HIERARCHY) *For each $s \in C_s$, let R^s to be the set of* representative valuations *that have distinct values in $OPTa^s(\cdot)$. Formally, $R^s = \{0\} \cup \{v \in V \mid OPTa^s(v) > OPTa^s(v-1)\}$. For each $v \in R^s$, let $L^s(v) = \{v' \in V \mid OPTa^s(v) = OPTa^s(v')\}$ be the set of valuations that share the same value in $OPTa^s(\cdot)$ with v.*

By Condition 4.1, for $s \in C_s$, a valuation $v \in V \cap R^s$ if and only if v binds the Border's constraint. From Condition 4.1 and the construction of LP (OPTGrad), we have the following claim about the number of constraints in LP (OPTGrad) for $s \in C_s$:

CLAIM 4.1. *If $s \in C_s$ satisfies Condition 4.1, the LP (OPTGrad) corresponding to supply s contains exactly $|V| + 1$ constraints (other than $a(0) = 0$).*

Since LP (OPTGrad) contains $|V|$ variables and all constraints are linearly independent, if $s \in C_s$ satisfies Condition 4.1, then $g^s(\cdot)$ does not bind exactly one constraint in LP (OPTGrad) corresponding to supply s. We first show three types of constraints that $g^s(\cdot)$ must bind.

LEMMA 4.3. *If $s \in C_s$ satisfies Condition 4.1, then $g^s(\cdot)$ binds*

- *the budget constraint $v_{\max} \cdot a(v_{\max}) - \sum_{v=1}^{v_{\max}-1} a(v) \le 0$;*
- *the Border's constraint corresponding to $R^s_{min} = \min(R^s \setminus \{0\})$;*
- *$a(v) \le a(v+1)$ for $v + 1 < \max(L^s(0))$.*

Proof. For the sake of contradiction, assume $g^s(\cdot)$ does not bind the budget constraint. Then, $g^s(\cdot)$ must bind all the other constraints. However, note that LP (OPTGrad) contains a constraint $a(v) \le a(v+1)$ if and only if $OPTa^s(v) = OPTa^s(v+1)$ and LP (OPTGrad) contains a Border's constraint corresponding to valuation v if and only if v binds the Border's constraint for supply s. Therefore, $g^s(\cdot) = OPTa^s(\cdot)/s$, which must violate the budget constraint.

In the previous argument, if consider $a'(\cdot)$ that binds all constraints except the budget constraint, such $a'(\cdot)$ violates the budget constraint. For the sake of contradiction, assume $g^s(\cdot)$ does not bind Border's constraint corresponding to $v = R^s_{\min}$, but binds all other constraints. Note that, for such $g^s(\cdot)$, we have for all $v' < v$, $g^s(v') = 0 = a'(v')$ and for all $v' > \max L^s(R^s_{\min})$, $g^s(v') = a'(v')$. To make sure that $g^s(\cdot)$ binds the budget constraint, we must have $g^s(R^s_{\min}) > a'(R^s_{\min})$. However, since $a'(\cdot)$ binds the Border's constraint corresponding to $R^s_{\min}$, such $g^s(\cdot)$ must violate the Border's constraint corresponding to $R^s_{\min}$.

Finally, for the sake of contradiction, assume $g^s(\cdot)$ does not bind $a(v) \le a(v+1)$ for $v + 1 < \max(L^s(0))$. Let $\overline{v} = \max(L^s(0))$. Note that for all $v + 1 \le v' < \overline{v}$, $g^s(v') = g^s(v+1)$ and for all $v' < v+1$, $g^s(v') = 0$ since at most one constraint does not bind. Consider $g'(\cdot) = g^s(\cdot)$ except $g'(v+1) = g'(v+1) - \Delta$ and $g'(\overline{v}) = g'(\overline{v}) + \Delta$ with $\Delta < g'(v+1)$. It can be verified that $g'(\cdot)$ is a feasible solution to the LP (OPTGrad) corresponding to supply s. We show that $g'(\cdot)$ has higher revenue than $g^s(\cdot)$. Note that the difference of the revenue between $g'(\cdot)$ and $g^s(\cdot)$ is

$$\begin{aligned}&-f(v+1)\nu(v+1)\Delta+f(\overline{v})\nu(\overline{v})\Delta\\=&\ \Delta\left(-f(v+1)\nu(v+1)+f(\overline{v})\nu(\overline{v})\right)\end{aligned}$$

Thus, it suffices to show $f(\overline{v})\nu(\overline{v})-f(v+1)\nu(v+1)>0$. If $f(v+1)\geq f(\overline{v})$, since $OPTa^s(v+1)=OPT(\overline{v})=0$, by Condition 4.1, $\nu(v+1)\leq\nu(\overline{v})\leq 0$. Therefore, we have $f(\overline{v})\nu(\overline{v})-f(v+1)\nu(v+1)>0$. If $f(v+1)<f(\overline{v})$, we have

$$\begin{aligned}&f(\overline{v})\nu(\overline{v})-f(v+1)\nu(v+1)\\=&\ f(\overline{v})\cdot\overline{v}+F(\overline{v})-f(v+1)\cdot(v+1)-F(v+1)>0.\end{aligned}$$

By Lemma 4.3, the remaining candidate constraints that $g^s(\cdot)$ does not bind are:

1. the Border's constraint corresponding to $v\in R^s\setminus\{0,R^s_{\min}\}$.
2. $a(v)\leq a(v+1)$ for $v+1=\max(L^s(0))$.
3. $a(v)\leq a(v+1)$ for $OPTa^s(v)=OPTa^s(v+1)>0$.

We rule out the third possibility using the induction.

CONDITION 4.2. *We say supply $s\in C_s$ satisfies Condition 4.2 if $g^s(\cdot)$ binds $a(v)\leq a(v+1)$ for all $OPTa^s(v)=OPTa^s(v+1)>0$.*

Since the virtual valuation is monotonically increasing, $OPTa^{s^B}(v)<OPTa^{s^B}(v+1)$ if $\nu(v)>0$. Therefore, there exists no constraint $a(v)\leq a(v+1)$ for $OPTa^s(v)=OPTa^s(v+1)>0$, and thus, s^B satisfies Condition 4.2.

LEMMA 4.4. *$s=s^B$ satisfies Condition 4.2.*

4.1.2 Structure of the Gradients.

Combining Lemma 4.3 and Condition 4.2, we have the following characterization of $g^s(\cdot)$ for $s\in C_s$:

CLAIM 4.2. *For $s\in C_s$ satisfying Condition 4.1 and 4.2, the constraint that $g^s(\cdot)$ does not bind is:*

1. *the Border's constraint corresponding to some $v^*\in R^s\setminus\{0,R^s_{min}\}$, or*
2. *$a(v^*)\leq a(v^*+1)$ for $v^*+1=\max(L^s(0))$.*

This leads to two types of gradients.

LEMMA 4.5. (TYPE I GRADIENT) *For $s\in C_s$ satisfying Condition 4.1 and 4.2, if $g^s(\cdot)$ does not bind the Border's constraint corresponding to $v^*\in R^s\setminus\{0,R^s_{min}\}$, then $g^s(\cdot)=OPTa^s(\cdot)/s$ except for $v'\in L^s(v^*)\cup L^s(v^p)$. Here, $v^p=\sup\{v'\in R^s\mid v'<v^*\}$ is the maximum representative valuation smaller than v^*. We have $g^s(v^p)>g^s(v^*)$ and*

- *for all $v'\in L^s(v^*)$, we have $g^s(v')=g^s(v^*)$ and for all $v'\in L^s(v^p)$, we have $g^s(v')=g^s(v^p)$;*
- *$g^s(\cdot)$ binds the Border's constraint corresponding to v^p and the budget constraint.*

LEMMA 4.6. (TYPE II GRADIENT) *For $s\in C_s$ satisfying Condition 4.1 and 4.2, if $g^s(\cdot)$ does not bind $a(v^*)\leq a(v^*+1)$ for $v^*+1=\max(L^s(0))$, then $g^s(\cdot)=OPTa^s(\cdot)/s$ except for $g^s(v^*+1)$, which chooses a value to ensure that $g^s(\cdot)$ binds the budget constraint.*

To prove these two lemmas, note that in both cases, there are many Border's constraints and monotonicity constraints ($a(v)\leq a(v+1)$) that are tight by both $g^s(\cdot)$ and $OPTa^s(\cdot)$. It turns out that we can show that $g^s(\cdot)$ and $OPTa^s(\cdot)/s$ must share the same value for most valuations. After settling down most values in $g^s(\cdot)$, we can derive the remaining values from the rest of the constraints that $g^s(\cdot)$ binds.

Proof. [Proof of Lemmas 4.5 and 4.6.] First, note that if $a(\cdot)$ binds Border's constraint corresponding to $v=\max(R^s)$ and for all $v'\in L^s(v)$, $a(v')=a(v)$, then for all $v'\in L^s(v)$, we have

$$\begin{aligned}&n\cdot\sum_{v'\geq v}f(v')\cdot a(v')=1-(1-\sum_{v'\geq v}f(v'))^n\\\Rightarrow&\ \forall v'\geq v, a(v')=\frac{1-(1-\sum_{v'\geq v}f(v'))^n}{n\cdot\sum_{v'\geq v}f(v')}\end{aligned}$$

Moreover, if $a(\cdot)$ binds Border's constraint corresponding to $v\in R^s$ but $v<\max(R^s)$ and $\hat{v}=\inf\{v'\in R^s\mid v'>v\}$, we have

$$n\cdot\sum_{v'\geq v}f(v')\cdot a(v')=1-(1-\sum_{v'\geq v}f(v'))^n$$

and

$$n\cdot\sum_{v'\geq\hat{v}}f(v')\cdot a(v')=1-(1-\sum_{v'\geq\hat{v}}f(v'))^n$$

Subtracting the above two equations, we have

$$\begin{aligned}&n\cdot\sum_{v\leq v'<\hat{v}}f(v')\cdot a(v')\\=&\left(1-(1-\sum_{v'\geq v}f(v'))^n\right)-\left(1-(1-\sum_{v'\geq\hat{v}}f(v'))^n\right)\end{aligned}$$

Moreover, since for all $v'\in L^s(v)$, $a(v')=a(v)$, then for all $v\leq v'<\hat{v}$, we have

$$a(v')=\frac{(1-\sum_{v'\geq\hat{v}}f(v'))^n-(1-\sum_{v'\geq v}f(v'))^n}{n\cdot\sum_{v\leq v'<\hat{v}}f(v')}.$$

Finally, if $a(\cdot)$ satisfies that for all $v' \in L^s(0)$, $g^s(v') = g^s(0)$, then $g^s(v') = 0$ for all $v' \in L^s(0)$.

By Condition 4.1 and LP (OPTGrad), we can conclude that if $g^s(\cdot)$ does not bind Border's constraint corresponding to v^*, then $g^s(\cdot) = OPTa^s(\cdot)/s$ except for $v' \in L^s(v^*) \cup L^s(v^p)$. As for $v' \in L^s(v^*) \cup L^s(v^p)$, by Claim 4.2, $g^s(\cdot)$ binds $a(v') \le a(v'+1)$ for $OPTa^s(v') = OPTa^s(v'+1)$. Therefore, we have for all $v' \in L^s(v^*)$, $g^s(v') = g^s(v^*)$ and for all $v' \in L^s(v^p)$, $g^s(v') = g^s(v^p)$.

To bind the budget constraint, for the sake of contradiction, assume $g^s(v^p) \le g^s(v^*)$. First note that $g^s(v^*) \le OPTa^s(v^*)$ since $g^s(\cdot)$ does not bind Border's constraint corresponding to v^*. Moreover, by the fact that $OPTa^s(\cdot)$ is non-decreasing and $g^s(\cdot) = OPTa^s(\cdot)$ except for $v' \in L^s(v) \cup L^s(v^p)$, we have for all $v' \in V$, $g^s(v') \le g^s(v_{\max})$, which implies that $g^s(\cdot)$ violates the budget constraint.

The proof for the Type II gradient follows along almost the same lines, and is omitted for brevity.

4.1.3 Induction.

We finish the characterization of the gradients by showing that for all $s \in C_s$, s satisfies both Conditions 4.1 and 4.2 by an induction. Assume for all $s \in \{s' \le \overline{s} \mid s' \in C_s\}$, supply s satisfies Condition 4.1 and 4.2. The base case is true for $\overline{s} = s^B$ by Lemmas 4.2 and 4.4.

Recall that LP (OPTGrad) contains exactly $|V|+1$ constraints for $s \in C_s$. For Type I gradients, note that the gradient binds all other constraints except the Border's constraint corresponding to v^*. Therefore, as supply increases by Δ, LP (OPTGrad) corresponding to $s+\Delta$ contains exactly $|V|$ constraints while the gradient binds exactly these $|V|$ constraints. Therefore, the gradient does not change until an additional constraint is tight. Let the next critical supply be $\hat{s} = \inf\{s' \in C_s \mid s' > \overline{s}\}$. The next lemma demonstrates that the next tight constraint is $OPTa^{\hat{s}}(v^*-1) = OPTa^{\hat{s}}(v^*)$ for Type I gradients. By a similar argument, for Type II gradients, the next tight constraint is the Border's constraint corresponding to the valuation v^*. It immediately follows that $\hat{s}$ satisfies Condition 4.1.

LEMMA 4.7. *If $g^{\overline{s}}(\cdot)$ is a Type I gradient, $OPTa^{\hat{s}}(v^*-1) = OPTa^{\hat{s}}(v^*)$; If $g^{\overline{s}}(\cdot)$ is a Type II gradient, then v^*+1 binds the Border's constraint for supply $\hat{s}$. Moreover, $\hat{s}$ satisfies Condition 4.1.*

Proof. Note that for $0 < \Delta < \hat{s} - s$, by induction hypothesis and Claim 4.2, $OPTa^{s+\Delta}(\cdot)/(s+\Delta) = (OPTa^s(\cdot) + \Delta \cdot g^s(\cdot))/(s+\Delta)$ binds all the constraints in LP (OPTGrad) corresponding to supply s except the constraint that $g^s(\cdot)$ does not bind. Therefore, in the LP (OPTGrad) corresponding to supply $s+\Delta$, it contains exactly $|V|$ constraints. By Lemma 4.5 and Lemma 4.6, $g^s(\cdot)$ binds exactly these $|V|$ constraints. Henceforth, $g^s(\cdot)$ is the optimal solution of the LP (OPTGrad) corresponding to supply $s+\Delta$. Thus, $g^s(\cdot)$ remains to be the gradient until one more constraint is added to LP (OPTGrad).

By Lemma 4.5, for a Type I gradient, no additional Border's constraint would be tight. Because a tight constraint must come from $R^s \setminus \{0\}$. However, all of them are already tight except v^*, while the gradient does not violate the Border's constraint corresponding to v^*. Therefore, the additional constraint must be $a(v^*-1) = a(v^*)$ when $OPTa^{\hat{s}}(v^*-1) = OPTa^{\hat{s}}(v^*)$. Similarly, by Lemma 4.6, for Type II gradients, the additional constraint would be either the Border's constraint corresponding to v^*+1 or $a(v^*+1) = a(v^*+2)$. However, notice that v^*+2 always binds the Border's constraints for $s+\Delta$. If $OPTa^{\hat{s}}(v^*+1) = OPTa^{\hat{s}}(v^*+2)$, the Border's constraints corresponding to v^*+1 must already be violated. Therefore, for Type II gradient, the additional constraint is the Border's constraint corresponding to v^*+1. Finally, it immediately follows that $\hat{s}$ satisfies Condition 4.1.

To finish the induction, we show that $\hat{s}$ satisfies Condition 4.2: the gradient $g^{\hat{s}}(\cdot)$ must bind the constraint $a(v) < a(v+1)$ for $OPTa^{\hat{s}}(v) = OPTa^{\hat{s}}(v+1) > 0$.

LEMMA 4.8. *$\hat{s}$ satisfies Condition 4.2.*

The proof is based on a contradiction argument. Assume $g^{\hat{s}}$ does not bind $a(v) < a(v+1)$ for $OPTa^{\hat{s}}(v) = OPTa^{\hat{s}}(v+1) > 0$. Note that since $OPTa^{\hat{s}}(v) = OPTa^{\hat{s}}(v+1) > 0$, it implies that there exists a supply $s' < \hat{s}$ such that its gradient $g^{s'}(\cdot)$ satisfies $g^{s'}(v) > g^{s'}(v+1)$. We argue that $g^{s'}(\cdot)$ generates more revenue than $g^{\hat{s}}(\cdot)$, and moreover, for sufficiently small $\varepsilon > 0$, $(1-\varepsilon)g^{s'}(\cdot) + \varepsilon \cdot g^{\hat{s}}(\cdot)$ is a feasible solution to LP (OPTGrad). This contradicts the fact that $g^{\hat{s}}(\cdot)$ is the optimal solution of LP (OPTGrad).

To formally prove that the next critical supply $\hat{s}$ satisfies Condition 4.2, we require the following lemma, which demonstrates that the Type I gradient $g^s(\cdot)$ not only satisfies Border's constraints appearing in the LP (OPTGrad) corresponding to s but also all other Border's constraints.

LEMMA 4.9. *For a gradient $g(\cdot)$, if for every v, either $g(\cdot)$ binds the Border's constraint corresponding to v or $g(v) = g(v-1)$, then $g(\cdot)$ satisfies Border's constraints for $V' = \{v \in V \mid v' \ge v\}$ for all $v \in V$.*

Proof. Similar to Definition 4.3, define R as the set of representative valuations of $g(\cdot)$ and For each $v \in R$, let $L(v)$ be the set of valuations that share the same value in $g(\cdot)$ with v.

For $v \in R \setminus \{0\}$, define $F_L = \sum_{v' \in L(v)} f(v')$, $F_R = \sum_{v' > \max(L(v))} f(v')$, and $C = \sum_{v' > \max(L(v))} f(v')g(v')$. Then, by the fact that $g(\cdot)$ satisfies Border's constraint for v, we have

$$F_L \cdot g(v) \leq \frac{1-(1-F_L-F_R)^n}{n} - C$$

and the fact that $g(\cdot)$ satisfies Border's constraint for $\hat{v} = \inf\{v' \in R \mid v' > v\}$, we have

$$0 \cdot g(v) \leq \frac{1-(1-0-F_R)^n}{n} - C$$

(if $\hat{v}$ does not exists, the above inequality still holds since $F_R = 0$ and $C = 0$).

Since $1-(1-x-F_R)^n$ is a concave function in terms of x, we have for any $0 \leq x \leq F_L$,

$$x \cdot g(v) \leq \frac{1-(1-x-F_R)^n}{n} - C.$$

Notice that for all $v < v' < \hat{v}$, we have $0 \leq \sum_{v' \leq v'' < \hat{v}} f(v'') \leq F_L$. Therefore, $g(\cdot)$ satisfies Border's constraints for $V' = \{v \in V \mid v' \geq v\}$ for all $v \in V$.

For a Type I gradient $g^s(\cdot)$, we say $g^s(\cdot)$ is identified by v^* if $g^s(\cdot)$ does not bind Border's constraint corresponding to v^*. Now, we are ready to show that $\hat{s}$ satisfies Condition 4.2.

Proof. [Proof of Lemma 4.8.] For the sake of contradiction, assume $g^{\hat{s}}(v) < g^{\hat{s}}(v+1)$ for $OPTa^{\hat{s}}(v) > 0$. Assume $v \in L^{\hat{s}}(v_R)$, i.e., the representative valuation that shares the same value as v in $OPTa^{\hat{s}}(\cdot)$ is v_R. Since $g^{\hat{s}}(\cdot)$ binds all other constraints, we have for $v' \in L^{\hat{s}}(v_R)$, $g^{\hat{s}}(v') = g^{\hat{s}}(v)$ if $v' \leq v$ and $g^{\hat{s}}(v') = g^{\hat{s}}(v+1)$ if $v' > v$.

By the fact that Lemma 4.5 and Lemma 4.6 hold for $\{s' \in C_s \mid s' < \hat{s}\}$, there exists supply s^* such that $g^{s^*}(\cdot)$ is identified by $v+1$. First, $g^{s^*}(\cdot)$ satisfies the budget constraint. By Lemma 4.9, $g^{s^*}(\cdot)$ satisfies all Border's constraints that appears in LP (OPTGrad) corresponding to supply s. Moreover, by Lemma 4.5, $g^{s^*}(\cdot)$ satisfies $a(v') \leq a(v'+1)$ for all $v' \neq v$. Thus, with sufficiently small $\varepsilon > 0$, $(1-\varepsilon) \cdot g^{\hat{s}}(\cdot) + \varepsilon \cdot g^{s^*}$ is still a feasible solution of LP (OPTGrad) corresponding to supply s.

We argue that $g^{\hat{s}}(\cdot)$ is a feasible solution of LP (OPTGrad) corresponding to supply s^*. First $g^{\hat{s}}(\cdot)$ satisfies the budget constraint. Moreover, by Lemma 4.7, LP (OPTGrad) corresponding to supply s^* contains less monotonicity constraints $a(v) \leq a(v+1)$, and by Lemma 4.9, $g^{\hat{s}}(\cdot)$ satisfies all Border's constraints in LP (OPTGrad) corresponding to supply s^*. Therefore, since $g^{s^*}(\cdot)$ is the optimal solution of LP (OPTGrad) corresponding to supply s^*, $g^{s^*}(\cdot)$ has higher revenue than $g^{\hat{s}}(\cdot)$.

Therefore, $(1-\varepsilon) \cdot g^{\hat{s}}(\cdot) + \varepsilon \cdot g^{s^*}$ has higher revenue than $g^{\hat{s}}(\cdot)$, contradicting the fact that $g^{\hat{s}}(\cdot)$ is the optimal solution of LP (OPTGrad) corresponding to supply $\hat{s}$.

Now, we are ready to bound the number of critical supplies. Note that a Type I gradient $g^s(\cdot)$ is a gradient that does not bind the Border's constraint corresponding to some valuation v^*. By Lemma 4.7, the gradient changes only when v^* shares the same value as v^*-1 in $OPTa^{\hat{s}}(\cdot)$. Moreover, by Lemma 4.8, v^* and v^*-1 will share the value for all $s' > \hat{s}$. Furthermore, only a Type II gradient can introduce a new Border's constraint to LP (OPTGrad) by Lemma 4.7. In summary, each valuation can be a new valuation to bind the Border's constraint at most once. Once it pools with another valuation and shares the same value in the allocation rule, these valuations remain the same for all future supplies. Therefore, there are at most $2|V|$ different gradients.

COROLLARY 4.1. $|C_s| \leq 2|V|$.

To efficiently obtain C_s, we can compute the gradient $g^s(\cdot)$ first by LP (OPTGrad) and then compute the next time when the gradient changes by Lemma 4.7.

4.2 Optimal Competitive Auction. Since there are at most $2|V|$ different gradients, there are at most $2|V|$ *critical* supplies that we need to check in the linear program. More precisely, we can restrict our attention to the auction in which the gradients are the same for the supplies between two critical supplies. Let $A = \{s \in C_s \mid s \leq S\} \cup \{S\}$ and let $s_p = \sup\{s' < s \mid s' \in A \cup \{0\}\}$ be the critical supply right before s. Denote their difference $d(s) = s - s_p$. Then, we can compute an interim supply-monotone auction with a tight competitive ratio using LP (onlineOPT).

The correctness of the algorithm and the time complexity are summarized in the following theorem.

THEOREM 4.1. *Given f, B, n and the maximum supply S, LP (onlineOPT) computes a supply-monotone auction with optimal competitive ratio in time $O(\text{poly}(|V|))$. Such an auction can be represented by a convex combinations of at most $O(|V|^3)$ supply-monotone hierarchical mechanisms.*

Proof. For the sake of contradiction, assume the LP (onlineOPT) corresponding to set $A' \supset A$ results in a solution $\hat{a}^s(\cdot)$, $\hat{c}^s(\cdot)$ for $s \in A'$ with better competitive ratio τ'. Let $a^s(\cdot) = \hat{a}^s(\cdot)$ for all $s \in A$. For convenience, define $s'_p = \sup\{s'' < s' \mid s'' \in A' \cup \{0\}\}$ for $s' \in A'$. For $s \in A$, let

$$\hat{c}^s(\cdot) = \frac{1}{s - s_p} \sum_{s_p < s' \leq s, s' \in A'} (s' - s'_p)\hat{c}^s(\cdot).$$

(onlineOPT)

$$
\begin{array}{llll}
\max & \tau & & \\
s.t. & \sum_{v\in V} f(v)\nu(v)a^s(v) \geq \tau \cdot \sum_{v\in V} f(v)\nu(v)OPTa^s(v) & \forall s\in A & \text{(Competitive Ratio)} \\
 & a^s(v) \leq a^s(v+1) & \forall s\in A, v\in V & \text{(Incentive Compatible)} \\
 & v_{\max}\cdot a^s(v_{\max}) - \sum_{v=1}^{v_{\max}-1} a^s(v) \leq B & \forall s\in A & \text{(Budget)} \\
 & a^s(v) \leq \sum_{k\in A, k\leq s} d(k)\cdot c^k(v) & \forall s\in A, v\in V & \text{(Supply-monotone I)} \\
 & n\cdot \sum_{v\in V'} f(v)c^s(v) \leq 1-(1-\sum_{v\in V'} f(v))^n & \forall V'\subseteq V, s\in A & \text{(Supply-monotone II)} \\
 & a^{s_p}(v) \leq a^s(v) & \forall v\in V, s\in A & \text{(Supply-monotone III)} \\
 & a^s(v), c^s(v) \geq 0 & \forall s\in A, v\in V &
\end{array}
$$

It can be verified that $a^s(\cdot)$ and $c^s(\cdot)$ forms a feasible solution for the LP (onlineOPT) corresponding to A (ignoring the constraints for the competitive ratio). Finally, we claim that such a solution has competitive ratio at least τ' since

$$
\begin{aligned}
\tau' &= \min_{s\in A'} \frac{\sum_{v\in V} f(v)\nu(v)a^s(v)}{\sum_{v\in V} f(v)\nu(v)OPTa^s(v)} \\
&\leq \min_{s\in A} \frac{\sum_{v\in V} f(v)\nu(v)a^s(v)}{\sum_{v\in V} f(v)\nu(v)OPTa^s(v)}
\end{aligned}
$$

Finally, since there are at most $2|V|$ critical supplies, by Theorem 3.3, the auction can be represented by a convex combinations of at most $4|V|^3+1$ supply-monotone hierarchical mechanisms.

Remark: The algorithm presented in this section is for continuous supply. For discrete supply, our algorithm works by setting $A = \{s' \mid \exists s \in C_s, s' = \lceil s \rceil \text{ or } s' = \lfloor s \rfloor\} \cup \{S\}$.

4.3 Asymptotic Value of the Competitive Ratio.

A natural question is to estimate the cost of making the auction supply-monotone, i.e., bound the revenue loss due to supply monotonicity. Denote $OPTa_n^s(\cdot)$ be the optimal offline auction when there are n bidders and $g_n^s(\cdot)$ be its gradient. Note that a necessary condition for $OPTa_n^s(\cdot)$ to satisfy supply-monotonicity is that $g_n^s(v) \geq 0$ for all s and v. The next theorem demonstrates that if $g_n^s(v) \geq 0$ for all s and v as n goes to ∞, then $OPTa_n^s(\cdot)$ satisfies supply monotonicity asymptotically as n goes to ∞. Therefore, the competitive ratio goes to 1 as n goes to ∞.

THEOREM 4.2. *If there exists n_0' such that for all $n > n_0'$, $g_n^s(v) \geq 0$ for all s and v, then for any $\varepsilon > 0$, there exists an n_0 such that for all $n > n_0$, $a_n^s(\cdot) = (1-\varepsilon)\cdot OPTa_n^s(\cdot)$ is supply-monotone.*

Proof. If $g_n^s(\cdot)$ is a Type II gradient, which does not bind the constraint $a(v^*) \leq a(v^*+1)$ for $v^* = \max(L^s(0)) - 1$. By Lemma 4.6, for all $v' > v^*+1$, $g_n^s(v') = OPTa_n^s(v')/s$. Therefore, $g_n^s(\cdot)$ satisfies Border's constraints for $v' > v^*+1$. Note that for all $v' \leq v^*+1$, $OPTa_n^s(v') = 0$. Therefore, given a valuation profile $\boldsymbol{v} = (v_1, \cdots, v_n)$, all first s units are unallocated if $v_i \leq v^*+1$ for all i. Moreover, the gradient changes when v^*+1 first binds the Border's constraints while v^*+2 also binds the Border's constraint. Therefore, the probability that bidders with valuation v^*+1 receive the units is exactly the probability that the maximum valuation of a valuation profile is v^*+1, and these units are unallocated. Therefore, we can fulfill the interim allocation for valuation v^*+1 by allocating the unallocated units. More precisely, we will allocate the units to bidders with valuation v^*+1 if the valuation profile satisfies $\max v_i = v^*+1$,

If $g_n^s(\cdot)$ is a Type I gradient, identified by v^*, let $v^p = \sup\{v' \in R^s \mid v' < v^*\}$. Let $F_L = \sum_{v'\in L^s(v^p)} f(v')$, $F_n = \sum_{v'\in L^s(v^*)} f(v')$, and $F_R = \sum_{v' > \max(L^s(v^*))} f(v')$. Note that by setting $a^s(\cdot) = (1-\varepsilon)OPTa_n^s(\cdot)$, with Δ increment of supply, the difference between the $a^{s+\Delta}(\cdot) - a^s(\cdot)$ is $(1-\varepsilon)\Delta \cdot g_n^s(\cdot)$. Our plan is to construct an allocation rule that satisfies Border's theorem (Theorem 2.1) and is at least $(1-\varepsilon)g_n^s(\cdot)$ for all valuation $v \in V$.

Let $a'(\cdot)$ be the allocation rule that given a valuation profile, uniformly allocate item to valuations in $L^s(v^p)$ if exists; otherwise, throw the item away. Therefore, for $v' \in L^s(v^p)$,

$$
a'(v') = \frac{1-(1-F_L)^n}{n\cdot F_L};
$$

otherwise, $a'(v') = 0$. Moreover, let $b'(\cdot)$ be the allocation rule that is the same as $OPTa_n^s(\cdot)/s$ except for the valuations in $L^s(v_p) \cup L^s(v^*)$: $b'(\cdot)$ prioritizes the valuations in $L^s(v_p)$. More precisely, for $v' \in L^s(v_p)$, by Lemma 4.5 and Border's theorem (Theorem 2.1),

$$
b'(v') = \frac{(1-F_R)^n - (1-F_L-F_R)^n}{n\cdot F_L},
$$

while for $v' \in L^s(v^*)$,

$$b'(v') = \frac{(1-F_R-F_L)^n - (1-F_L-F_n-F_R)^n}{n \cdot F_n}$$

Note that if for $v' \in L^s(v^*), b'(v') \le g_n^s(v')$, we have for $v' \in L^s(v_p)$, $b'(v') \ge g_n^s(v')$. Then, we have $g_n^s(\cdot)$ satisfies Border's theorem. For now on, we assume $v' \in L^s(v^*), b'(v') \ge g_n^s(v')$.

We show that for any $\varepsilon > 0$, there exists an n_0 such that for all $n > n_0$

$$\varepsilon \cdot a'(\cdot) + (1-\varepsilon) \cdot b'(\cdot) \ge (1-\varepsilon) g_n^s(\cdot)$$

By Lemma 4.5, we have

$$\forall v' \notin L^s(v_p) \cup L^s(v^*), b'(v') = OPTa_n^s(v')/s = g_n^s(v')$$

As $v' \in L^s(v_p)$, we have $\varepsilon \cdot a'(v') + (1-\varepsilon) \cdot b'(v') = \varepsilon \frac{1-(1-F_L)^n}{n \cdot F_L} + (1-\varepsilon) \frac{(1-F_R)^n - (1-F_L-F_R)^n}{n \cdot F_L}$. Therefore,

$$\lim_{n\to\infty} \varepsilon \cdot a'(v') + (1-\varepsilon) \cdot b'(v') = \lim_{n\to\infty} \frac{\varepsilon + (1-\varepsilon) \cdot (1-F_R)^n}{n \cdot F_L}$$

Since $g_n^s(\cdot) \ge 0$ for all $v \in V$, the extreme case is that for all $v' \in L^s(v^*)$, $g_n^s(v') = 0$. By Border's theorem (Theorem 2.1), we have

$$g_n^s(v') \le \frac{(1-F_R)^n - (1-F_L-F_n-F_R)^n}{n \cdot F_L}$$

Therefore,

$$\lim_{n\to\infty} (1-\varepsilon) \cdot g_n^s(v') \le \lim_{n\to\infty} (1-\varepsilon) \cdot \frac{(1-F_R)^n}{n \cdot F_L}$$

Thus, for any combination of F_L, F_n, F_R, there exists an $n_0(F_L, F_n, F_R)$ such that for all $n > n_0(F_L, F_n, F_R)$,

$$\varepsilon \cdot a'(\cdot) + (1-\varepsilon) \cdot b'(\cdot) \ge (1-\varepsilon) \cdot g_n^s(\cdot)$$

Since there are only finitely many combinations of F_L, F_n, F_R, if let $n_0^* = \max_{F_L, F_n, F_R} n_0(F_L, F_n, F_R)$, then for all $n > n_0^*$, we have

$$\forall s, \varepsilon \cdot a'(\cdot) + (1-\varepsilon) \cdot b'(\cdot)/s \ge (1-\varepsilon) g_n^s(\cdot).$$

5 Conclusion

In this paper, we provided a succinct and efficiently implementable characterization of supply-monotonicity and designed an efficient algorithm to compute the auction for symmetric bidders in Bayesian settings. We also provide evidence that the revenue loss caused by supply-monotonicity is relatively small. Experimental evidence (see Appendix B) suggests that even for modest number of bidders, the competitive ratio is quite high (larger than 0.95 for all our experiments). A natural direction for future work is to show that for any number of bidders, the revenue loss caused by supply monotonicity is small. Another interesting direction is to extend our results to asymmetric bidders and/or private budgets. Our succinct characterizations extend to asymmetric bidders, but the revenue-optimizing auction uses the structure of the offline optimal auction for symmetric bidders. Private budgets are more challenging, in that they create an additional layer of possible misreporting for strategic gains, and as such, appear to be beyond current techniques in this line of research.

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Appendix

A Supply-monotone Auctions with identical (but not necessarily independent) buyers

A.1 Necessary and Sufficient Condition. If bidder i's valuation is independently drawn from distribution f_i over V, i.e., bidders are independent but not necessarily identical, we have the following characterization of feasibility.

THEOREM A.1. *For an independent set of bidders, a monotonically non-decreasing interim allocation rule $a_i^s(\cdot)$ for $1 \le s \le S$ and $1 \le i \le n$ is supply-monotone if and only if for all $V_i^S \subseteq V_i^{S-1} \subseteq \cdots \subseteq V_i^1 \subseteq V$,*

$$\sum_{s=1}^{S} \sum_{i} \sum_{v_i \in V_i^s} f_i(v_i)(a_i^s - a_i^{s-1})(v_i) \le \sum_{s=1}^{S} (1 - \prod_{i} (1 - \sum_{v_i \in V_i^s} f_i(v_i)))$$

The proof of this theorem is identical to the main proof of Theorem 3.1 by replacing all the i.i.d. distribution $f(v_i)$ by $f_i(v_i)$. Hence, we skip this proof for brevity.

A.2 Efficient Verification. To obtain an efficient verification, following a similar argument in Section 3.2, we have:

THEOREM A.2. *For an independently distributed set of bidders and a monotonically non-decreasing interim allocation rule $a_i^s(\cdot)$ for $s = 1, \cdots, S$, it is supply-monotone if and only if there exists $c_i^s : V \to \mathbb{R}$ for all i and $s = 1, \cdots, S$ that is feasible for LP (Supply-monotone-non-iid).*

(Supply-monotone-non-iid)

max	0		
s.t.	$a_i^s(v_i) \le \sum_{k=1}^{s} c_i^k(v_i)$	$\forall v_i \in V, s \le S$	(Supply Monotone I)
	$\sum_i \sum_{v_i \in V_i'} f_i(v_i) c_i^s(v_i) \le 1 - \prod_i (1 - \sum_{v_i \in V_i'} f_i(v_i))$	$\forall V_i' \subseteq V, s \le S$	(Supply Monotone II)
	$c_i^s(v_i) \ge 0$	$\forall i, s \le S$	

Although there are exponentially many constraints, Cai et al. [9] extends Theorem 2.2 to independent but not necessary identical cases by introducing *virtual reduced form* $\hat{a}_i(v_i) = a_i(v_i) \cdot \sum_{v_i': a_i(v_i') \le a_i(v_i)} f_i(v_i')$.

THEOREM A.3. (CAI ET AL. [9]) *Let $E_i^\beta = \{v_i \in V \mid \hat{a}_i(v_i) \ge \beta\}$ for $\beta \in [0,1]$. Then, $a_i(\cdot)$ is feasible if and only if for all β,*

$$\sum_i \sum_{v_i \in E_i^\beta} f_i(v_i) a_i(v_i) \le 1 - \prod_i (1 - \sum_{v_i \in E_i^\beta} f_i(v_i)).$$

Note that given an allocation rule $a_i(\cdot)$, there are at most $n \cdot |V|$ different combinations of E_i^β. Therefore, we can check the feasibility in time $O(\text{poly}(n, |V|))$. As a result, we can construct an efficient separation oracle to identify a violating constraint for LP (Supply-monotone-non-iid). Thus, we can determine whether an interim allocation rule is supply-monotone in $O(\text{poly}(n, |V|, S))$ time.

A.3 Efficient Implementation. For non-identical buyers, Cai et al. [9] characterize that an interim allocation rule $a_i(\cdot)$ can be represented by a convex combination of $n \cdot |V| + 1$ hierarchical mechanisms.

DEFINITION A.1. *A hierarchical mechanism consists of a function $H : V \times [n] \to \{0, 1, \cdots, |V|\}$. On bid vector $(v_1, \cdots, v_n)$, if $H(v_i, i) = 0$ for all i, the mechanism throws the unit away; otherwise, the unit is awarded uniformly at random to a bidder in* $\arg\max_i H(v_i, i)$.

Similar to the i.i.d. setting, we generalize their definition to *supply-monotone hierarchical mechanisms.*

DEFINITION A.2. *A supply-monotone hierarchical mechanism consists of a set of functions $H^s : V \times [n] \to \{0, 1, \cdots, |V|\}$ and $G^s : V \times [n] \to \{1, \cdots, S\}$ for the s-th unit. On bid vector $(v_1, \cdots, v_n)$, for the s-th unit, if $H^s(v_i, i) = 0$ for all i, the mechanism throws the unit away; otherwise, the unit is divided uniformly at random to bidders in* $\arg\max_i H^s(v_i, i)$ *and allocated to bidder i when the $G^s(v_i, i)$-th unit arrives.*

A supply-monotone hierarchical mechanism is a hierarchical mechanism that has different timings to allocate the units for different valuations. In the supply monotone hierarchical mechanism, when the s-th unit appears, the mechanism computes the fraction of the unit to be assigned to each bidder according to the function H^s. However, the actual allocation to a bidder i only happens when the $G^s(v_i, i)$-th unit appears in the future.

THEOREM A.4. *A supply-monotone mechanism can be represented by a convex combination of at most $n \cdot |V| \cdot S^2 + 1$ supply-monotone hierarchical mechanisms for independently distributed bidders.*

Proof. Similar to the i.i.d. setting, we can decompose the interim allocation rule $a_i^s(\cdot)$ as $a_i^s(v) = \sum_{k=1}^{s} \gamma_i^s(k, v)$ where $\gamma_i^s(k, v)$ represents the expected number of units that a bidder receives from the k-th unit when the supply is s.

Let $\pi_i^k : \{1, \cdots, |V|\} \to V$ be an ordering over V such that for all $1 \le j < |V|$, $\gamma_i^S(k, \pi_i^k(j)) \le \gamma_i^S(k, \pi_i^k(j+1))$. By an extension of Theorem 2.2 to non identical cases, $\gamma_i^S(k, \cdot)$ is feasible if and only if

$$\sum_i \sum_{j \ge l_i} f_i(\pi_i^k(j)) \cdot \gamma_i^S(k, \pi_i^k(j)) \le 1 - \prod_i (1 - \sum_{j \ge l_i} f_i(\pi_i^k(j))), \ \forall 1 \le l_i \le |V|.$$

Consider the following set of feasibility constraints:

$$\theta_i^s(k, \pi_i^k(j)) \le \theta_i^{s+1}(k, \pi_i^k(j))$$
$$\theta_i^S(k, \pi_i^k(j)) \le \theta_i^S(k, \pi_i^k(j+1))$$

and

$$\sum_i \sum_{j \ge l_i} f_i(\pi_i^k(j)) \cdot \theta_i^S(k, \pi_i^k(j)) \le 1 - \prod_i (1 - \sum_{j \ge l_i} f_i(\pi_i^k(j)))$$

while $\theta_i^s(k, \pi_i^k(j)) \ge 0$ for all s, i, j, k. First, note that the above constraints define a polytope and the constructed $\gamma_i^s(\cdot, \cdot)$ is inside the polytope. Moreover, Cai et al. [9] demonstrates that for any corner of

$\lvert V\rvert$	2	3	4	5	6	7	8	9	10
τ_1	1.0	0.976	0.976	0.978	0.982	0.984	0.986	0.987	0.988
τ_2	1.0	0.988	0.989	0.983	0.981	0.981	0.981	0.981	0.983
τ_3	1.0	0.977	0.980	0.976	0.980	0.983	0.985	0.987	0.987
τ_4	1.0	0.990	0.988	0.987	0.987	0.987	0.987	0.987	0.987
τ_5	1.0	0.975	0.970	0.975	0.977	0.978	0.980	0.981	0.981

Table 1: τ_1: Uniform f: $f(i) = 1/|V|$; τ_2: Linearly decreasing f: $f(i) \propto |V| - i + 1$; τ_3: Linearly increasing f: $f(i) \propto i$; τ_4: Exponentially decreasing f: $f(i) \propto 2^{|V|-i}$; τ_5: Exponentially increasing f: $f(i) \propto 2^i$.

the polytope, $\theta_i^S(k, \cdot)$ corresponds to a hierarchical mechanism.

As for $\theta^s(k, \pi_i^k(j))$ with $s < S$, at a corner of the polytope, there must exists a $G_i^k(\pi_i^k(j)) \geq k$ such that for all $s < G_i^k(\pi_i^k(j))$, $\theta_i^s(k, \pi_i^k(j)) = 0$ and for all $s \geq G_i^k(\pi_i^k(j))$, $\theta_i^s(k, \pi_i^k(j)) = \theta_i^S(k, \pi_i^k(j))$. Therefore, any corner of the polytope corresponds to a supply-monotone hierarchical mechanism.

Finally, since there are totally $n \cdot |V| \cdot S^2$ variables, the polytope is of $(n \cdot |V| \cdot S^2)$-dimension. By Carathéodory theorem, $\gamma_i^s(\cdot, \cdot)$ can be represented by at most $n \cdot |V| \cdot S^2 + 1$ corners of the polytope.

An efficient algorithm to sample a supply-monotone hierarchical mechanism can be obtained by applying the algorithm developed by [9]. As for the payment, we can maintain BIC and BIR by simply charging the interim payment $p_i^s(v_i)$ for a bidder i with valuation v_i no matter which supply-monotone hierarchical mechanism is sampled.

B Experiments

We conducted experiments to empirically determine the revenue loss due to supply monotonicity on several distributions with monotone hazard ratio. In the experiments, we set $B = 100$, $n = 2$ (since $n = 1$ is trivial), and the maximum $S = \infty$. The results are summarized in Table 1.