Online Truthful Mechanisms for Combinatorial Auctions

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Abstract

We study combinatorial auctions in online environments with an objective to maximize the social welfare. In this problem, items arrive on each day and must be sold before the closing date of the market which is unknown to the seller and the buyers. We consider two settings, one where items must be sold immediately on arrival (called ImmediateSale), and the other where items can be stored and sold later (called DeferredSale).

In the ImmediateSale setting where item must be sold immediately on arrival, we demonstrate a surprising separation between submodular and XOS valuations: on one hand, we show there is no online mechanism that can achieve $\Omega((m/\log m)^{1/3})$ approximation for XOS (and subadditive) valuations in a Bayesian setting, where m is the total number of items. Our lower bound holds even if we do not require truthfulness and/or efficiency of the mechanism. On the other hand, for submodular valuations, we give a $O(\log m)$-competitive mechanism for adversarially chosen valuations and an $O(1)$-competitive mechanism in a Bayesian setting. Both of our mechanisms are efficient and universally truthful.

In the DeferredSale setting where the items can be stored and sold later, we provide a reduction that can convert the state-of-the-art offline mechanism into an online mechanism while preserving the approximation ratio for subadditve valuations.

1 Introduction

Combinatorial auctions (see, e.g., [4]) are a central object of study in the field of algorithmic game theory. In a combinatorial auction, we are given a set U of m items and a set of n buyers with respective valuation functions $(v_1, \cdots, v_n)$ defined on all subsets of items. The goal is to design an auction that allocates the items to the buyers $S = (S_1, \ldots, S_n)$ (i.e., buyer i receives the subset of items S_i where $S_i \cap S_j = \emptyset$) such that the social welfare, defined as $\sum_i v_i(S_i)$, is maximized. In a seminal work, Dobzinski et al. [11] provided the first *truthful*[1] and efficient mechanism that approximates the social welfare to a factor of $O(\sqrt{m})$ for general monotone combinatorial valuations and $O(\log^2 m)$ when restricted to XOS valuations. In the last decade, welfare-optimal combinatorial auction design has been extensively studied and the approximation ratio has been improved to $O(\log m \log \log m)$ for subadditive valuations [6] and $O((\log \log m)^3)$ [1] for XOS valuations. This line of work establishes a relatively clear picture of combinatorial auctions in *static* settings.

Motivated by real-world applications, there has been growing interest in the design of auctions for *dynamic* settings. For instance, in marketplaces such as Amazon and eBay, the items arrive online over time and neither the platforms (seller) nor the buyers have knowledge of items that will appear in the future. Therefore, a natural question arises: *can we design combinatorial auctions in an online environment?* In this paper, we address this question by studying online combinatorial auction design. We consider an environment where there is a fixed set of buyers throughout the entire time horizon while the items arrive in an online manner.

In an online environment, a natural requirement is that once an item is sold to a buyer, the seller cannot retrieve the item and reallocate it later. In terms of when an items can be sold, we consider two possibilities: DeferredSale and ImmediateSale. In both environments, the closing date of the market is unknown to both the seller and the buyers. However, in the DeferredSale environment, the seller is allowed to keep the items and sell them later, as long as the market is not yet closed, while in the more restrictive ImmediateSale environment, the seller must sell the items immediately on arrival. The motivation for ImmediateSale is apparent — it models scenarios where items cannot be put in an inventory, such as perishable items. For DeferredSale, if the valuation of the items remained unchanged even after the market closes, then this environment is equivalent to offline auctions since the seller can sell all items after the market closes. But, in many scenarios, this is not the case. For instance, consider a market for a particular generation of smartphones. Once the next generation of phones is released, the previous phones lose substantial value, and hence, it is not a viable strategy to wait for the market to close. The DeferredSale environment, where sales can be deferred but only as long as the market remains open, models these scenarios.

For the buyers' valuations, we consider both the prior-free setting and the Bayesian setting. In both settings, the buyers' valuations for the future goods are chosen by an adversary in an adaptive manner while being consistent with the valuations for the existing goods. In the Bayesian setting, the seller can additionally get access to the distributions of the buyers' valuations over the existing goods. Since neither the seller nor the buyers have any prior knowledge about the future, we consider *myopic* buyers when reasoning about their strategic behaviors. Our task is to design *truthful* and *efficient* mechanisms maximizing social welfare in the online setting.

[1] We use truthful and incentive compatible interchangeably in this paper to mean that a buyer cannot profit by lying about her preferences.

1.1 Our Results

In this paper, we focus on complement-free buyers, i.e., buyers with subadditive valuations. As subclasses of subadditive valuations, we consider the widely studied XOS valuations, and as a further subclass, submodular valuations. Moreover, we assume that in addition to *value queries*, where a buyer is asked to report her value of a particular set of items, the seller is allowed to use *demand queries* to ask for a buyer's favorite bundle given a vector of prices for each item.

Recall that m denotes the total number of items and n is the number of buyers. In the more restrictive ImmediateSale setting where items must be sold immediately, we demonstrate a sharp separation between XOS and submodular valuations. For XOS buyers (and by generalization also for subadditive buyers), we provide an impossibility result showing that no randomized mechanism can achieve an $o(n)$ approximation when $m = \Omega(n^3 \log n)$ in the Bayesian setting (Section 3). When $m = \Theta(n^3 \log n)$, this impossibility result implies an $\Omega\left((m/\log m)^{1/3}\right)$ lower bound on the approximation factor of online combinatorial auctions with XOS buyers. Interestingly, this lower bound holds even if we do not require truthfulness and/or efficiency of the mechanism, and also in the symmetric setting where all the buyers have exactly the same valuation distributions. In contrast, for submodular valuations, we design an online mechanism achieving an $O(\log m)$-approximation in the prior-free setting and an online mechanism achieving an 8-approximation in the Bayesian setting, thereby providing a sharp separation with the XOS scenario (Section 4). To the best of our knowledge, our result is the first one that demonstrates a separation between XOS and submodular valuations in truthful welfare-maximizing combinatorial auction design.

In the DeferredSale environment where items can be sold in later stages, we give a reduction that can convert the state-of-the-art mechanisms for the offline setting to the DeferredSale environment while preserving the approximation ratio. As a result, the state-of-the-art mechanisms for subadditive [6, 12] and XOS [1, 12] buyers for both the prior-free setting and the Bayesian setting generalize to our online setting immediately. Our results are summarized in Table 1. We have also provided existing results in the offline setting for comparison.

		ImmediateSale	DeferredSale	Offline
Prior-free	Subadditive	$\Omega\left((m/\log m)^{1/3}\right)$	$O(\log m \log\log m)$	$O(\log m \log\log m)$ [6]
	XOS	$\Omega\left((m/\log m)^{1/3}\right)$	$O((\log\log m)^3)$	$O((\log\log m)^3)$ [1]
	Submodular	$O(\log m)$	$O((\log\log m)^3)$	$O((\log\log m)^3)$ [1]
Bayesian	Subadditive	$\Omega\left((m/\log m)^{1/3}\right)$	$O(\log m)$	$O(\log m)$ [12]
	XOS	$\Omega\left((m/\log m)^{1/3}\right)$	≤ 8	2 [12]
	Submodular	≤ 8	≤ 8	2 [12]

Table 1: Summary of our results. We include the results for offline versions of our problem from previous work for reference.

1.2 Technical Overview

In the ImmediateSale environment, we demonstrate a sharp separation between XOS and submodular valuations in terms of the approximation ratio. XOS and submodular valuations are often considered to be at the same level of complexity in welfare-maximizing combinatorial auction design, since most techniques developed for submodular valuations can be directly generalized to apply for XOS valuations (e.g., [11, 8]). This is mostly due to the fact that prior work relies heavily on a standard revenue-utility decomposition argument (see, e.g., [16, 12, 13]). In the ImmediateSale setting, this revenue-utility decomposition argument fails in a fundamental way — in fact, our lower bound implies that no auction works for XOS valuations in this setting. Our online mechanisms, instead, rely on a carefully designed online pricing scheme and an associated novel argument on revenue-utility decomposition for submodular valuations.

Our reduction in the DeferredSale setting provides a framework to convert the state-of-the-art offline mechanisms to an online mechanism while preserving the approximation ratio. Compared to the offline setting, the main difficulty in our online setting comes from the unknown closing date. For example, in the prior-free setting, when there is a dominant buyer who contributes substantially to the total welfare, we can no longer implement a second price auction over the grand bundle in an online environment. A seemingly immediate adaptation is to run a second price auction in each round, which works for submodular valuations but fails when the buyers valuations are XOS or subadditive. Instead, we propose using a second price auction with reserve set by the current welfare for each round. When there is no dominant buyer, intuitively, we design a scheme to implement the state-of-the-art offline mechanism every time when the (estimated) optimal welfare doubles. The challenging part in the analysis is that the buyers may have already purchased some items in the past, which will lower their interests for purchasing items that come in the future since their valuations are subadditive. To overcome this hurdle, we generalize the revenue-utility decomposition argument to the DeferredSale setting for subadditive valuations.

1.3 Related Work

Initiated by the seminal work of Dobzinski et al. [11], offline truthful combinatorial auctions have been extensively studied in the last decade. For general monotone valuations with demand queries, Dobzinski et al. [11] gave an $O(\sqrt{m})$-approximation, which matches the communication complexity lower bound by Nisan [21]. Restricted to complement-free buyers, the first nontrivial $O(\log^2 m)$ upper bound for XOS valuations was also given by Dobzinski et al. [11]. Dobzinski [6] later improved the upper bound to $O(\log m \log\log m)$ for subadditive buyers. For XOS buyers, Krysta and Vöcking [18] obtained an upper bound of $O(\log m)$ that betters the more general bound for subadditive buyers. Later, Dobzinski [8] further improved this bound to $O(\sqrt{\log m})$ for XOS buyers. In a very recent paper, Assadi and Singla [1] gave an $O((\log\log m)^3)$-approximate mechanism by combining existing techniques with a novel learning procedure, which iteratively estimates the supporting prices of individual items. No super-constant lower bound is known in this setting. Instead of both demand and value queries, if one were restricted only to value queries, Dobzinski et al. [10] gave an $O(\sqrt{m})$ upper bound for submodular buyers, which is matched by information-theoretic [7] and complexity-theoretic [9] lower bounds.

From a pure algorithmic point of view, the problem of computing a welfare maximizing combinatorial allocation has also been extensively studied. For submodular valuations, Vondrák [22] gave an $(e/(e-1))$-approximation using value queries only, with a matching lower bound by Mirrokni et al. [20]. With demand queries, a slightly better upper bound was given by Feige and Vondrák [15], while the best known lower bound is $((2e)/(2e-1))$ [9]. For XOS and subadditive buyers, Feige [14] gave an $(e/(e-1))$-approximation and a 2-approximation respectively, using both value and demand queries.

Another line of research closely related to this paper considers an online setting with sequentially arriving buyers and b identical copies of each item, which was initiated by Bartal et al. [3] and Awerbuch et al. [2]. In particular, Krysta and Vöcking [18] gave truthful mechanisms that are $O(m^{1/(b+1)}\log(bm))$-competitive for general buyers for any $b \geq 1$, and $O(\log m)$-competitive for XOS buyers when $b = 1$. In Bayesian settings where the distributions of buyers' valuations is known, the model with buyers arriving online can be viewed as a combinatorial variant of prophet inequalities. In this setting, Feldman et al. [16] gave a truthful $((2e)/(e-1))$-competitive mechanism for XOS buyers, which was later improved to 2-competitive by Dütting et al. [12]. Ehsani et al. [13] further showed that the ratio improves to $e/(e-1)$ when buyers arrive in a uniformly random order. One of our main results generalize these bounds to our online setting.

2 Preliminaries

We consider a setting with n buyers and one seller. The goods arrive online in T stages and we let the set of newly arrived goods at stage t be B^t. The entire set of goods is $U = \bigcup_{t=1}^{T} B^t$. For convenience, we will use the notation $U^{(t,t')} = \bigcup_{\tau=t}^{t'} B^\tau$ to represent the items arriving between stage t and stage t'. Let $m_t = |U^{(1,t)}|$ be the total number of items in the first t stages. As usual, we use $-i$ to indicate the buyers other than buyer i.

The buyers' valuations are combinatorial and in line with the literature, we assume the valuations are normalized, i.e., $v_i(\emptyset) = 0$ for all buyers i, and monotone, i.e., $v_i(S) \geq v_i(S')$ for all $S \supseteq S'$. For convenience, we give each item an index j in a chronological order of its arrival. More precisely, the items in B^t are indexed between $(|U^{(1,t-1)}| + 1)$ and $|U^{(1,t)}|$. Moreover, for item j, we will write $v_i(j) = v_i(\{j\})$ for short. We focus on subadditive valuations in this paper.

Definition 1 (Subadditive Valuation)**.** *A valuation v is sub-additive if for every bundle S and S' such that $S \cap S' = \emptyset$, we have $v(S) + v(S') \geq v(S \cup S')$.*

Among the class of subadditive valuations, there are two classes of valuations that are used heavily in the literature: XOS and submodular valuations.

Definition 2 (Additive and XOS Valuation)**.** *A valuation v is additive if for every bundle $S \subseteq U$, we have $v(S) = \sum_{j \in S} v(\{j\})$. A valuation v is XOS if there exist additive valuations $a_1, \cdots, a_q$ such that for every bundle $S \subseteq A$, we have $v(S) = \max_r a_r(S)$.*

For an XOS valuation v with associated additive valuations $a_1, \cdots, a_m$, each a_r is called a *clause* of v. If $a^* \in \arg\max_{a_r} a_r(S)$, then we say a^* is a *maximizing clause* of S and $a^*(\{j\})$ is the *supporting price* of good j in this maximizing clause.

Definition 3 (Submodular Valuation)**.** *A valuation v is submodular if for every bundle S and S', we have $v(S) + v(S') \geq v(S \cup S') + v(S \cap S')$.*

It is well-known that submodularity implies XOS, and XOS implies subadditivity. The marginal valuation of a buyer on an additional bundle S' given that she already has a bundle S is represented by $v(S' \mid S) = v(S') - v(S)$.

2.1 Online Environments

Given a submodular valuation v over $U^{(1,t)}$ and a submodular valuation v' over $U^{(1,t+1)}$, we say v' is *extendable* from v if for all $S \subseteq U^{(1,t)}$, $v'(S) = v(S)$; similarly for XOS and subadditive valuations. In a prior-free environment, we consider an environment where the valuations are selected by an adversary. In other words, the buyers' valuations for future goods can be arbitrary but must be extendable from the valuation over the existing goods.

In the Bayesian setting, given a distribution F_i^t of submodular valuations over $U^{(1,t)}$ with support $\mathcal{V}^t$ and a distribution F_i^{t+1} of submodular valuations over $U^{(1,t+1)}$ with support $\mathcal{V}^{t+1}$, we say F_i^{t+1} is *extendable* from F_i^t if there exists a collection of subsets $Q_v \subseteq \mathcal{V}^{t+1}$ for each $v \in \mathcal{V}^t$ such that (1) Q_v is non-empty; (2) for all $v' \in Q_v$, v' is extendable from v; and (3) $\sum_{v' \in Q_v} \Pr[v'|F_i^{t+1}] = \Pr[v|F_i^t]$. Suppose F_i^{t+1} is extendable from F_i^t and let the buyer's valuation over $U^{(1,t)}$ be v. Then, the buyer's valuation v' over $U^{(1,t+1)}$ is randomly drawn from Q_v such that the probability of $v' \in Q_v$ is $\frac{\Pr[v'|F_i^{t+1}]}{\Pr[v|F_i^t]}$. In a Bayesian environment, we assume that F_i^t is publicly known and independent across buyers for all t and the buyers' distributions for the future goods can be arbitrarily chosen but must be extendable from the distributions over the existing goods.

Let V be some set of valuations. We use $\langle f, p\rangle$ to denote an n-bidder deterministic online mechanism for combinatorial auctions. $f_i^t : V^n \to 2^{U^{(1,t)}}$ is the allocation function that maps the valuation profile $\vec{v} = (v_1, \cdots, v_n)$ to a subset of goods, indicating the set of goods allocated to buyer i in the first t rounds. An allocation rule is valid if for all t, and two different buyers i, i', we have $f_i^t(\vec{v}) \cap f_{i'}^t(\vec{v}) = \emptyset$. Moreover, in an online auction, once a good is sold, the seller cannot retrieve the good and reallocate it in the future, i.e., $\forall t,\ f_i^t(\vec{v}) \subseteq f_i^{t+1}(\vec{v})$. The payment function $p_i^t : V^n \to \mathbb{R}$ maps the valuation profile to buyer i's cumulative payment for the first t rounds.

We consider two different online environments depending on whether the seller is required to sell the goods immediately: DeferredSale and ImmediateSale. When the seller is required to sell the goods immediately (ImmediateSale), it formally requires that for $j \in B_t$, we have $j \in f_i^{t'}(\vec{v})$ for all $t' \geq t$ if and only if $j \in f_i^t(\vec{v})$.

2.2 Universally Truthful Mechanisms

Since the buyers are myopic, incentive compatibility only concerns the current stage without taking the future into account. In both the prior-free and Bayesian settings, we are interested in designing universally truthful mechanism.

Definition 4. *A deterministic mechanism $\langle f, p\rangle$ is truthful if for every stage t, every buyer i, and any valuations $v_i, v_i' \in V$, and $\vec{v}_{-i} \in V^{n-1}$, we have*

$$v_i\big(f_i^t(v_i, \vec{v}_{-i})\big) - p_i^t(v_i, \vec{v}_{-i}) \geq v_i\big(f_i^t(v_i', \vec{v}_{-i})\big) - p_i^t(v_i', \vec{v}_{-i}).$$

A randomized mechanism $\langle f, p\rangle$ is universally truthful if it is a probability distribution over truthful deterministic mechanisms.

2.3 Competitive Ratio

Let S_i^t be the set of items allocated to buyer i at the end of stage t. We will use the vectorized symbol $\vec{S}^t = (S_1^t, \cdots, S_n^t)$ to represent the overall allocation at the end of stage t. For convenience, we will use $A^t = \bigcup_{i=1}^n S_i^t$ to represent the set of items sold in the first t stages. The welfare with respect to an allocation S is denoted by $v(\vec{S}) = \sum_i v_i(S_i)$. For a set of items U', the welfare-optimal allocation with respect to a valuation profile $\vec{v}$ is represented by $\mathrm{OPT}(U', \vec{v}) = \big(\mathrm{OPT}_1(U', \vec{v}), \cdots, \mathrm{OPT}_n(U', \vec{v})\big)$. We will drop $\vec{v}$ from the notation when it is clear from the context. The performance of our mechanism is measured by its competitive ratio:

Definition 5 (Competitive Ratio)**.** *For a set V of valuations, in a prior-free setting, we say an online mechanism $\langle f, p\rangle$ is κ-competitive if for any $(v_1, \cdots, v_n) \in V^n$ and $1 \leq t \leq T$:*

$$\kappa \cdot \mathbb{E}\left[v\left(f_1^t(U^{(1,t)}), \cdots, f_n^t(U^{(1,t)})\right)\right] \geq v\left(\mathrm{OPT}(U^{(1,t)})\right)$$

where the expectation is taken over the randomness of the online mechanism. Moreover, in the Bayesian setting, we say an online mechanism $\langle f, p\rangle$ is κ-competitive if for any independent distributions $F_1, \cdots, F_n \in \Delta(V)$ and $1 \leq t \leq T$:

$$\kappa \cdot \mathbb{E}_{\vec{v}}\left[v\left(f_1^t(U^{(1,t)}), \cdots, f_n^t(U^{(1,t)})\right)\right] \geq \mathbb{E}_{\vec{v}}\left[v\left(\mathrm{OPT}(U^{(1,t)})\right)\right]$$

where the expectation is additionally taken over $\vec{v}$ randomly drawn from the prior $\prod_i F_i$.

3 Lower bound for XOS valuations with Immediate Sale

In an environment where the seller must immediately sell the items on arrival (ImmediateSale), we demonstrate a sharp contrast between XOS and submodular valuations. In this section, we show that for XOS valuations in the ImmediateSale environment, no (randomized) truthful mechanism is $o\left((m_T/\log m_T)^{1/3}\right)$-competitive, even in the Bayesian setting (the same lower bound naturally holds in the prior free case as well). Our lower bound is information-theoretic, which means that it holds even if we do not require the mechanism to be truthful or efficient. Moreover, our construction works even when the buyers are symmetric, i.e., all the buyers have the same valuation distributions.

Theorem 1. *When the buyers' valuations are XOS, in an* ImmediateSale *environment, no randomized mechanism is $o(n)$-competitive for $m_T = \Omega(n^3 \log n)$, even if all buyers have i.i.d. valuations whose prior is public knowledge.*

We provide the high level idea about our construction, while the full proof is deferred to Appendix A. We consider an online environment with T stages with a single new item at each stage, i.e., item j arrives at stage j. For each buyer, we will construct an XOS valuation with n clauses. For ease of presentation, we represent buyer k's XOS valuation by a matrix Z_k such that row i corresponds to the i-th clause and column j corresponds to the j-th item, i.e., $Z_k(i,j)$ is the value of item j in the i-th clause of buyer k. On the arrival of item j, for each buyer k, we add a new column in Z_k as follows: pick a row i uniformly at random (and independent of any other choice) and assign $Z_k(i,j) = 1$ and $Z_k(i',j) = 0$ for all $i' \neq i$.

We first argue the performance of the optimal offline allocation on these valuations. Let c_k be the k-th clause of buyer k, i.e., the k-th row in Z_k. We will allocate item j to any buyer k with $c_k(j) = 1$; and if such a buyer does not exist, then we allocate item j arbitrarily. Notice that for each pair (k,j) of buyer k and item j, we have $\Pr[c_k(j) = 1] = 1/n$, and therefore, $\Pr[\exists k,\ c_k(j) = 1] \geq 1 - e^{-1} = \Omega(1)$. By linearity of the expectation, the expected welfare of the optimal offline allocation is $\Omega(T)$.

We are left to bound the welfare generated by an online algorithm. In order to build intuition, let us first make the simplifying, but false, assumption that the online algorithm cannot observe the realization of $Z_k(i,j)$ for all buyers k and clauses i when a new item j arrives. Now, suppose that the online algorithm assigns s_k items in total to buyer k. Note that for each of these s_k items, exactly one clause chosen uniformly at random has a valuation of 1, and all other clauses have valuation of 0, for buyer k. If we think of the n clauses as bins, and a valuation of 1 for each of the s_k items as balls being thrown uniformly at random into the bins, then the clause with the maximum valuation for these s_k items corresponds to the bin with the most balls. Using this correspondence, a simple calculation then shows that the welfare of the online algorithm summed over all the buyers concentrates around T/n, thereby giving us the lower bound we are after.

But, our simplifying assumption is false because the adversary must reveal the realization of $Z_k(i,j)$ for all clauses i and buyers k when item j arrives. Recall that our goal, for any buyer k, is to extend one clause with 1 and the other $n-1$ clauses with 0s without revealing which clause got a 1. To this end, on the arrival of item j, we create a temporary matrix Z'_k with $2n$ rows such that both the $(2i-1)$-th and $(2i)$-th rows of Z'_k are copies of the i-th row of the current Z_k. For each i, we will assign $Z'_k(2i-1,j) = 1$ and $Z'_k(2i,j) = 0$, and present Z'_k to the online algorithm at stage j. After the end of stage j, we pick an index $i \in \{1, 2, \ldots, n\}$ uniformly at random, and reconstruct Z_k as follows: $Z_k(i,\cdot) = Z'_k(2i-1,\cdot)$ while for all $i' \neq i$, we have $Z_k(i',\cdot) = Z'_k(2i',\cdot)$. Such a procedure successfully hides the random choice of i at stage j since Z'_k does not contain any information about the random choice. We "discard" the remaining n clauses in Z'_k by giving them valuations of 0 for all items henceforth. Details of this construction are given in Appendix A.

4 Mechanisms for Submodular Valuations with Immediate Sale

In this section, we provide online mechanisms for submodular valuations (full proofs in Appendix B).

4.1 Prior-free Setting

Our online mechanism for the prior-free setting PriorFreeOnline consists of a second price auction and a random fixed-price auction. Before the first stage, the seller flips a fair coin and runs the *second price auction* if it is heads, and the *random fixed-price auction* otherwise.

Second Price Auction. At stage t, we will run a second price auction without a reserve price for the bundle B^t that arrives at stage t. Notice that the second price auction is deterministic and dominant-strategy incentive-compatible, and thus, at stage t, each buyer i will truthfully report her marginal valuation over the bundle: $v_i(B^t \mid S_i^{t-1})$. (Recall that S_i^{t-1} is the set of items allocated to buyer i till stage $t-1$.)

Random fixed-price Auction. We first divide the buyers into two groups: STAT and MECH where the group for each buyer is chosen independently and uniformly at random. For convenience, let $\mathcal{O}(U', \mathcal{C})$ be the set of all possible allocations for items U' and buyers $\mathcal{C} \subseteq [n]$ and

$$\mathrm{OPT}_{\mathcal{C}}^{(t_1,t_2)} = \underset{\vec{S} \in \mathcal{O}(U^{(t_1,t_2)}, \mathcal{C})}{\arg\max} \sum_i v_i(S_i)$$

represent the welfare maximizing allocation of items between stage t_1 and t_2 to buyers $\mathcal{C}$. As a shorthand, let $\mathrm{OPT}^{(t_1,t_2)} = \mathrm{OPT}_{[n]}^{(t_1,t_2)}$ be the welfare maximizing allocation to all buyers. In addition, let $\mathrm{OPT}^{(t_1,t_2)}|_{\mathcal{C}}$ be the optimal allocation for all buyers restricted to buyers in $\mathcal{C}$, such that $\mathrm{OPT}_i^{(t_1,t_2)}|_{\mathcal{C}} = \mathrm{OPT}_i^{(t_1,t_2)}$ if $i \in \mathcal{C}$ and otherwise, $\mathrm{OPT}_i^{(t_1,t_2)}|_{\mathcal{C}} = \emptyset$. Note that in general, $\mathrm{OPT}_{\mathcal{C}}^{(t_1,t_2)} \neq \mathrm{OPT}^{(t_1,t_2)}|_{\mathcal{C}}$. For each stage t, we maintain an estimate est_t of the optimal welfare of allocating $U^{(1,t)}$ to buyers in STAT, using the following result from [14]:

Theorem 2 (Feige [14])**.** *For complement-free valuations, there exists an efficient 2-approximation estimation algorithm for the optimal welfare using demand queries and value queries. The estimate* est_t *obtained from the algorithm satisfies*

$$\frac{1}{2} v\left(\mathrm{OPT}^{(1,t)}|_{\mathrm{STAT}}\right) \leq \mathrm{est}_t \leq v\left(\mathrm{OPT}^{(1,t)}\right). \tag{1}$$

Given the estimate est_t, let P_t be a set of prices such that

$$P_t = \left\{ \left(\frac{\mathrm{est}_t}{c \cdot m_t^2}\right), \left(\frac{\mathrm{est}_t}{\frac{c}{2} \cdot m_t^2}\right), \cdots, \left(\frac{c}{2} \cdot m_t^2 \cdot \mathrm{est}_t\right), \left(c \cdot m_t^2 \cdot \mathrm{est}_t\right) \right\}. \tag{2}$$

where c is a sufficiently large constant and $m_t = |U^{(1,t)}|$. Intuitively, the size of P_t is $O(\log m_t)$ and the price grows geometrically such that the j-th price is $\frac{\mathrm{est}_t}{c \cdot m_t^2} \cdot 2^{j-1}$. We are now ready to describe our random fixed-price auction for the buyers in MECH (See Algorithm 1). Notice that for a fixed price p^t, the fixed-price auction is truthful at stage t.

Theorem 3. PriorFreeOnline *is universally truthful and* $O(\log m_T)$*-competitive.*

Our analysis uses submodularity in both the second price auction and the random fixed-price auction. We consider two situations depending on whether a *dominant* buyer exists. Buyer i is a dominant buyer if $v_i\left(U^{(1,T)}\right) \geq \frac{v\left(\mathrm{OPT}^{(1,T)}\right)}{10^4 \log m_T}$. When there exists a dominant buyer, it is easy to show that the welfare of running the second price auction is at least the valuation of the dominant buyer i^* over the entire bundle, i.e., $v_{i^*}\left(U^{(1,T)}\right)$, which immediately yields a $O(\log m_T)$ approximation.

for *each stage t* **do**
  Set p^t to be a price drawn from P_t uniformly at random
  Let $M = B^t$
  for *each buyer $i \in$ MECH in some arbitrary order* **do**
    Let $D_i = \arg\max_{S \in M} v_i(S \mid S_i^{t-1}) - p^t|S|$
    Allocate D_i to buyer i ($S_i^t = S_i^{t-1} \cup D_i$) and charge her $p^t|D_i|$
    $M = M \setminus D_i$
  end
end

Algorithm 1: Online Random Fixed-price Auction with Immediate Sale

Lemma 1. *For a set V of submodular valuations, when there exists a dominant buyer, the second price auction yields welfare at least $\frac{v\left(\mathrm{OPT}^{(1,T)}\right)}{10^4 \log m_T}$.*

From now on, we assume that there is no dominant buyer. To begin with, we will consider the following set

$$\mathcal{T} = \left\{ (t_1, t_2) \mid t_1 \le t_2 \le T,\, v\left(\mathrm{OPT}^{(t_1,t_2)}\right) \ge \frac{v\left(\mathrm{OPT}^{(1,T)}\right)}{256} \right\}.$$

Intuitively, $(t_1, t_2) \in \mathcal{T}$ if the optimal welfare restricted to the items appearing between stages t_1 and t_2 is a constant fraction of the optimal welfare over all items. We first provide a lemma showing that for any $(t_1, t_2) \in \mathcal{T}$, both $v\left(\mathrm{OPT}^{(t_1,t_2)}|_{\mathrm{STAT}}\right)$ and $v\left(\mathrm{OPT}^{(t_1,t_2)}|_{\mathrm{MECH}}\right)$ are a constant fraction of $v\left(\mathrm{OPT}^{(t_1,t_2)}\right)$.

Lemma 2. *For any $(t_1, t_2) \in \mathcal{T}$, with probability at least $1 - \frac{1}{m_T}$, we have*

$$\min\left\{ v\left(\mathrm{OPT}^{(t_1,t_2)}|_{\mathrm{STAT}}\right), v\left(\mathrm{OPT}^{(t_1,t_2)}|_{\mathrm{MECH}}\right) \right\} \ge \frac{1}{4} v\left(\mathrm{OPT}^{(t_1,t_2)}\right). \tag{3}$$

To analyze the performance of our algorithm, we will use $v\left(\mathrm{OPT}^{(t^*,T)}\right)$ as a benchmark where t^* is chosen from Lemma 3 such that $v\left(\mathrm{OPT}^{(t^*,T)}\right)$ is at least half of $v\left(\mathrm{OPT}^{(1,T)}\right)$.

Lemma 3. *There is some $t^* < T$ such that for any $t \ge t^*$, $v\left(\mathrm{OPT}^{(1,t)}\right) \ge \frac{v\left(\mathrm{OPT}^{(1,T)}\right)}{2}$, and $v\left(\mathrm{OPT}^{(t^*,T)}\right) \ge \frac{v\left(\mathrm{OPT}^{(1,T)}\right)}{2}$.*

Note that Lemma 3 implies that (a) $(t^*, T) \in \mathcal{T}$, and (b) for any $t \ge t^*$, $(1, t) \in \mathcal{T}$. Therefore, Lemma 2 can be applied to all these intervals. The key lemma we are going to establish next is that with $\Omega\left(\frac{1}{\log m_T}\right)$ probability, the item goes to the market with a desirable price constructed from additive valuation functions that approximates the submodular valuations.

Definition 6 (Point-wise Approximation [5])**.** *A set V of valuations can be point-wise β-approximated by additive valuations if for any $v \in V$ and $S \subseteq U^{(1,T)}$, v can be point-wise β-approximated at S by an additive valuation v' such that*

$$\beta_t \cdot v'(S) \ge v(S) \quad \textit{and} \quad \forall S' \subseteq U^{(1,T)},\ v'(S') \le v(S').$$

It is well-known that submodular valuations are point-wise 1-approximated by additive valuations (see, e.g., [14]). However, an 1-approximated additive valuation v' is not enough for our analysis since the smallest non-zero entry $v_{\min} > 0$ could be arbitrarily small such that we can no longer guarantee the random price is within $[c_1 v_{\min}, c_2 v_{\min}]$ for some constants $0 < c_1 < c_2 < 1$ with $\Omega\left(\frac{1}{m_T}\right)$ probability. To overcome this difficulty, we provide a way to trim the additive valuations.

Lemma 4. *There exist additive valuations $(v'_1, \cdots, v'_n)$ such that:*

- $v'_i(S) \leq v_i(S)$ *for any buyer i and $S \subseteq U^{(1,T)}$;*
- $v'\left(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}}\right) \geq \frac{v\left(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}}\right)}{10} \geq \frac{v\left(\mathrm{OPT}^{(t^*,T)}\right)}{40}$*;*
- *if $j \notin \mathrm{OPT}_i^{(t^*,T)}|_{\mathrm{MECH}}$, $v'_i(j) = 0$;*
- *for $j \in \mathrm{OPT}_i^{(t^*,T)}|_{\mathrm{MECH}}$, if $v'_i(j) > 0$, then $v'_i(j) \geq \frac{v'\left(\mathrm{OPT}_i^{(t^*,T)}|_{\mathrm{MECH}}\right)}{2j^2}$.*

We construct $(v'_1, \cdots, v'_n)$ satisfying the properties defined in Lemma 4. Let the supporting price of item j be $p_j = v'_i(j)$ for $j \in \mathrm{OPT}_i^{(t^*,T)}$. We say an item $j \in B^t$ is a *hit-item* if the random price p^t satisfies $\frac{1}{4}p_j \leq p^t \leq \frac{1}{2}p_j$.

Lemma 5. *For an item $j \in B^t$ with $p_j > 0$ and $t \geq t^*$, with probability $\Omega\left(\frac{1}{\log m_t}\right)$, $\frac{1}{4}p_j \leq p^t \leq \frac{1}{2}p_j$.*

Let G be the set of hit-items in $U^{(t^*,T)}$ and SOLD be the set of items that are sold. Notice that if $j \in G \cap \mathrm{SOLD}$, it contributes revenue p_j to the welfare. All that remains to show is that the buyers' utilities can capture the welfare generated by the unsold items. Recall that S_i^T is a set of items allocated to buyer i at the end of stage T.

Lemma 6. $\sum_i v_i\left(S_i^T\right) \geq \frac{1}{2}\sum_{G \setminus \mathrm{SOLD}} p_j$.

Proof. Let $\gamma_i = \left(\mathrm{OPT}_i^{(t^*,T)} \cap G\right) \setminus \mathrm{SOLD}$. Consider the telescoping sum representation of $v_i\left(\gamma_i \mid S_i^T\right)$:

$$v_i\left(\gamma_i \mid S_i^T\right) = \sum_{t \geq t^*} v_i\left(\gamma_i \cap B^t \mid \gamma_i \cap U^{(1,t-1)} \cup S_i^T\right).$$

Because of the submodularity of v_i and the fact that $S_i^t \subseteq S_i^T \subseteq \gamma_i \cap U^{(1,t-1)} \cup S_i^T$,

$$v_i\left(\gamma_i \mid S_i^T\right) \leq \sum_{t \geq t^*} v_i\left(\gamma_i \cap B^t \mid S_i^t\right).$$

Now consider the behavior of buyer i at time t. After buying $S_i^t \setminus S_i^{t-1}$ at stage t, buyer i could still choose to buy $\gamma_i \cap B^t$, which would give her $v_i\left(\gamma_i \cap B^t \mid S_i^t\right)$ value with payment $\sum_{j \in \gamma_i \cap B^t} p^t$. The only reason that buyer i does not do so is that her marginal gain is at most her payment, i.e., $v_i\left(\gamma_i \cap B^t \mid S_i^t\right) \leq \sum_{j \in \gamma_i \cap B^t} p^t$. So given that $p^t \leq p_j/2$, we have

$$v_i\left(\gamma_i \mid S_i^T\right) \leq \sum_{t \geq t^*} v_i\left(\gamma_i \cap B^t \mid S_i^t\right) \leq \sum_{t \geq t^*}\sum_{j \in \gamma_i \cap B^t} p^t \leq \frac{1}{2}\sum_{j \in \gamma_i} p_j = \frac{1}{2}v'_i(\gamma_i).$$

Therefore, buyer i's value is at least

$$v_i\left(S_i^T\right) = v_i\left(S_i^T \cup \gamma_i\right) - v_i\left(\gamma_i \mid S_i^T\right) \geq v_i(\gamma_i) - v_i\left(\gamma_i \mid S_i^T\right) \geq v_i(\gamma_i) - \frac{1}{2}v'_i(\gamma_i) \geq \frac{1}{2}\sum_{j \in \gamma_i} p_j. \quad \square$$

We can now prove Theorem 3 by combining the contributions from items that are sold and items that are not sold. The detailed proof is deferred to Appendix B.

4.2 Bayesian Setting

In this section, we extend our results to a Bayesian setting, where the buyers' valuations are drawn independently from prior distributions that are common knowledge. We design an efficient and universally truthful mechanism which guarantees $\frac{1}{8}$ of the the optimal welfare in expectation.

Our plan is to first establish an *offline benchmark* that guarantees a constant fraction of the welfare produced by the optimal offline allocation algorithm. We then show that our online truthful mechanism can approximate the offline benchmark with a constant ratio.

Offline benchmark and supporting prices. For an offline allocation algorithm $\mathcal{A}$, let $\mathcal{A}(U,\vec{v}) = (\mathcal{A}_1(U,\vec{v}),\ldots,\mathcal{A}_n(U,\vec{v}))$ be the allocation $\mathcal{A}$ generates with items U and valuations v as input. Consider the greedy allocation rule $\mathcal{A}$ defined inductively, as follows.

$$\mathcal{A}\big([j],\vec{v}\big)_i = \begin{cases} \mathcal{A}_i\big([j-1],\vec{v}\big)\cup\{j\}, & \text{if } i = \operatorname{argmax}_{i'} v_{i'}\Big(\{j\}\mid \mathcal{A}_{i'}\big([j-1],\vec{v}\big)\Big) \\ \mathcal{A}_i\big([j-1],\vec{v}\big), & \text{otherwise} \end{cases}$$

where $[j] = \{1,\cdots,j\}$ and ties are broken in any consistent manner. In other words, $\mathcal{A}$ allocates items in a greedy manner such that item j is allocated to the buyer with the largest marginal value for item j. It is known that $\mathcal{A}$ always produces a 2-approximation of the optimal offline allocation.

Lemma 7 ([17, 19])**.** *For any U and $\vec{v}\in V^n$, $v\big(\mathcal{A}(U,\vec{v})\big) \geq \frac{1}{2}v\big(\mathrm{OPT}(U,\vec{v})\big)$.*

We define the supporting prices with respect to $\vec{v}$ for item j from the greedy allocation $\mathcal{A}$:

$$\mathrm{SP}_j\left(\mathcal{A}(U^{(1,T)},\vec{v}),\vec{v}\right) = \sum_i \mathbb{I}\left[j\in\mathcal{A}_i\big(U^{(1,T)},\vec{v}\big)\right]\cdot v_i\Big(\{j\}\mid\mathcal{A}_i\big([j-1],\vec{v}\big)\Big).$$

That is, SP_j is the marginal value of j for the buyer who receives j according to the greedy allocation $\mathcal{A}$. These prices *support* the welfare generated by $\mathcal{A}$ in the following sense:

Lemma 8. *The supporting prices $\mathrm{SP}_j\Big(\mathcal{A}\big(U^{(1,T)},\vec{v}\big),\vec{v}\Big)$ satisfy for any buyer i,*

$$\sum_{j\in\mathcal{A}_i\left(U^{(1,T)},\vec{v}\right)} \mathrm{SP}_j\Big(\mathcal{A}\big(U^{(1,T)},v\big),\vec{v}\Big) = v_i\Big(\mathcal{A}_i\big(U^{(1,T)},\vec{v}\big)\Big).$$

Moreover, for any buyer i and $S\subseteq\mathcal{A}_i\big(U^{(1,T)},\vec{v}\big)$, we have $\sum_{j\in S}\mathrm{SP}_j\Big(\mathcal{A}\big(U^{(1,T)},\vec{v}\big),\vec{v}\Big)\leq v_i(S)$.

Notice that in the greedy allocation algorithm $\mathcal{A}$, for $j\in B^t$, we have $\mathbb{I}\left[j\in\mathcal{A}_i\big(U^{(1,T)},\vec{v}\big)\right] = \mathbb{I}\left[j\in\mathcal{A}_i\big(U^{(1,t)},\vec{v}\big)\right]$, which implies that $\mathrm{SP}_j\Big(\mathcal{A}_i\big(U^{(1,t)},\vec{v}\big),\vec{v}\Big) = \mathrm{SP}_j\Big(\mathcal{A}_i\big(U^{(1,T)},\vec{v}\big),\vec{v}\Big)$. For each item $j\in B^t$, we will set p_j to be half of the expectation of $\mathrm{SP}_j\Big(\mathcal{A}_i\big(U^{(1,t)},\vec{v}\big),\vec{v}\Big)$, i.e.,

$$p_j = \frac{1}{2}\cdot\mathbb{E}_{\vec{v}}\left[\mathrm{SP}_j\Big(\mathcal{A}_i\big(U^{(1,t)},\vec{v}\big),\vec{v}\Big)\right].$$

We are now ready to present our online mechanism BayesianOnline for the Bayesian environment (Algorithm 2). We emphasize that it is crucial to approach the buyers in the same ordering for all stages and we choose the natural order $\{1,\cdots,n\}$ for ease of presentation.

Our mechanism is truthful since for each stage, the posted-price auction is truthful. To implement the mechanism, notice that for $j\in B^t$, $p_j = \frac{1}{2}\cdot\mathbb{E}_{\vec{v}}\left[\mathrm{SP}_j\Big(\mathcal{A}_i\big(U^{(1,t)},\vec{v}\big),\vec{v}\Big)\right]$ only depends on $F_1^t,\cdots,F_n^t$, which are already known to the mechanism upon the arrival of item j.

for *each stage* t **do**
  Let $M = B^t$
  for *each buyer* i *in the ordering of* $\{1, \cdots, n\}$ **do**
    Let $D_i = \arg\max_{S \in M} v_i(S \mid S_i^{t-1}) - \sum_{j \in S} p_j$
    Allocate D_i to buyer i ($S_i^t = S_i^{t-1} \cup D_i$) and charge her $\sum_{j \in D_i} p_j$
    $M = M \setminus D_i$
  end
end

Algorithm 2: BayesianOnline: Online Posted-price Auction with Immediate Sale

Theorem 4. BayesianOnline *is universally truthful and* 8*-competitive.*

For convenience, let $\vec{S}^T(\vec{v})$ represent the allocation after stage T by our online mechanism BayesianOnline when the realized valuation profile is $\vec{v}$. To prove Theorem 4, we first generalize Lemma 6. Fix a buyer i's valuation v_i and consider two arbitrary valuation profiles $\vec{v} = (v_i, \vec{v}_{-i})$ and $\vec{v}' = (v_i, \vec{v}'_{-i})$. Let $W_i(\vec{v}, \vec{v}') = \mathcal{A}_i(U^{(1,T)}, \vec{v}') \cap S_i^T(\vec{v})$ and $Y_i(\vec{v}, \vec{v}') = \mathcal{A}_i(U^{(1,T)}, \vec{v}') \setminus \left(\bigcup_{i' \leq i} S_{i'}^T(\vec{v})\right)$.

Lemma 9. *For any buyer* i *and two valuation profiles* $\vec{v} = (v_i, \vec{v}_{-i})$ *and* $\vec{v}' = (v_i, \vec{v}'_{-i})$

$$v_i\left(S_i^T(\vec{v})\right) \geq v_i\Big(W_i(\vec{v}, \vec{v}') \cup Y_i(\vec{v}, \vec{v}')\Big) - \sum_{j \in Y_i(\vec{v}, \vec{v}')} p_j.$$

Proof. For ease of presentation, let $S_i^T = S_i^T(\vec{v})$, $W_i = W_i(\vec{v}, \vec{v}')$, and $Y_i = Y_i(\vec{v}, \vec{v}')$. Then, we have

$$v_i\left(S_i^T\right) = v_i\left(S_i^T \cup Y_i\right) - v_i\left(Y_i \mid S_i^T\right) = v_i\left(S_i^T \cup Y_i\right) - \sum_t v_i\left(Y_i \cap B^t \mid S_i^T \cup \left(Y_i \cap U^{(1,t-1)}\right)\right)$$

where the last inequality is due to the telescoping sum representation of $v_i\left(Y_i \mid S_i^T\right)$. Notice that $S_i^T \cap U^{(1,t)} \subseteq S_i^T \subseteq S_i^T \cup \left(Y_i \cap U^{(1,t-1)}\right)$ and we have

$$\begin{aligned} v_i\left(S_i^T\right) &= v_i\left(S_i^T \cup Y_i\right) - \sum_t v_i\left(Y_i \cap B^t \mid S_i^T \cup \left(Y_i \cap U^{(1,t-1)}\right)\right) \\ &\geq v_i\left(S_i^T \cup Y_i\right) - \sum_t v_i\left(Y_i \cap B^t \mid S_i^T \cap U^{(1,t)}\right). \end{aligned}$$

Since buyer i did not purchase bundle $Y_i \cap B^t$ when she has already purchased $S_i^T \cap U^{(1,t)}$, the price for acquiring $Y_i \cap B^t$ must be larger than her marginal value. Therefore, we have

$$v_i\left(S_i^T\right) \geq v_i\left(S_i^T \cup Y_i\right) - \sum_t v_i\left(Y_i \cap B^t \mid S_i^T \cap U^{(1,t-1)}\right) \geq v_i\left(S_i^T \cup Y_i\right) - \sum_t \sum_{j \in Y_i \cap B^t} p_j.$$

We finish the proof by noticing that $\sum_t \sum_{j \in Y_i \cap B^t} p_j = \sum_{j \in Y_i} p_j$ and $v_i\left(S_i^T \cup Y_i\right) \geq v_i\left(W_i \cup Y_i\right)$ since $W_i \subseteq S_i^T$. □

Lemma 9 implies that we can lower bound $v_i\left(S_i^T(\vec{v})\right)$ as follows:

$$v_i\left(S_i^T(\vec{v})\right) \geq \mathbb{E}_{\vec{v}'_{-i}}\left[v_i\Big(W_i(\vec{v}, \vec{v}') \cup Y_i(\vec{v}, \vec{v}')\Big) - \sum_{j \in Y_i(\vec{v}, \vec{v}')} p_j\right],$$

where $\vec{v}'_{-i} \sim \prod_{i' \neq i} F_i^T$. We are now ready to prove Theorem 4.

Proof of Theorem 4. Let $\mathrm{SOLD}(\vec{v})$ be the set of items that are sold in BayesianOnline when the realized valuation profile is $\vec{v}$. Moreover, let $\mathrm{SOLD}_i(\vec{v}_{-i})$ be the set of items that are sold to some buyer in $[i-1]$ when the realized valuation profile for other buyers is $\vec{v}_{-i}$. Notice that if item j is sold, then item j contributes revenue p_j to the welfare. Therefore, for any $\vec{v}$, we have

$$\mathbb{E}_{\vec{v}}\left[v\left(\vec{S}^T(\vec{v})\right)\right] \geq \sum_j \Pr_{\vec{v}}\left[j \in \mathrm{SOLD}(\vec{v})\right] \cdot p_j = \frac{1}{2}\sum_j \Pr_{\vec{v}}\left[j \in \mathrm{SOLD}(\vec{v})\right] \cdot \mathbb{E}_{\vec{v}}\left[\mathrm{SP}_j\Big(\mathcal{A}\big(U^{(1,T)}, \vec{v}\big), \vec{v}\Big)\right].$$

We now proceed to show that for any $\vec{v}$,

$$\mathbb{E}_{\vec{v}}\left[v\left(\vec{S}^T(\vec{v})\right)\right] \geq \frac{1}{2}\sum_j \Pr_{\vec{v}}\left[j \notin \mathrm{SOLD}(\vec{v})\right] \cdot \mathbb{E}_{\vec{v}}\left[\mathrm{SP}_j\Big(\mathcal{A}\big(U^{(1,T)}, \vec{v}\big), \vec{v}\Big)\right].$$

From Lemma 9, we have

$$\begin{aligned}
&\mathbb{E}_{\vec{v}}\left[v_i\left(S_i^T(\vec{v})\right)\right] \\
\geq\ & \mathbb{E}_{v_i,\vec{v}_{-i},\vec{v}'_{-i}}\left[v_i\Big(W_i(\vec{v},\vec{v}') \cup Y_i(\vec{v},\vec{v}')\Big) - \sum_j \mathbb{I}\big[j \in Y_i(\vec{v},\vec{v}')\big] \cdot p_j\right] \\
\geq\ & \mathbb{E}_{v_i,\vec{v}_{-i},\vec{v}'_{-i}}\Bigg[\sum_j \mathbb{I}\Big[j \in \mathcal{A}_i\big(U^{(1,T)}, v_i, \vec{v}'_{-i}\big)\Big] \cdot \mathbb{I}\Big[j \notin \mathrm{SOLD}_i(\vec{v}_{-i})\Big] \cdot \mathrm{SP}_j\Big(\mathcal{A}\big(U^{(1,T)}, v_i, \vec{v}'_{-i}\big), v_i, \vec{v}'_{-i}\big)\Big) \\
&\qquad - \sum_j \mathbb{I}\Big[j \in \mathcal{A}_i\big(U^{(1,T)}, v_i, \vec{v}'_{-i}\big)\Big] \cdot \mathbb{I}\Big[j \notin \mathrm{SOLD}_i(\vec{v}_{-i})\Big] \cdot p_j\Bigg] \\
=\ & \sum_j \Pr_{\vec{v}_{-i}}\Big[j \notin \mathrm{SOLD}_i(\vec{v}_{-i})\Big] \cdot \mathbb{E}_{v_i,\vec{v}'_{-i}}\Big[\mathbb{I}\Big[j \in \mathcal{A}_i\big(U^{(1,T)}, v_i, \vec{v}'_{-i}\big)\Big] \cdot \Big(\mathrm{SP}_j\big(\mathcal{A}(U^{(1,T)}, v_i, \vec{v}'_{-i}), v_i, \vec{v}'_{-i}\big) - p_j\Big)\Big] \\
\geq\ & \sum_j \Pr_{\vec{v}}\Big[j \notin \mathrm{SOLD}(\vec{v})\Big] \cdot \mathbb{E}_{\vec{v}}\Big[\mathbb{I}\Big[j \in \mathcal{A}_i\big(U^{(1,T)}, \vec{v}\big)\Big] \cdot \Big(\mathrm{SP}_j\big(\mathcal{A}(U^{(1,T)}, \vec{v}), \vec{v}\big) - p_j\Big)\Big].
\end{aligned}$$

where the second inequality follows that $W_i(\vec{v},\vec{v}') \cup Y_i(\vec{v},\vec{v}') = \mathcal{A}_i(U^{(1,T)}, \vec{v}') \setminus \big(\bigcup_{i'<i} S_{i'}(\vec{v})\big)$, Lemma 8, and $W_i(\vec{v},\vec{v}') \cup Y_i(\vec{v},\vec{v}') \supseteq Y_i(\vec{v},\vec{v}')$, while the third inequality follows the independence between $\mathrm{SOLD}_i(\vec{v}_{-i})$ and $(v_i, \vec{v}'_{-i})$. Summing over i, we have

$$\begin{aligned}
\sum_i \mathbb{E}_{\vec{v}}\left[v_i\left(S_i^T(\vec{v})\right)\right] &\geq \sum_j \Pr_{\vec{v}}\Big[j \notin \mathrm{SOLD}(\vec{v})\Big] \cdot \mathbb{E}_{\vec{v}}\left[\sum_i \mathbb{I}\Big[j \in \mathcal{A}_i\big(U^{(1,T)}, \vec{v}\big)\Big] \cdot \Big(\mathrm{SP}_j\big(\mathcal{A}(U^{(1,T)}, \vec{v}), \vec{v}\big) - p_j\Big)\right] \\
&= \sum_j \Pr_{\vec{v}}\Big[j \notin \mathrm{SOLD}(\vec{v})\Big] \cdot \mathbb{E}_{\vec{v}}\Big[\mathrm{SP}_j\Big(\mathcal{A}\big(U^{(1,T)}, \vec{v}\big), \vec{v}\Big) - p_j\Big] \\
&= \frac{1}{2}\sum_j \Pr_{\vec{v}}\Big[j \notin \mathrm{SOLD}(\vec{v})\Big] \cdot \mathbb{E}_{\vec{v}}\Big[\mathrm{SP}_j\Big(\mathcal{A}\big(U^{(1,T)}, \vec{v}\big), \vec{v}\Big)\Big].
\end{aligned}$$

Putting the two parts for sold items and unsold items together, we have

$$\mathbb{E}_{\vec{v}}\left[v\left(\vec{S}^T(\vec{v})\right)\right] \geq \frac{1}{4}\sum_j \mathbb{E}_{\vec{v}}\left[\mathrm{SP}_j\Big(\mathcal{A}\big(U^{(1,T)}, \vec{v}\big)\Big)\right] = \frac{1}{4}\mathbb{E}_{\vec{v}}\left[v\Big(\mathcal{A}\big(U^{(1,T)}, \vec{v}\big)\Big)\right] \geq \frac{1}{8}\mathbb{E}_{\vec{v}}\left[v\Big(\mathrm{OPT}\big(U^{(1,T)}\big)\Big)\right]$$

where the equality follows Lemma 8 and the last inequality follows Lemma 7. □

5 Online Mechanism with Deferred Sale

In this section, we describe our reduction from the DeferredSale environment to the classical offline environment. The only condition required by the reduction is that the offline mechanism needs to be *approximately monotone*, which roughly says that if we give buyers some items before the mechanism starts, then the (expected) welfare after running the mechanism is not much smaller than the welfare from running the mechanism without the initial items. This condition holds for most, if not all, existing mechanisms for subadditive (including XOS) buyers. As long as the condition holds, our reduction preserves the approximation ratio of the offline mechanism up to a constant factor in the DeferredSale environment.

5.1 The Reduction

We first state the requirement of our reduction.

Definition 7 (Approximate Monotonicity). *A truthful mechanism $\mathcal{M}$, which maps a set of items U and valuations $\vec{v}$ to a (randomized) allocation $\mathcal{M}(U,\vec{v})$ is approximately monotone for class $\mathcal{V}$ of valuations, if there exists a constant $C > 0$, such that for any $\vec{U}^0 = \{U_1^0, \cdots, U_n^0\}$ and U with $U \cap U_i^0 = \emptyset$ for all i, and $\vec{v} \in \mathcal{V}^n$ where the domain of v_i is over $\bigcup_{i=1}^n U_i^0 \cup U$,*

$$\mathbb{E}\left[\sum_i v_i\Big(\mathcal{M}_i(U,\vec{v})\Big)\right] \le C \cdot \mathbb{E}\left[\cdot \sum_i v_i\Big(U_i^0 \cup \mathcal{M}_i(U, v_{i|U_i^0})\Big)\right].$$

where $v_{i|U_i^0}(S) = v_i(S \cup U_i^0) - v_i(U_i^0)$ for all i and $S \subseteq U$.

As discussed above, the condition can be interpreted as giving items U_i^0 to buyer i for free before running $\mathcal{M}$ does not hurt the expected welfare more than a multiplicative constant factor. While this interpretation makes the condition appear trivially true, we emphasize that it is technically non-trivial to prove an offline mechanism is approximately monotone since $v_{i|U_i^0}$ is not necessarily a member of the class $\mathcal{V}$, e.g., when $\mathcal{V}$ is the class of XOS valuations. We show that fortunately, most existing mechanisms, including those inducing the desired competitive ratios in the literature, are in fact approximately monotone. The proofs are deferred to Appendix C.3.

5.1.1 Prior-free setting

We are now ready to give our reduction in the prior-free setting. Before the first stage, the seller flips a fair coin. If the result is heads, she implements *the second price auction with reserve*; otherwise if the result is tails, she runs an estimation scheme and makes repeated calls upon the offline mechanism.

Second Price Auction with Reserve At stage t, we will run a second price auction for the entire bundle of unsold items $U^{(1,t)} \setminus A^{t-1}$ with a reserve price set to the total welfare of the allocated items, i.e., $v(\vec{S}^{t-1})$. Notice that the second price auction is deterministic and dominant-strategy incentive-compatible, and thus, at stage t, each buyer i will truthfully report her marginal valuation over the bundle: $v_i(U^{(1,t)} \setminus A^{t-1} \mid S_i^{t-1})$. Once the bundle is sold to buyer i at stage t, the seller can update the total welfare of the allocated items to $v(\vec{S}^t) = v(\vec{S}^{t-1}) + v_i(U^{(1,t)} \setminus A^{t-1} \mid S_i^{t-1})$.

The Estimation Scheme We first divide the buyers into two groups, STAT and MECH, where the group for each buyer is chosen independently and uniformly at random. For each stage t, we maintain an estimate est_t of the optimal welfare of allocating $U^{(1,t)}$ to agents in STAT, using the 2-approximation algorithm for subadditive valuations by Feige [14] (see Theorem 2). The estimation scheme works in the following way:

- Initialize $k = 0$, $\text{est}_0 = 0$, and $t_0 = 0$.
- At each time t, compute est_t. If $\text{est}_t \geq 8\text{est}_{t_k}$, set $k = k + 1$ and $t_k = t$; call the offline mechanism with items $U^{(t_{k-1}+1,t)}$ and buyers MECH.

In words, we implement the offline mechanism on a new batch of items when the current estimate est_t is at least 8 times the estimate when the previous allocation happened. Intuitively, this guarantees that there is high enough welfare to be allocated in the current batch of items and at the same time, the welfare loss is low, if the market terminates with these items unallocated.

5.1.2 Bayesian setting

For the Bayesian setting, there is no need to flip a coin to implement two different mechanisms. In fact, it suffices to implement the estimation scheme only:

The Estimation Scheme For each stage t, we compute an estimate est_t of the expected optimal welfare of allocating $U^{(1,t)}$ from the prior. The estimation scheme works in the following way:

- Initialize $k = 0$, $\text{est}_0 = 0$, and $t_0 = 0$.
- At each time t, compute est_t. If $\text{est}_t \geq 2\text{est}_{t_k}$, set $k = k + 1$ and $t_k = t$; call the offline mechanism with items $U^{(t_{k-1}+1,t)}$, and all the buyers.

In words, we implement the offline mechanism on a new batch of items when the current estimate est_t is at least twice the estimate when the previous allocation happened.

5.2 Analysis

In the prior-free setting, the second price auction with reserve is clearly incentive-compatible. As for the estimation scheme in the prior-free setting, observe that the scheme queries only buyers in STAT, who get no items whatsoever, and therefore will answer all queries truthfully. On the other hand, the offline mechanism interacts only with buyers in MECH, whom the estimation scheme does not query at all. These buyers, being myopic, will act truthfully as long as the offline mechanism itself is truthful. Therefore, the estimation scheme is also incentive-compatible, and thus, the combination of these two subroutines is universally truthful. A similar argument can demonstrate that the estimation scheme in the Bayesian setting is also universally truthful.

The following theorem, which is the main result of this section, translates approximation guarantees in the classical offline setting to the DeferredSale environment. The only requirement, as stated above, is that the offline mechanism must be approximately monotone, which is indeed satisfied by almost all existing results, and in particular, by the state-of-the-art mechanisms for subadditive and XOS buyers, respectively,

Theorem 5. *For a set $\mathcal{V}$ of complement-free valuations, suppose there exists a truthful offline $\beta(m_T)$-approximate mechanism for valuations with m_T items and the offline mechanism is approximately monotone. Then, there exists a truthful online $\beta(m_T)$-competitive mechanism with* DeferredSale.

The rest of this section is devoted to the proof of Theorem 5. The proof for the Bayesian setting follows the fact that the state-of-art mechanisms are approximately monotone and an argument presented below for the estimation scheme similar to the prior-free setting. We will focus on the prior-free setting from now on.

Note that it suffices to show that for a fixed end of horizon T, the expected welfare generated by our reduction is at least $\Omega\left(\frac{1}{\beta(m_t)}\right)$ fraction of the welfare of the optimal allocation. Our analysis is divided into two parts depending on whether a *dominant* buyer exists: buyer i is dominant if $v_i(U^{(1,T)}) \geq \frac{v\left(\mathrm{OPT}(U^{(1,T)})\right)}{10^4}$.

Lemma 10. *When there exists a dominant buyer, the second price auction with reserve guarantees* $\Omega\,(1)$ *fraction of the optimal welfare.*

From now on, we will focus on the case in which there is no dominant buyer. Moreover, let K be the final value of k at the end of the estimation scheme, which is the number of calls made to the offline mechanism. The estimation scheme divides items into batches, and runs one auction for each batch. The approximation guarantee of the offline mechanism then applies with respect to the welfare supported by these individual batches. We first need one of these batches to be large enough to support a *constant* fraction of the welfare given by the offline optimal allocation. To this end, we consider batches which overlap the time interval $[T_1, T_2]$, on which the optimal welfare from the prefix of items $U^{(1,t)}$ for $t \in [T_1, T_2]$ grows from $\frac{v(\mathrm{OPT}^{(1,T)})}{1000}$ to $\frac{v(\mathrm{OPT}^{(1,T)})}{100}$. Subadditivity guarantees that the optimal welfare from $U^{(T_1,T_2)}$ is a constant fraction of $v(\mathrm{OPT}^{(1,T)})$. By standard concentration bounds, this welfare is distributed almost equally into STAT and MECH. As a result, est_{T_1} and est_{T_2} are within a constant factor of each other, and there are only a constant number of batches overlapping $[T_1, T_2]$, since est can only increase so much. Thus, the largest batch among these provides a constant fraction of $v(\mathrm{OPT}^{(1,T)})$ to buyers in MECH. This intuition is formalized in Lemma 11.

Lemma 11. *Suppose there is no dominant agent. That is, for any agent i,*

$$v_i(U^{(1,T)}) \leq v(\mathrm{OPT}^{(1,T)})/10^5.$$

With constant probability, there is some k such that

$$v(\mathrm{OPT}_{\mathrm{MECH}}^{(t_{k-1}+1,t_k)}) = \Theta(v(\mathrm{OPT}^{(1,T)})),$$

so that the batch supports enough welfare.

Now we focus on this constant-approximate batch guaranteed by Lemma 11. We argue that the approximation guarantee of the offline mechanism still holds for this batch, so the welfare from this batch alone is a good approximation of the offline optimal welfare. While this may appear trivially true, we note that by the time the offline mechanism is called, the buyers already possess some items, which may lower their interest in purchasing new items. Such a change of their behavior has a potential to ruin the welfare guarantee. This, however, will not happen if the offline mechanism is approximately monotone, which concludes the proof of Theorem 5.

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A Omitted Proofs of Section 3

A.1 Proof of Theorem 1

Proof. Assume that there is one item arriving per stage. We construct the prior distribution by designing a scheme to generate the collection of additive valuations randomly, proceeding from the first item to the last item. At stage t, a clause is either *outstanding* or *done*. If a clause c is done at stage t, then for all $j > t$, $c(j) = 0$. Our scheme maintains a set of n outstanding clauses at every stage.

Suppose at the end of stage t, the n outstanding clauses are $\{c_1^t, \dots, c_n^t\}$. Upon the arrival of the $(t+1)$-th item at stage $(t+1)$, we create $2n$ new clauses $c_1'^{t+1}, \dots, c_n'^{t+1}$ and $c_1''^{t+1}, \dots, c_n''^{t+1}$. Clause $c_a'^{t+1} = [c_a^t, 1]$ is obtained by simply appending 1 to the end of c_a^t, and $c_a''^{t+1} = [c_a^t, 0]$ is obtained by appending 0 to c_a^t. After the mechanism allocates the $(t+1)$-th item, the adversary flips a coin to choose uniformly at random some $i^{t+1} \in \{1, \cdots, n\}$. The new outstanding clauses are then

$$c_a^{t+1} = \begin{cases} c_a'^{t+1}, & a = i^{t+1} \\ c_a''^{t+1}, & \text{otherwise} \end{cases}.$$

We now analyze this construction after T stages. Let $Z_{k,i}$ be the i-th outstanding clause at time T in buyer k's valuation. First observe that the offline optimal allocation can achieve welfare

$\Omega(T)$. This is because for each item j and any buyer k, with probability $\frac{1}{n}$, $Z_{k,k}(j) = 1$, and with probability

$$\left(1 - \frac{1}{n}\right)^n \approx 1 - \frac{1}{e},$$

there is some buyer, denoted k_j,[2] whose valuation satisfies $Z_{k_j,k_j}(j) = 1$. If such a buyer does not exist, let $k_j = 0$.

Knowing the future realization of all valuations, the offline optimal allocation could assign item j to buyer k_j, and discard j (possibly by assigning j to an arbitrary buyer) when $k_j = 0$. Whenever $k_j \neq 0$, item j contributes 1 to the total welfare. By the definition of XOS valuations, the expected welfare would be

$$\begin{aligned}\mathbb{E}\left[\sum_{k\in[n]} v(\{j \mid k_j = k\})\right] &\geq \mathbb{E}\left[\sum_{k\in[n]} Z_{k,k}(\{j \mid k_j = k\})\right] = \mathbb{E}\left[\sum_{k\in[n]} \sum_{j:k_j=k} Z_{k,k}(j)\right] \\ &= \mathbb{E}\left[\sum_j \mathbb{I}[k_j \neq 0]\right] = \sum_j \left(1 - \frac{1}{n}\right)^n = \Omega(T).\end{aligned}$$

Now consider the welfare obtained by any online mechanism. We upper bound the welfare by upper bounding the value of each buyer separately.

Fix some buyer k_0, random indices i^t for all t, and outstanding clauses $\{c_a^t\}$ drawn for k_0 for all t, and a realized allocation $S_{k_0}^T$ generated by the online mechanism. Let t_ℓ be the ℓ-th item assigned to k_0 for $1 \leq \ell \leq |S_{k_0}^T|$. When item t_ℓ is assigned to k_0, the value of clause $c_a^{t_\ell}(S_{k_0}^{t_\ell})$ increases by 1 if and only if $i^{t_\ell} = a$, which happens with probability exactly $\frac{1}{n}$. Therefore,

$$\mathbb{E}[c_a^T(S_{k_0}^T)] = \frac{|S_{k_0}^T|}{n}. \tag{4}$$

Moreover, by the Chernoff bound,

$$\Pr\left[c_a^T(S_{k_0}^T) - \frac{|S_{k_0}^T|}{n} \geq \sqrt{|S_{k_0}^T| \log n}\right] \leq \exp(-2\log n) = \frac{1}{n^2}.$$

Taking union bound over the outstanding clauses after stage T, we have

$$\max_a c_a^T(S_{k_0}^T) \leq \frac{|S_{k_0}^T|}{n} + \sqrt{|S_{k_0}^T| \log n} \tag{5}$$

with probability at least $1 - \frac{1}{n}$, and with probability at most $\frac{1}{n}$, $\max_a c_a^T(S_{k_0}^T)$ is at most $|S_{k_0}^T|$. Moreover, notice that we have

$$\mathbb{E}[v_{k_0}(S_{k_0}^T)] \leq \mathbb{E}[\max_a c_a^T(S_{k_0}^T) + 1].$$

The extra 1 in the inequality is due to the fact that the maximizing clauses at stage T might be one of the clauses created at stage T rather than one of the outstanding clause after stage T. Nevertheless, note that the difference between their valuations is at most 1. Using (4) and (5), we have

$$\mathbb{E}[v_{k_0}(S_{k_0}^T)] \leq \frac{n-1}{n} \cdot \left(\frac{|S_{k_0}^T|}{n} + \sqrt{|S_{k_0}^T| \log n}\right) + \frac{1}{n} \cdot |S_{k_0}^T| + 1 \leq \frac{2|S_{k_0}^T|}{n} + \sqrt{|S_{k_0}^T| \log n} + 1.$$

[2] If there are multiple such buyers, let k_j be the one with the smallest index, or simply any of them.

Finally, we sum over k_0:

$$\mathbb{E}[v(\vec{S}^T)] \leq \sum_{k_0} \left(\frac{2|S_{k_0}^T|}{n} + \sqrt{|S_{k_0}^T| \log n} + 1 \right) \leq \frac{2T}{n} + \sqrt{Tn \log n} + n.$$

As long as $m_T = T = \Omega(n^3 \log n)$, the above inequality implies that $\mathbb{E}[v(\vec{S}^s)] = O\left(\frac{T}{n}\right)$. $\square$

B Omitted Proofs of Section 4

B.1 Proof of Lemma 1

Proof. Let buyer i^* be one of the dominant buyers. We show that the welfare is at least $v_{i^*}(U^{(1,T)})$. For convenience, let i_t be the buyer winning the second price auction at time t. We have

$$v(\vec{S}^T) = \sum_i v_i(S_i^T) = \sum_t v_{i_t}(B^t \mid S_{i_t}^{t-1}).$$

Because of the property of second price auctions, the marginal value of the winner is no less than that of any other buyer, so we have

$$v(\vec{S}^T) \geq \sum_t v_{i^*}(B^t \mid S_{i^*}^{t-1}) \geq \sum_t v_{i^*}(B^t \mid U^{t-1}) = v_{i^*}(U^{(1,T)}),$$

where the second inequality follows from subadditivity of v_{i^*} and the fact that $S_{i^*}^{t-1} \subseteq U^{t-1}$. $\square$

B.2 Proof of Lemma 2

Proof. Fix $t_1 \leq t_2 \leq T$ where $(t_1, t_2) \in \mathcal{T}$. First note that for any i,

$$v_i(U^{(t_1,t_2)}) \leq v_i(U^{(1,T)}) \leq \frac{v(\mathrm{OPT}^{(1,T)})}{10^4 \log m_T} \leq \frac{v(\mathrm{OPT}^{(t_1,t_2)})}{24 \log m_T}.$$

where the second inequality follows the fact that there is no dominant buyer and the last inequality is due to the definition of $\mathcal{T}$. Let $X_i = \mathbf{1}[i \in \mathrm{STAT}]$. Observe that $\{X_i v_i(\mathrm{OPT}_i^{(t_1,t_2)})\}_i$ are independent random variables, where $X_i v_i(\mathrm{OPT}_i^{(t_1,t_2)})$ is in range $[0, v_i(\mathrm{OPT}_i^{(t_1,t_2)})] \subseteq [0, \frac{v(\mathrm{OPT}^{(t_1,t_2)})}{24 \log m_T}]$, and

$$\mathbb{E}\left[\sum_i X_i v_i(\mathrm{OPT}_i^{(t_1,t_2)})\right] = \frac{1}{2} v(\mathrm{OPT}^{(t_1,t_2)}).$$

Applying Hoeffding's inequality, we have

$$\begin{aligned}
& \Pr\left[\left|v(\mathrm{OPT}^{(t_1,t_2)}|_{\mathrm{MECH}}) - \mathbb{E}[v(\mathrm{OPT}^{(t_1,t_2)}|_{\mathrm{MECH}})]\right| \geq \frac{1}{4}v(\mathrm{OPT}^{(t_1,t_2)})\right] \\
= & \Pr\left[\left|\sum_i X_i v_i(\mathrm{OPT}_i^{t_1,t_2}) - \frac{1}{2}v(\mathrm{OPT}^{(t_1,t_2)})\right| \geq \frac{1}{4}v(\mathrm{OPT}^{(t_1,t_2)})\right] \\
\leq & \exp\left(-\frac{2\cdot\left(\frac{1}{4}v(\mathrm{OPT}^{(t_1,t_2)})\right)^2}{\sum_i v_i(\mathrm{OPT}_i^{(t_1,t_2)})^2}\right) \\
\leq & \exp\left(-\frac{1}{8}\cdot\frac{v(\mathrm{OPT}^{(t_1,t_2)})}{24\log m_T\cdot(v(\mathrm{OPT}^{(t_1,t_2)})/(24\log m_T))^2}\right) \\
= & \exp(-3\log m_T) = \frac{1}{m_T^3}.
\end{aligned}$$

Exactly the same argument implies the same concentration for $v(\mathrm{OPT}^{(t_1,t_2)}|_{\mathrm{STAT}})$.

Observe that there are at most $\binom{T}{2} \leq \binom{m_T}{2} \leq \frac{1}{2}m_T^2$ pairs of t_1 and t_2 satisfying (3). Taking union bound over all such pairs and STAT and MECH, we have that (3) holds with probability at least $1 - \frac{1}{2}m_T^2 \cdot 2 \cdot \frac{1}{m_T^3} = 1 - \frac{1}{m_T}$. □

B.3 Proof of Lemma 3

Proof. Let t^* be the earliest stage such that $v(\mathrm{OPT}^{(1,t^*)}) \geq \frac{v(\mathrm{OPT}^{(1,T)})}{2}$. Notice that $t^* \leq T$. The first property follows immediately from the definition of t^*. For the second property, because of the optimality of $\mathrm{OPT}^{(t^*,T)}$ and subadditivity of v_i,

$$v(\mathrm{OPT}^{(t^*,T)}) \geq \sum_i v_i(\mathrm{OPT}_i \cap U^{(t^*,T)}) \geq v(\mathrm{OPT}^{(1,T)}) - \sum_i v_i(\mathrm{OPT}_i \cap U^{(1,t^*-1)}).$$

Again, because of the optimality of $\mathrm{OPT}^{(1,t^*-1)}$,

$$v(\mathrm{OPT}^{(t^*,T)}) \geq v(\mathrm{OPT}^{(1,T)}) - v(\mathrm{OPT}^{(1,t^*-1)}) \geq \frac{1}{2}v(\mathrm{OPT}^{(1,T)}).$$ □

B.4 Proof of Lemma 4

Proof. Let v_i'' be any additive valuation that point-wise 1-approximates v_i at $\mathrm{OPT}_i^{(t^*,T)}|_{\mathrm{MECH}}$. We construct v_i' such that $v_i'(j) = v_i''(j)$ if $j \in \mathrm{OPT}_i^{(t^*,T)}|_{\mathrm{MECH}}$ and $v_i''(j) \geq \frac{v''(\mathrm{OPT}_{\mathrm{MECH}}^{(t^*,T)})}{2j^2}$, while all other entries are simply 0.

Clearly v_i' satisfies the first, third and fourth properties in the lemma. For the second property, notice that we have

$$\begin{aligned}
v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}}) &= \sum_{i\in\mathrm{MECH}}\sum_{j\in\mathrm{OPT}_i^{(t^*,T)}} v_i'(j) \\
&\geq \sum_{i\in\mathrm{MECH}}\sum_{j\in\mathrm{OPT}_i^{(t^*,T)}} \left(v_i''(j) - \frac{v''(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{2j^2}\right) \\
&\geq v''(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}}) - \sum_{1\leq j\leq m}\frac{v''(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{2j^2}.
\end{aligned}$$

Recall that

$$\sum_{1\le j\le\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

Therefore we have

$$v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}}) \ge v''(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}}) - \frac{\pi^2 v''(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{12} \ge \frac{v(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{10}. \quad \square$$

B.5 Proof of Lemma 5

Proof. Observe that $|P^t| = O(\log m_t)$, so each price is chosen with probability $\Omega\left(\frac{1}{\log m_t}\right)$. It suffices to show that there exists $p \in P^t$ satisfying $\frac{1}{4}p_j \le p \le \frac{1}{2}p_j$, which is equivalent to showing that

$$\frac{\mathrm{est}^t}{c\cdot m_t^2} \le \frac{v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{4m_t^2} \le \frac{v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{4j^2} \le \frac{1}{2}p_j, \tag{6}$$

where the second inequality follows the fact that $j \le m_t$ and

$$c\cdot m_t^2 \cdot \mathrm{est}_t \ge \frac{v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{4} \ge \frac{1}{4}p_j. \tag{7}$$

Let $c = 2048$. For the first inequality in (6), since $\mathrm{est}_t \le v(\mathrm{OPT}^{(1,T)})$ and $v(\mathrm{OPT}^{(1,T)}) \le 2v(\mathrm{OPT}^{(t^*,T)})$ (Lemma 3),

$$\frac{\mathrm{est}_t}{2048m_t^2} \le \frac{v(\mathrm{OPT}^{(1,T)})}{2048m_t^2} \le \frac{v(\mathrm{OPT}^{(t^*,T)})}{1024m_t^2}.$$

Now by Lemma 2 and Lemma 4, we have $v(\mathrm{OPT}^{(t^*,T)}) \le 4v(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})$ and $v(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}}) \le 10v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})$, so

$$\frac{\mathrm{est}_t}{2048m_t^2} \le \frac{v(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{256m_t^2} \le \frac{v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{25.6m_t^2} \le \frac{v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{4m_t^2}.$$

For the first inequality in (7), because $\mathrm{est}_t \ge v(\mathrm{OPT}^{(1,t)}|_{\mathrm{STAT}})$ and $v(\mathrm{OPT}^{(1,t)}|_{\mathrm{STAT}}) \ge \frac{1}{4}v(\mathrm{OPT}^{(1,t)})$ (Lemma 2),

$$2048m_t^2\cdot\mathrm{est}_t \ge 1024m_t^2\cdot v(\mathrm{OPT}^{(1,t)}|_{\mathrm{STAT}}) \ge 256m_t^2\cdot v(\mathrm{OPT}^{(1,t)}).$$

Again by Lemma 4, $v(\mathrm{OPT}^{(1,t)}) \ge \frac{1}{2}v(\mathrm{OPT}^{(1,T)})$, so

$$2048m_t^2\cdot\mathrm{est}_t \ge 128m_t^2\cdot v(\mathrm{OPT}^{(1,T)}) \ge \frac{v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}})}{4}. \quad \square$$

B.6 Proof of Theorem 3

Proof. We consider each buyer separately. For buyer $i \in \mathrm{MECH}$, his welfare contribution to $\mathrm{OPT}^{(t^*,T)}$ can be divided into three different parts:

- those allocated to i, $\alpha_i = S_i^T \cap (\mathrm{OPT}_i^{(t^*,T)} \cap G)$,
- those allocated to some other agent, $\beta_i = (\cup_{i'\ne i} S_{i'}^T) \cap (\mathrm{OPT}_i^{(t^*,T)} \cap G)$, and

- those not allocated to anyone, $\gamma_i = (\mathrm{OPT}_i^{(t^*,T)} \cap G) \setminus (\cup_{i'} S_{i'}^T)$.

We first bound the revenue of the seller, which is at least

$$\mathrm{REV} \geq \sum_{i \in \mathrm{MECH}} \sum_{j \in \alpha_i \cup \beta_i, j \in B^t} p^t \geq \frac{1}{4} \sum_{i \in \mathrm{MECH}} \sum_{j \in \alpha_i \cup \beta_i} p_j.$$

where the last inequality follows $p^t \geq \frac{1}{4} p_j$.

With Lemma 6, we are now ready to bound the total welfare:

$$\begin{aligned}
v(\vec{S}^T) &\geq \max\left\{\mathrm{REV}, \sum_{i \in \mathrm{MECH}} v_i(S_i^T)\right\} \\
&\geq \max\left\{\frac{1}{8} \sum_{i \in \mathrm{MECH}} \sum_{j \in \alpha_i \cup \beta_i} p_j, \frac{1}{2} \sum_{i \in \mathrm{MECH}} \sum_{j \in \gamma_i} p_j\right\} \\
&\geq \frac{4}{5} \cdot \frac{1}{8} \sum_{i \in \mathrm{MECH}} \sum_{j \in \alpha_i \cup \beta_i} p_j + \frac{1}{5} \cdot \frac{1}{2} \sum_{i \in \mathrm{MECH}} \sum_{j \in \gamma_i} p_j \\
&= \frac{1}{10} \sum_{i \in \mathrm{MECH}} \sum_{j \in \mathrm{OPT}_i^{(t^*,T)} \cap G} p_j \\
&= \frac{1}{10} \sum_{j \in G} p_j. \qquad \square
\end{aligned}$$

The welfare guaranteed by the protocol is at least

$$\mathbb{E}[v(\vec{S}^T)] \geq \frac{1}{10} \mathbb{E}_G \left[\sum_{j \in G} p_j\right] = \frac{1}{10} \sum_{j : p_j > 0} \Pr[j \in G] \cdot p_j.$$

Since $\Pr[j \in G] = \Omega(1/\log m_t)$,

$$\begin{aligned}
\mathbb{E}[v(\vec{S}^T)] &\geq \frac{1}{10} \sum_{j : p_j > 0} \Pr[j \in G] \cdot p_j \\
&\geq \frac{1}{10} \sum_{j : p_j > 0, j \in B^t} \Omega\left(\frac{1}{\log m_t}\right) \cdot p_j && \text{(Lemma 5)} \\
&\geq \frac{1}{10} \sum_{j : p_j > 0} \Omega\left(\frac{1}{\log m_T}\right) \cdot p_j && (m_t \leq m_T) \\
&= \Omega\left(\frac{1}{\log m_T}\right) \cdot v'(\mathrm{OPT}^{(t^*,T)}|_{\mathrm{MECH}}) && \text{(definition of } p_j\text{)} \\
&= \Omega\left(\frac{1}{\log m_T}\right) \cdot v(\mathrm{OPT}^{(1,T)}) && \text{(Lemma 4 and Lemma 2)}
\end{aligned}$$

which concludes the proof of Theorem 3.

B.7 Proof of Lemma 8

Proof. Observe that for any i and $j \in \mathcal{A}_i(U^{(1,T)}, v)$,

$$\begin{aligned}\mathrm{SP}_j(\mathcal{A}(U^{(1,T)}, v)) &= \sum_{i'} \mathbb{I}[j \in \mathcal{A}_{i'}(U^{(1,T)}, \vec{v})] v_{i'}(\{j\} \mid \mathcal{A}_{i'}([j-1], v)) \\ &= v_i(\{j\} \mid \mathcal{A}_i([j-1], v)) \\ &= v_i(\mathcal{A}([j], \vec{v}) \mid \mathcal{A}([j-1], \vec{v})).\end{aligned}$$

The first part of the lemma then follows by summing over j. For the second part, by submodularity

$$v_i(S) = \sum_{j \in S} v_i(\{j\} \mid S \cap [j-1]) \geq \sum_{j \in S} v_i(\{j\} \mid \mathcal{A}_i(U^{(1,S)}, \vec{v}) \cap [j-1]) = \sum_{j \in S} \mathrm{SP}_j(\mathcal{A}). \qquad \square$$

C Omitted Proofs of Section 5

C.1 Proof of Lemma 10

Proof. Let i^* be a dominant buyer. For convenience, let t_k be the k-th stage in which the bundle is sold in the auction, i.e., there exists a buyer i such that $v_i(U^{(1,t)} \setminus A^{t-1} | S_i^{t-1}) \geq v(\vec{S}^{t-1})$ for $t = t_k$. We show inductively that for every t_k, the welfare $v(\vec{S}^{t_k})$ satisfies $v(\vec{S}^{t_k}) \geq \frac{v_{i^*}(U^{(1,t_k)})}{2}$.

By our definition of t_k, we have $A^{t_k} = U^{(1,t_k)}$ and $A^{t_k - 1} = U^{(1,t_{k-1})}$. Assume that at t_k, the bundle $U^{(t_{k-1}+1,t_k)}$ is allocated to agent i_k. Since the bundle is sold, we have

$$v_{i_k}(U^{(t_{k-1}+1,t_k)} \mid S_{i_k}^{t_{k-1}}) \geq v(\vec{S}^{t_{k-1}}) \geq v_{i^*}(S_{i^*}^{t_{k-1}}). \tag{8}$$

On the other hand, since buyer i_k wins the second price auction, her bid must be at least the bid submitted by buyer i^*:

$$v_{i_k}(U^{(t_{k-1}+1,t_k)} \mid S_{i_k}^{t_{k-1}}) \geq v_{i^*}(U^{(t_{k-1}+1,t_k)} \mid S_{i^*}^{t_{k-1}}). \tag{9}$$

Combining (8) and (9), we have

$$2v_{i_k}(U^{(t_{k-1}+1,t_k)} \mid S_{i_k}^{t_{k-1}}) \geq v_{i^*}(U^{(t_{k-1}+1,t_k)}) \geq v_{i^*}(U^{(t_{k-1}+1,t_k)} \mid U^{(1,t_{k-1})}), \tag{10}$$

where the last inequality is by subadditivity. Therefore, we have:

$$\begin{aligned}v(\vec{S}^{t_k}) &= v(\vec{S}^{t_{k-1}}) + v_{i_k}(U^{(t_{k-1}+1,t_k)} \mid S_{i_k}^{t_{k-1}}) \\ &\geq \frac{1}{2} v_{i^*}(U^{(1,t_{k-1})}) + \frac{1}{2} v_{i^*}(U^{(t_{k-1}+1,t_k)} \mid U^{(1,t_{k-1})}) \qquad \text{(induction hypothesis and (10))} \\ &= \frac{1}{2} v_{i^*}(U^{(1,t_k)}).\end{aligned}$$

We will finish our proof by showing that at stage T,

$$v(\vec{S}^T) \geq \frac{1}{5} v_{i^*}(U^{(1,T)}) \geq \frac{1}{5} \cdot \frac{v\left(\mathrm{OPT}(U^{(1,T)})\right)}{10^4}.$$

Let t_K is the last stage in which the bundle is sold in the auction. If $t_K = T$, then the above inequality from induction implies

$$v(\vec{S}^T) = v(\vec{S}^{t_K}) \geq \frac{1}{2} v_{i^*}(U^{(1,t_K)}) = \frac{1}{2} v_{i^*}(U^{(1,T)}).$$

Otherwise, at time T no item is allocated. Using the induction hypothesis and the property of a second-price auction with reserve, we have that

$$v(\vec{S}^T) \geq \max\left\{\frac{1}{2}v_{i^*}(U^{(1,t_K)}), v_{i^*}(U^{(t_K+1,T)} \mid S_{i^*}^{t_K})\right\}.$$

Therefore,

$$\begin{aligned}
v(\vec{S}^T) &\geq \frac{4}{5}\cdot\frac{1}{2}v_{i^*}(U^{(1,t_K)}) + \frac{1}{5}v_{i^*}(U^{(t_K+1,T)} \mid S_{i^*}^{t_K}) \\
&= \frac{1}{5}v_{i^*}(U^{(1,t_K)}) + \frac{1}{5}v_{i^*}(S_{i^*}^{t_K}) + \frac{1}{5}v_{i^*}(U^{(1,t_K)} \mid S_{i^*}^{t_K}) + \frac{1}{5}v_{i^*}(U^{(t_K+1,T)} \mid S_{i^*}^{t_K}).
\end{aligned}$$

Using monotonicity of the valuation functions, we get

$$\begin{aligned}
v(\vec{S}^T) &\geq \frac{1}{5}v_{i^*}(U^{(1,t_K)}) + \frac{1}{5}v_{i^*}(S_{i^*}^{t_K}) + \frac{1}{5}v_{i^*}(U^{(t_K+1,T)} \mid S_{i^*}^{t_K}) \\
&= \frac{1}{5}v_{i^*}(U^{(1,t_K)}) + \frac{1}{5}v_{i^*}(U^{(t_K+1,T)}) \\
&\geq \frac{1}{5}v_{i^*}(U^{(1,T)}), \quad \text{by subadditivity of the valuation functions.}
\end{aligned}$$

□

C.2 Proof of Lemma 11

Proof. Let

$$T_1 = \min\{t \mid v(\mathrm{OPT}^{(1,t)}) \geq v(\mathrm{OPT}^{(1,T)})/1000\},$$

and

$$T_2 = \min\{t \mid v(\mathrm{OPT}^{(1,t)}) \geq v(\mathrm{OPT}^{(1,T)})/100\}.$$

Observe that $T_1 \leq T_2 \leq T$, and it is possible that $T_1 = T_2$ or $T_2 = T$ or $T_1 = T_2 = T$. Also, by the Hoeffding bound, with constant probability, simultaneously for all $t \in \{T_1, T_2, T\}$,

$$\begin{aligned}
0.2v(\mathrm{OPT}^{(1,t)}) &\leq \min(v(\mathrm{OPT}^{(1,t)}|_{\mathrm{STAT}}), v(\mathrm{OPT}^{(1,t)}|_{\mathrm{MECH}})) \\
&\leq \max(v(\mathrm{OPT}^{(1,t)}|_{\mathrm{STAT}}), v(\mathrm{OPT}^{(1,t)}|_{\mathrm{MECH}})) \leq 0.8v(\mathrm{OPT}^{(1,t)}).
\end{aligned}$$

We condition on this from now on.

Note that there are only a constant number of batches ending between T_1 and T, inclusively. This is simply because $\mathrm{est}^{T_1} = \Omega(v(\mathrm{OPT}^{(1,T)}))$, and $\mathrm{est}^{t_K} \leq v(\mathrm{OPT}^{(1,T)})$. We argue that one of these batches satisfies the conditions of the proposition.

We first show there is enough welfare between T_1 and T_2 (inclusively) for agents in MECH.

$$\begin{aligned}
v(\mathrm{OPT}_{\mathrm{MECH}}^{(T_1,T_2)}) &\geq v(\mathrm{OPT}_{\mathrm{MECH}}^{(1,T_2)}) - v(\mathrm{OPT}_{\mathrm{MECH}}^{(1,T_1-1)}) && \text{(subadditivity of OPT)} \\
&\geq v(\mathrm{OPT}^{(1,T_2)}|_{\mathrm{MECH}}) - v(\mathrm{OPT}^{(1,T_1-1)}) \\
& && \text{(optimality and monotonicity w.r.t. agents of OPT)} \\
&\geq 0.2v(\mathrm{OPT}^{(1,T_2)}) - v(\mathrm{OPT}^{(1,T_1-1)}) && \text{(concentration at } T_2\text{)} \\
&\geq \frac{0.2}{100}v(\mathrm{OPT}^{(1,T)}) - \frac{1}{1000}v(\mathrm{OPT}^{(1,T)}) && \text{(choice of } T_1 \text{ and } T_2\text{)} \\
&= \Omega(v(\mathrm{OPT}^{(1,T)})).
\end{aligned}$$

Now intuitively, the remaining issue is that maybe the final unclosed batch starts before T_2 (inclusively), and contains most of the above welfare. We show that this is impossible. In particular, there must be a batch ending after T_2 (inclusively). Suppose otherwise, i.e., $t_K < T_2$. We show that $\mathrm{est}^T \geq 8\mathrm{est}^{t_K}$, leading to a contradiction. In fact,

$$\begin{aligned}
\mathrm{est}^T &\geq \frac{1}{2} v(\mathrm{OPT}^{(1,T)}|_{\mathrm{STAT}}) && \text{(2-approximation)}\\
&\geq 0.1 v(\mathrm{OPT}^{(1,T)}) && \text{(concentration at } T\text{)}\\
&\geq 8 \times \frac{v(\mathrm{OPT}^{(1,T)})}{100} && \\
&\geq 8 \times v(\mathrm{OPT}^{(1,T_2-1)}) && \text{(choice of } T_2\text{)}\\
&\geq 8 \times \mathrm{est}^{T_2-1} && \text{(definition of } \mathrm{est}^t\text{)}\\
&\geq 8 \times \mathrm{est}^{t_K}. && (t_K < T_2 \text{ and monotonicity of } \mathrm{est}^t)
\end{aligned}$$

Now we know:

- $v(\mathrm{OPT}^{(T_1,T_2)}_{\mathrm{MECH}}) = \Theta(v(\mathrm{OPT}^{(1,T)}))$, and
- there are only $O(1)$ batches overlapping $[T_1, T_2]$, whose indices are $k_1, \ldots, k_2$ where $t_{k_1-1} < T_1 \leq t_{k_1}$, $t_{k_2} \geq T_2$, and $k_2 - k_1 = O(1)$.

We only need to show that for some $k \in \{k_1, \ldots, k_2\}$,

$$v(\mathrm{OPT}^{(t_{k-1}+1,t_k)}_{\mathrm{MECH}}) = \Theta(v(\mathrm{OPT}^{(1,T)})).$$

By subadditivity and monotonicity w.r.t. items of OPT,

$$\sum_{k \in \{k_1,\ldots,k_2\}} v(\mathrm{OPT}^{(t_{k-1}+1,t_k)}_{\mathrm{MECH}}) \geq v(\mathrm{OPT}^{(t_{k_1-1}+1,t_{k_2})}_{\mathrm{MECH}}) \geq v(\mathrm{OPT}^{(T_1,T_2)}_{\mathrm{MECH}}).$$

Since there are only $O(1)$ summands, for some k,

$$v(\mathrm{OPT}^{(t_{k-1}+1,t_k)}_{\mathrm{MECH}}) = \Theta(v(\mathrm{OPT}^{(T_1,T_2)}_{\mathrm{MECH}})) = \Theta(v(\mathrm{OPT}^{(1,T)})).$$

This is our desired batch. □

C.3 Approximate Monotonicity of Mechanisms Based on Posted-Price Auctions

In this section, we argue that if the offline mechanism is "essentially based on posted-price auctions and standard revenue-surplus arguments," then the mechanism is approximately monotone. The argument presented here applies in particular for the $O(\log m \log\log m)$-approximate mechanism for subadditive buyers [6] and the $O((\log\log m)^3)$-approximate mechanism for XOS buyers [1].

For brevity we refrain from unnecessarily repeating the entire arguments of the offline mechanisms. The key property we need to prove is that in a posted-price auction, if enough "under-priced" items remain unsold, then the allocation supports reasonably large welfare, *no matter what items buyers already possess before the auction.* This can be formalized as the following lemma.

Lemma 12. *Given a set of buyers $\mathcal{C}$ with valuations $\vec{v}$, suppose buyer $i \in \mathcal{C}$ already has items S_i^0. Consider a posted-price auction that is run with items U and prices p_j for $j \in U$ as input and after the auction, buyer i has items $S_i^0 \cup S_i^1$.*

Let OPT *be an allocation maximizing the welfare* $\sum_{i\in\mathcal{C}} v_i(\mathrm{OPT}_i \cup S_i^0)$*. Suppose* $\{q_j\}_{j\in U}$ *satisfy: for any* i *and* $T' \subseteq \mathrm{OPT}_i$, $\sum_{j\in T'} q_j \le v_i(T')$.

Let $T \subseteq U$ *be a set of items satisfying:* T *is not sold in the auction, and for any* $j \in T$, $p_j \le \frac{1}{2}q_j$*, then*

$$\sum_{i\in\mathcal{C}} v_i(S_i^0 \cup S_i^1) \ge \frac{1}{4}\sum_{j\in T} q_j.$$

Before proving the lemma, we briefly discuss the offline counterpart of Lemma 12 and the connection between them. $\{q_j\}$ in the lemma can be viewed as supporting prices for OPT, and T is the unsold set of items whose prices are sufficiently smaller compared to the supporting prices. In the offline environment, when the posted-price auction happens, no buyer has any item, i.e., $S_i^0 = \emptyset$. In such cases, it is easy to show that the outcome of the auction satisfies

$$\sum_{i\in\mathcal{C}} v_i(S_i^0 \cup S_i^1) \ge \frac{1}{2}\sum_{j\in T} q_j.$$

The intuition is that the unsold items provided an option for all buyers, which would guarantee each buyer some surplus (i.e., value minus payment). The buyers, however, did not choose this option, so it must be the case that the buyers chose something more desirable, which gave them only larger surplus. The above lemma essentially says, even if the buyers already have some items before the auction, this bound can only be worse by a factor of 2.

Proof of Lemma 12. For each $i \in \mathcal{C}$, we show

$$v_i(S_i^0 \cup S_i^1) \ge \frac{1}{4}\sum_{j\in T\cap \mathrm{OPT}_i} q_j.$$

The lemma then follows by summing over i. By purchasing $T \cap \mathrm{OPT}_i$ instead of S_i^1, the marginal utility of i is at least

$$\sum_{j\in T\cap \mathrm{OPT}_i} (q_j - p_j) - v_i(S_i^0) \ge \sum_{j\in T\cap \mathrm{OPT}_i} \frac{1}{2}q_j - v_i(S_i^0),$$

which lower bounds i's value $v_i(S_i^0 \cup S_i^1)$. On the other hand, by monotonicity, i's value is at least $v_i(S_i^0)$. Putting the two bounds together,

$$\begin{aligned} v_i(S_i^0 \cup S_i^1) &\ge \max\left\{\sum_{j\in T\cap \mathrm{OPT}_i} \frac{1}{2}q_j - v_i(S_i^0), v_i(S_i^0)\right\} \\ &\ge \frac{1}{2}\sum_{j\in T\cap \mathrm{OPT}_i} \frac{1}{2}q_j - v_i(S_i^0) + \frac{1}{2}v_i(S_i^0) \\ &= \frac{1}{4}\sum_{j\in T\cap \mathrm{OPT}_i} q_j. \end{aligned}$$

□

On may check that given the above lemma, the entire arguments in [6] and [1] remain valid even with free items dispensed beforehand.