An Improved Prophet Inequality for Combinatorial Welfare Maximization with Subadditive Agents

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Abstract
We give an $O(\log m / \log \log m)$-competitive prophet inequality for combinatorial welfare maximization with subadditive agents, improving over the $O(\log m)$ upper bound. Our policy is computationally efficient given sample access to the prior and demand queries. As a byproduct, we show that essentially the same policy is $O(1)$-competitive for combinatorial welfare maximization with submodular or fractionally subadditive (a.k.a. XOS) agents.

1 Introduction

Prophet inequalities are a classical topic in stopping theory. The problem is neat and natural: an agent plays a game, where there are $n$ boxes, each containing a reward (e.g., some amount of cash). The agent cannot see through the boxes to know precisely the amounts of cash inside each box. However, she has the prior knowledge that the amounts are drawn independently for each box, and fortunately, knows the distributions according to which the amounts are drawn. Now nature opens the boxes one by one. Upon seeing the inside of each box, the agent gets to make a choice: she can either (1) take the cash in the box and leave, or (2) let the current box expire (which means she gains nothing from the current box and it disappears), in which case the game proceeds with the remaining boxes. What is the maximum expected amount of cash the agent can get, and how to achieve that?

Quite surprisingly, the agent can guarantee half the amount of reward that a prophet is able to get, who sees through the boxes and therefore always picks the box with the largest reward [20, 21]. Moreover, the agent can achieve this by executing a simple threshold-based policy: accept the first box containing a reward exceeding a pre-calculated amount. The existence of such 2-approximate policies lead to the name “prophet inequalities”.

Prophet inequalities were recently rediscovered in computer science and economics. Since then, they have been drawing increasing interest in both fields. Hajiaghayi et al. [18] observe the connection between prophet inequalities and a pricing problem in auctions. They formulate the problem in the following equivalent way: A seller has an indivisible item to sell. $n$ buyers arrive one by one, each of which has a value for the item, drawn independently from a distribution known to the seller. When a buyer arrives, the seller learns the value of the buyer (or otherwise negotiates with the buyer), and decides whether to sell the item to the buyer. The buyer then leaves forever (with the item if sold to the buyer). The goal of the seller is to maximize the utility of the buyer who receives the item, or just the total utility (i.e., the welfare) of all buyers, since all other buyers have utility 0. Here, each
buyer corresponds to a box in the classical formulation, and the value of the buyer represents the reward in the box. A threshold-based policy can then be translated directly into a take-it-or-leave-it offer — a buyer receives the item (i.e., she buys) iff her value exceeds the pre-calculated price, and so buying is preferred to not buying.

Given this connection, various forms of auctions have been considered in the prophet inequality context (see, e.g., the recent survey by Lucier [22]). Examples include (1) the case where the seller has \( k \) identical items to sell and each buyer wants only one of them (or equivalently, up to \( k \) boxes can be accepted) [18], (2) the setting with a knapsack style constraint, requiring that the total “weight” of sellers who get an item cannot exceed 1 [17], and (3) the setting where \( m \) possibly distinct items are available for sale, and each agent has a \( \text{combinatorial} \) (as opposed to additive) valuation function assigning every subset of items a value [16]. The third setting, known as \( \text{combinatorial welfare maximization} \), appears particularly interesting and general, as it nicely captures the potentially complex interaction between items. For instance, a Coke and a Pepsi substitute each other, in the sense that having a Coke or a Pepsi probably gives one roughly the same utility (say 1), whereas having both likely gives \( \text{strictly less} \) utility than the sum, 2, of the former values, since one can only drink so much at a time.

In this paper, we consider combinatorial welfare maximization in the prophet inequality context. We focus on \( \text{subadditive} \) (also known as \( \text{complement-free} \)) agents, who regard items only as substitutes but not complements to each other. This is a standard assumption, as when agents exhibit complementarity, strong lower bounds\(^1\) rule out the possibility of any interesting policy\(^2\).

### 1.1 Our Contribution

**Current landscape of the problem.** Feldman et al. [16] were the first to study combinatorial welfare maximization with rich valuations in the prophet inequality context. They give an existential 2-approximate policy and a computationally efficient \( (2e)/(e - 1) \)-approximate policy when agents are submodular or fractionally subadditive (which are strict subclasses of subadditive valuations). Dütting et al. [11] propose a powerful framework, unifying a number of prophet inequalities, and yielding a computationally efficient 2-approximate policy for the same class of valuations, which is optimal given a lower bound inherited from the single-item setting. These bounds, through a standard approximation result, extend directly to subadditive agents with a loss of factor \( O(\log m) \), where \( m \) is the number of items. The best known lower bound for subadditive agents, however, is again 2, leaving a huge gap in between. This gap, as acknowledged by Feldman et al. [16] and Dütting et al. [11], raises a curious question:

\[
\text{Can we do better than } O(\log m) \text{ for subadditive agents?}
\]

**The logarithmic barrier for subadditive agents.** The above question did not have an immediate answer. For subadditive agents, all existing results essentially build on the same argument: one first computes prices that \( O(\log m) \)-approximately “support” the optimal allocation. Such prices have the property, that if one posts the prices on the items, and let agents, one by one in some

\(^1\)E.g., Nisan [23] shows it is impossible to achieve a better ratio than \( \Omega(m^{1/2}) \), even in a much stronger query model, without any randomness.

\(^2\)Sometimes, by enforcing special structures on complements, it is possible to achieve nontrivial upper bounds for certain classes of valuations with complementarity (see, e.g., [13] [16] [11]). However, we do not consider this in the current paper.
arbitrary order, purchase their utility-maximizing bundles of items, then the resulting allocation $O(\log m)$-approximately maximizes the expected welfare. With these prices, one can implement an $O(\log m)$-approximate threshold-based policy, by running a posted-price auction with the supporting prices for each arriving agent, and allocating the set of items purchased to the agent. The bottleneck of this approach is that the $O(\log m)$ factor of approximate supporting prices is tight: there are subadditive valuations for which no $o(\log m)$-approximate supporting prices exist [2]. Therefore, any policy relying on supporting prices (including all currently existing results) cannot possibly give better ratios than $O(\log m)$. In fact, given the tightness of this $O(\log m)$ factor, one may even suspect that the right ratio for subadditive agents is precisely $\Theta(\log m)$. We show that this is not the case.

Our result. We give an $O(\log m / \log \log m)$-approximate prophet inequality for combinatorial welfare maximization with subadditive agents, breaking the foregoing logarithmic barrier. Our protocol is computationally efficient given: (1) sample access to the prior distributions, and (2) demand queries to the sample valuations. Unlike previous results, our policy is not based on pricing items via approximately supporting prices and running sequential auctions — which enables the policy to bypass the obstacle discussed above. As a byproduct, we show that our policy, with minor modification, is also $O(1)$-approximately optimal for submodular or fractionally subadditive agents. Our approach provides an alternative view of combinatorial welfare maximization in the prophet inequality context, which may be of independent interest.

Technical overview. Our policy works by rounding a standard LP relaxation of the welfare maximization problem. We show that an online rounding and tie-breaking procedure preserves in expectation a decent fraction of the welfare for all agents simultaneously. The main difficulty is to deal with incomplete information, since when handling one agent, the policy does not know the actual valuation of any agent yet to arrive. As a result, it is impossible to round online the solution to the LP with respect to all agents’ actual valuations. To this end, we create a partially fictitious LP for each agent, by chaining this agent’s actual valuation (which becomes available to the policy upon the agent’s arrival) with independently drawn dummy valuations for all other agents. By doing this, each agent is intuitively playing against the average case configuration of all other agents. Moreover, in each agent’s fictitious LP, her share of the fractional allocation gives her precisely the expected utility she would get in the actual optimal fractional allocation. So if we can round the fictitious LP solutions with mild loss, the resulting welfare will in fact be approximately optimal. To achieve this, the key observation is that by rounding the solutions to the fictitious LPs, with high probability each item is demanded by $O(\log m / \log \log m)$ agents. Conditioned on this event, we break ties online and (roughly speaking) allocate each item uniformly at random to one of the agents who demand it. Now since each agent receives each item in her demand set independently with probability $\Omega(\log \log m / \log m)$, subadditivity ensures that she gets an $\Omega(\log \log m / \log m)$ fraction of the value of her demand set. The competitive ratio then follows.

1.2 Additional Related Work

Their setting, while also being combinatorial, is different from combinatorial welfare maximization considered in this paper.

Another line of research consider revenue maximization in the prophet inequality context. We list a few results here. Blumrosen and Holenstein [3] give a constant factor policy in the single-item setting. When agents are unit-demand — that is, they only want a single item — Chawla et al. [5] give constant factor posted-price policies. Cai and Zhao [4] study truthful policies for combinatorial auctions with subadditive agents. Their goal, however, is to maximize the revenue of the policy, instead of the welfare as we consider.

Prior to Feldman et al., Chawla et al. [6] and Alaei [1] consider welfare maximization with unit-demand agents. Cohen-Addad et al. [7] further show that dynamic pricing achieves optimal welfare for unit-demand agents. Ehsani et al. [13] show that the ratio improves to $\frac{e}{e-1}$ for combinatorial welfare maximization with submodular or fractionally subadditive agents, if agents arrive in a random order.

2 Preliminaries

Notation. Throughout the paper, we use $n$ to denote the number of agents, and $m$ the number of items. In general, we use $i$ as the index of an agent, and $j$ the index of an item.

Combinatorial valuations. A combinatorial valuation function $f : 2^{[m]} \to \mathbb{R}^+$ maps any subset $S$ of the ground set $[m] = \{1, 2, \ldots, m\}$ to a nonnegative real number $f(S)$. In this paper, we consider valuation functions that are:

- monotone: $f$ is monotone iff for any $S \subseteq T \subseteq [m]$, $f(S) \leq f(T)$, and
- subadditive: $f$ is subadditive iff for any $S, T \subseteq [m]$, $f(S) + f(T) \geq f(S \cup T)$.

The following subclasses of valuation functions are also useful:

- additive valuations: $f$ is additive iff for any disjoint $S, T \subseteq [m]$, $f(S) + f(T) = f(S \cup T)$.
- submodular valuations: $f$ is submodular iff for any $S, T \subseteq [m]$, $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$.
- fractionally subadditive (or XOS) valuations: $f$ is fractionally subadditive iff there exist additive valuations $c^1, \ldots, c^\ell$, such that for any $S \subseteq [m]$, $f(S) = \max_{k \in [\ell]} c^k(S)$. Each such additive valuation $c^k$ is called a clause.

It is known that every additive valuation is submodular, every submodular valuation is fractionally subadditive, and every fractionally subadditive function is subadditive. While the definitions of these classes may appear somewhat confusing, in this paper, we only utilize certain properties of the respective classes in a blackbox manner, to a minimum extent.

Problem formulation. We formulate the problem in the following way: There are $n$ agents and $m$ items, and a prior $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$ for the valuations of the agents over the items. All agents are monotone and subadditive (resp., fractionally subadditive), i.e., for any agent $i$, any valuation function $f_i$ in the support of $\mathcal{F}_i$ is monotone and subadditive (resp., fractionally subadditive). Agents arrive one by one in an adversarial order. When agent $i$ arrives, we see the realization $f_i \sim \mathcal{F}_i$ of her
valuation (through query oracles discussed below), and must allocate irrevocably some items to the agent. The agent then takes the items and departs, and all items allocated to the agent becomes unavailable to subsequently arriving agents.

The goal is to maximize the expected (over the realization of the valuations and the randomness of the policy) welfare of all agents. In particular, we wish to compete against the offline optimal allocation (i.e., the prophet), which has unlimited computational power, and knows beforehand the realization of all agents’ valuations \( \{f_i\}_i \). The competitive ratio is defined to be the ratio between the expected welfare (denoted \( \text{OPT} \)) of the offline optimal allocation\(^3\) and the expected welfare produced by the online policy.

**Oracle access to valuation functions.** The representation of a combinatorial valuation function may be exponentially large in the number of items \( m \). Given this complexity, it is standard to assume that the policy may access valuation functions only through query oracles. In particular, the following two types of queries are commonly allowed (see, e.g., [14, 16, 11]):

- **value** queries: given a valuation function \( f \) and a set \( S \), return the value of \( S \), \( f(S) \).
- **demand** queries: given a valuation function \( f \) and prices \( \{p_j\}_{j \in [m]} \), return a utility-maximizing set (i.e., a demand set) with respect to \( f \) under the given prices. That is, the query returns a set \( S \) that maximizes \( f(S) - \sum_{j \in S} p_j \).

In this paper, we assume the policy has access to both kinds of queries.

**The welfare maximizing LP.** The following LP relaxation of the welfare maximization problem has been considered in [10, 24, 14]:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i \in [n]} \sum_{S \subseteq [m]} x_{i,S} f_i(S) \\
\text{s.t.} & \quad \sum_{S \subseteq [m]} x_{i,S} \leq 1 \quad \forall i \in [n] \\
& \quad \sum_{i \in [n], S \subseteq [m], j \in S} x_{i,S} \leq 1 \quad \forall j \in [m] \\
& \quad x_{i,S} \geq 0 \quad \forall i \in [n], S \subseteq [m].
\end{align*}
\]

One may interpret the LP in the following way: \( x_{i,S} \) stands for the probability that agent \( i \) receives bundle \( S \). The objective is therefore the total expected value of all agents, which is the expected welfare. The first constraint requires that each agent \( i \) receives at most 1 bundle, and the second requires that each item goes to at most 1 agent, in expectation.

It is known (see, e.g., [10, 24]) that the above LP can be solved with polynomially many value and demand queries to \( f_1, \ldots, f_n \). Let \( x_{i,S}(\{f'_\nu\}_\nu) \) (parameters omitted when clear from the context) denote the value of variable \( x_{i,S} \) in the optimal solution to the LP with respect to valuation functions \( \{f'_\nu\}_\nu \). Given \( \mathcal{F} \), denote the expected utility of agent \( i \) in this optimal solution by

\[
\text{LP}_i := \mathbb{E}_{\{f'_\nu\}_\nu \sim \mathcal{F}} \left[ \sum_{S \subseteq [m]} x_{i,S}(\{f'_\nu\}_\nu) \cdot f_i(S) \right],
\]

\(^3\)Note that the offline optimal allocation may be hard to find, computationally and / or information theoretically.

\(^4\)If there are multiple such solutions, let \( \{x_{i,S}(\{f'_\nu\}_\nu)\}_{i,S} \) be the one produced by the efficient LP solving algorithm which we use as a subroutine of the policy.
and the optimal objective value by
\[
\text{LP} := \sum_{i \in [n]} \text{LP}_i = \mathbb{E}_{\{f_i\}_i \sim \mathcal{F}} \left[ \sum_{i \in [n], S \subseteq [m]} x_{i,S}(\{f_i\}_i) \cdot f_i(S) \right].
\]
Note that for any prior \( \mathcal{F} \), we always have \( \text{LP} \geq \text{OPT} \). We will use the welfare maximizing LP and its solution as a building block of our policy in a blackbox manner.

3 The Policy

In this section, we present our policy for subadditive agents, which we later adapt to work for submodular or fractionally subadditive agents. The policy works for any arrival order of agents, but for ease of presentation, we assume agents arrive according to their indices. That is, agent 1 arrives first, followed by agent 2, etc.

We also let \( f_{-i} := (f_1, \ldots, f_{i-1}, f_{i+1}, f_n) \) (i.e., \( f_{-i} \) denotes the valuations of all agents but \( i \)) and use \( g_{-i} \) similarly. Below is our policy:

1. For each item \( j \), let counter \( c_j = 0 \), and generate \( r_j \) uniformly at random from \( \{1, \ldots, 100 \log m / \log \log m\} \).

2. Upon agent \( i \)'s arrival:
   
   (a) Let \( f_i \) be agent \( i \)'s realized valuation; draw dummy valuations \( g_{-i} \sim \mathcal{F}_{-i} \) for all other agents.\(^5\)
   
   (b) Solve the welfare maximizing LP with valuations \( (f_i, g_{-i}) \), and let \( \{x_{i,S}(f_i, g_{-i})\}_i \) be the solution.
   
   (c) Now view \( \{x_{i,S}\}_i \) as a distribution over sets of items.\(^6\) Draw set \( S_i \sim \{x_{i,S}\}_i \) from this distribution, where for any \( S \), \( \Pr[S_i = S] = x_{i,S} \). We say agent \( i \) demands set \( S_i \).
   
   (d) For each item \( j \in S_i \), let \( c_j \leftarrow c_j + 1 \) (i.e., increase the counter for item \( j \)), and give agent \( i \) item \( j \) iff \( c_j = r_j \) after the update (i.e., iff agent \( i \) is chronologically the \( r_j \)-th agent demanding item \( j \)).

4 The Competitive Ratio for Subadditive Agents

In this section, we bound the competitive ratio of the above policy when agents are subadditive. Formally, we show:

**Theorem 1.** When agents are subadditive, the above policy guarantees each agent \( i \) expected utility
\[
\Omega \left( \frac{\log \log m}{\log m} \cdot \text{LP}_i \right).
\]

**Proof.** We first consider the unconditional distribution of the set \( S_i \) demanded by any agent \( i \). Let \( y_{i,S} \) be the probability (over \( f_i, g_{-i}, \) and the random bits of the policy) that agent \( i \) demands set \( S \). That is,
\[
y_{i,S} := \Pr_{f_i \sim \mathcal{F}_i, g_{-i} \sim \mathcal{F}_{-i}}[S_i = S].
\]

\(^5\) Note that we abuse notation here, so for any \( i_1 \neq i_2 \), \( g_{-i_1} \) and \( g_{-i_2} \) are independent — they are not different parts of a same group of valuations.

\(^6\) It is possible that \( \sum_S x_{i,S} < 1 \), in which case with probability \( 1 - \sum_S x_{i,S} \), \( S_i = \emptyset \).
Note that $y_{i,S}$ is not random, and depends only on the prior $\mathcal{F}$. We first show that $\{y_{i,S}\}_{i,S}$ form a feasible solution to the welfare maximizing LP, regardless of the actual valuations of agents. This is well-defined, since fixing $n$ and $m$, the precise values of sets do not appear in any constraint of the welfare maximizing LP.

**Lemma 1.** $\{y_{i,S}\}_{i,S}$ satisfy:

- for any agent $i \in [n]$, $\sum_{S \subseteq [m]} y_{i,S} \leq 1$,
- for any item $j \in [m]$, $\sum_{i \in [n], S \subseteq [m], j \in S} y_{i,S} \leq 1$, and
- for any $i \in [n]$, $S \subseteq [m]$, $y_{i,S} \geq 0$.

**Proof.** Observe that according to the policy, for any $i$, $S$
\[ y_{i,S} = \mathbb{E}_{f_i \sim F, g_i \sim F - i} [x_{i,S}(f_i, g_{-i})] = \mathbb{E}_{(f_{i'}, g_{i'}) \sim F} [x_{i,S}({\{f_{i'}\}_{i'}})]. \]

In other words, $y_{i,S}$ is the expected value of variable $x_{i,S}$ in the optimal solution to the welfare maximizing LP, when valuations are distributed according to prior $F$. Now since for any realization of $\{f_{i'}\}_{i'}$, $\{x_{i,S}({\{f_{i'}\}_{i'}})\}_{i,S}$ satisfy the LP constraints, it follows from linearity of expectation that the expected values $\{y_{i,S}\}_{i,S}$ also satisfy the constraints. The lemma follows.

For any agent $i$ and item $j$, let $p^j_i$ be the probability that item $j$ is demanded by agent $i$. That is,
\[ p^j_i := \Pr_{f_i, g_{-i}} [j \in S_i] = \sum_{S : j \in S} y_{i,S}. \]

Feasibility of $y_{i,S}$ (Lemma 1) implies: for any $j \in [m]$,
\[ \sum_{i \in [n]} p^j_i = \sum_{i \in [n], S : j \in S} y_{i,S} \leq 1. \]

In other words, throughout the execution of the policy, no item is demanded by more than 1 agent in expectation.

Now fix agent $i$. We show that when $i$ arrives, with high probability (and independently of $f_i$, $g_{-i}$ and $\{r_j\}_{j}$) all items have been demanded by strictly less than $100 \log m / \log \log m$ agents. Recall the following fact about independent Bernoulli variables (see, e.g., [14]):

**Lemma 2.** For any $n \in \mathbb{Z}^+$, independent Bernoulli random variables $X_1, \ldots, X_n$ where $\mathbb{E} \left[ \sum_{i \in [n]} X_i \right] \leq 1$, and any $k \in \mathbb{Z}^+$,
\[ \Pr \left[ \sum_{i} X_i \geq k \right] \leq \frac{1}{k!} \leq (e/k)^k. \]

The lemma says, that if the sum of independent Bernoulli variables in expectation does not exceed 1, then the tail of this sum decays factorially fast. Now observe that $S_1, \ldots, S_{i-1}$, and therefore $\mathbb{I} [j \in S_1], \ldots, \mathbb{I} [j \in S_{i-1}]$ (where $\mathbb{I} [j \in S_{i'}]$ is the indicator random variable for the event $j \in S_{i'}$), are independent. This is because for any $i'$, $S_{i'}$ depends only on $f_{i'}$ and $g_{-i'}$. Also, since
\[ \mathbb{E} \left[ \sum_{i' < i} \mathbb{I} [j \in S_{i'}] \right] = \sum_{i' < i} \mathbb{P}[j \in S_{i'}] = \sum_{i' < i} p^j_{i'} \leq \sum_{i' \in [n]} p^j_{i'} \leq 1, \]
random variables \( \{ \mathbb{I}[j \in S_{i'}] \}_{i' < i} \) satisfy the conditions of Lemma 2. So for large enough \( m \), setting \( k = 100 \log m / \log \log m \), Lemma 2 implies that

\[
\Pr \left[ \sum_{i' < i} \mathbb{I}[j \in S_{i'}] < 100 \log m / \log \log m \right] \geq 1 - m^{-2}.
\]

In other words, with high probability, item \( j \) is demanded by less than \( 100 \log m / \log \log m \) agents before \( i \). Taking a union bound over the \( m \) items, with probability \( 1 - 1/m \), all items are demanded by less than \( 100 \log m / \log \log m \) agents before \( i \). Let this event be \( \mathcal{E}_i \).

Now we show that conditioned on this, \( i \)'s expected utility (over \( f_i, g_{-i}, \{ r_j \}_{j} \) and the random bits of the policy) is at least \( \frac{\text{LP}_i \log \log m}{100 \log m} \). The first step is to show that the expected value of \( S_i \) to agent \( i \) is at least \( \text{LP}_i \). By the optimality of \( \{ x_i, S_i(f_i, g_{-i}) \}_{i} \),

\[
\mathbb{E}_{f_i \sim F_i, g_{-i} \sim F_{-i}}[f_i(S_i)] = \mathbb{E}_{f_i \sim F_i, g_{-i} \sim F_{-i}} \left[ \sum_{S} x_i, S(f_i, g_{-i}) : f_i(S) \right] = \mathbb{E}_{\{ f_i', g_{-i}' \} \sim F} \left[ \sum_{S} x_i, S(f_i') : f_i(S) \right] = \text{LP}_i.
\]

Given that the expected value of \( S_i \) is at least \( \text{LP}_i \), we only need to round this fractional allocation while preserving a good fraction of its value. Conditioned on \( \mathcal{E}_i \), observe the following two facts:

- For any item \( j \), agent \( i \) receives \( j \) iff \( i \) is the \( r_j \)-th to demand \( j \). This happens with probability \( \log \log m / 100 \log m \).
- Whether agent \( i \) receives item \( j \) depends only on \( r_j \), which means the events that \( i \) receives each item are independent.

Recall the following fact about subadditive valuations (see, e.g., [14]):

**Lemma 3.** For subadditive \( f \) and any set of items \( S \), let \( T \) be such that for any \( j \in S \), \( j \in T \) independently with probability at least \( p \). Then

\[
\mathbb{E}[f(T)] \geq pf(S).
\]

So let \( T_i \) be the set of items \( i \) actually gets. Given the above two facts, conditioned on \( \mathcal{E}_i \), \( S_i \) and \( T_i \) satisfy the condition of Lemma 3 and we have

\[
\mathbb{E}[f_i(T_i) \mid \mathcal{E}_i] \geq \frac{\log \log m}{100 \log m} \cdot \mathbb{E}[f_i(S_i) \mid \mathcal{E}_i].
\]

Moreover, observe that \( f_i(S_i) \) depends only on \( f_i, g_{-i} \) and the random bits of the policy at agent \( i \). On the other hand, event \( \mathcal{E}_i \) depends only on \( \{ (f_{i'}, g_{-i'}) \}_{i' < i} \) and the random bits of the policy before agent \( i \). As a consequence, \( f_i(S_i) \) is independent of \( \mathcal{E}_i \). So \( \mathbb{E}[f_i(S_i) \mid \mathcal{E}_i] = \mathbb{E}[f_i(S_i)] \), and

\[
\mathbb{E}[f_i(T_i) \mid \mathcal{E}_i] \geq \frac{\log \log m}{100 \log m} \cdot \mathbb{E}[f_i(S_i) \mid \mathcal{E}_i] = \frac{\log \log m}{100 \log m} \cdot \mathbb{E}[f_i(S_i)] = \frac{\log \log m \cdot \text{LP}_i}{100 \log m}.
\]

Now since \( \Pr[\mathcal{E}_i] \geq 1 - 1/m \), clearly

\[
\mathbb{E}[f_i(T_i)] \geq \left( 1 - \frac{1}{m} \right) \cdot \frac{\log \log m \cdot \text{LP}_i}{100 \log m} = \Omega \left( \frac{\log \log m}{\log m} \cdot \text{LP}_i \right).
\]

\[\square\]
Given Theorem 1, the competitive ratio of the policy follows simply by summing over all agents: the expected welfare guaranteed by the policy is at least

$$\Omega\left(\sum_{i \in [n]} \frac{\log \log m}{\log m} \cdot \text{LP}_i\right) = \Omega\left(\frac{\log \log m}{\log m} \cdot \text{LP}\right) = \Omega\left(\frac{\log \log m}{\log m} \cdot \text{OPT}\right).$$

5 The Competitive Ratio for Fractionally Subadditive Agents

In this section, we adapt our policy to work for fractionally subadditive agents, and show that the competitive ratio improves to 4. All results in this section also apply to submodular agents, since every submodular valuation is fractionally subadditive.

The only modification we make is in step 1 of the policy: for fractionally subadditive agents, we draw $r_j$ uniformly at random from $\{1, 2\}$ instead of $\{1, \ldots, 100 \log m / \log \log m\}$. The intuition behind this is that fractionally subadditive valuations have stronger properties, such that we do not need items in agent $i$’s demand set $S_i$ to be assigned independently to $i$. Instead, as long as each item in $S_i$ goes to $i$ with a reasonable probability (possibly with arbitrary correlation), the resulting bundle will preserve a proportionally large fraction of the value of $S_i$. For the latter condition to hold, we no longer need to take a union bound over all items, which allows the probability of individual bad events to be constant instead of $1/m^2$. As a result, it suffices to break ties among a constant number of agents, which improves the competitive ratio to a constant.

We now bound the competitive ratio. We prove:

**Theorem 2.** When agents are fractionally subadditive, the modified policy guarantees each agent $i$ expected utility $\text{LP}_i/4$.

**Proof.** Observe that the only place where properties of subadditive valuations are used is Lemma 3. We first state an alternative and stronger fact for fractionally subadditive valuations (see, e.g., [14]), which replaces Lemma 3 in this proof.

**Lemma 4.** For fractionally subadditive $f$ and any set of items $S$, let $T$ be such that for any $j \in S$, $j \in T$ with probability at least $p$. Then

$$\mathbb{E}[f(T)] \geq pf(S).$$

The only difference between Lemmas 3 and 4 is that the latter does not require items to be in $T$ independently.

Recall that $y_{i,S} = \Pr[S_i = S]$ and $p_{i}^{j} = \Pr[j \in S_i]$. Since Lemma 1 still holds, for any $j \in [m],$

$$\sum_{i \in [n]} p_{i}^{j} \leq 1.$$

And still, for any agent $i \in [n],$

$$\mathbb{E}[f_i(S_i)] = \text{LP}_i.$$

We now show that conditioned on the choice of $S_i$, each item in $S_i$ ends up in $T_i$, the bundle agent $i$ actually receives, with probability at least $1/4$. Given this, the theorem follows immediately.

Consider agent $i$, and condition on the set $S_i$ demanded by $i$. Given Lemma 1, we only need to show that each item $j \in S_i$ is in $T_i$ with probability at least $1/4$. We break this into two events:
(1) $j$ is demanded by at most 1 agent before $i$, and (2) conditioned on the previous event, $i$ is the $r_j$-th agent demanding $j$. Note that both events are independent of $S_i$. For (1), consider variables \{$I[j \in S_{i'}] \mid i' < i$\}. By the Markov bound,
\[
\Pr \left[ \sum_{i' < i} I[j \in S_{i'}] \geq 2 \right] \leq \frac{1}{2} \mathbb{E} \left[ \sum_{i' < i} I[j \in S_{i'}] \right] = \frac{1}{2} \sum_{i' < i} p^j_{i'} \leq 1/2.
\]
In other words, (1) happens with probability at least 1/2. Now conditioned on (1), clearly (2) happens with probability 1/2, which means $j \in T_i$ with probability at least 1/4. This concludes the proof.

Given Theorem 2, the claimed competitive ratio of the modified policy again follows by summing over all agents.

6 Open Problems

A number of intriguing questions remain open. We name a few here:

- In the prophet inequality context, what is the (asymptotically) right competitive ratio for combinatorial welfare maximization with subadditive agents? In this paper, we improve the upper bound from $O(\log m)$ to $O(\log m / \log \log m)$, but the best known lower bound remains 2. Better bounds from either side would be interesting.

- It is open whether all prophet inequalities can be implemented using posted prices (see, e.g., [11]). While some state-of-the-art upper bounds (like the one presented in this paper) are not based on posted prices, there is no known separation between general policies and pricing-based ones. For combinatorial welfare maximization particularly, one may also ask whether general policies are more powerful than (static, anonymous) item-pricing schemes. We note that given our result and the $O(\log m)$ approximation factor for supporting prices, combinatorial welfare maximization with subadditive agents look like a promising candidate for such a separation.

- The logarithmic barrier for subadditive valuations discussed in this paper also exists for some other problems, including truthful combinatorial auctions [9], revenue maximization with multiple subadditive agents [4], the price-of-anarchy of simple auctions with subadditive agents [8], and combinatorial prophet inequalities with subadditive objectives [25]. It would be interesting if one could beat $O(\log m)$ for any of these problems.

References


\[7\text{Here, } m \text{ is the cardinality of the ground set of the subadditive set function in the respective problem.}\]


