**Research Problem**

The Maximum Acyclic Subgraph Problem

**Problem 4-1.** Consider the maximum acyclic subgraph problem discussed in Homework 2-1. Refer to that homework for several simple 0.5 approximations.

We can formulate linear ordering as a set of integer constraints. Let \( x_{ij} \) be set to 1 if \( i \) appears before \( j \) in the ordering and 0 otherwise. Show that the following set of constraints encode a valid linear ordering:

\[
\begin{align*}
    x_{ij} + x_{ji} & = 1 \quad \forall i, j \\
    x_{ij} + x_{jk} + x_{ki} & \leq 2 \quad \forall i, j, k \\
    x_{ij} & \in \{0, 1\} \quad \forall i, j
\end{align*}
\]

Let \( c_{ij} = 1 \) if there is an edge from \( i \) to \( j \) in \( G(V, E) \), and 0 otherwise. The objective function is therefore to maximize \( \sum_{i,j} c_{ij} x_{ij} \). The linear relaxation of the above integer program has an integrality gap of 0.5. For a proof, refer [1].

We can consider semidefinite relaxations for this problem. Let \( \mathbf{e} \) denote an arbitrary unit vector. Let \( \mathbf{x}_{ij} \) be a vector for every ordered pair \((i, j)\) in the graph. The goal of the SDP is to set \( \mathbf{x}_{ij} = \mathbf{e} \) if \( i \) comes before \( j \) in the ordering and to \(-\mathbf{e}\) otherwise. Show that the following is a valid SDP relaxation for this problem (note that \( \mathbf{e} \) is a variable and not a constant):

\[
\text{Maximize} \sum_{i,j} c_{ij} \frac{\mathbf{x}_{ij} \cdot \mathbf{e} + 1}{2}
\]

\[
\begin{align*}
    \| \mathbf{x}_{ij} + \mathbf{x}_{ji} \| & = 0 \quad \forall i, j \\
    \| \mathbf{x}_{ij} + \mathbf{x}_{jk} + \mathbf{x}_{ki} \| & = 1 \quad \forall i, j, k \\
    \| \mathbf{x}_{ij} \| & = 1 \quad \forall i, j \\
    \| \mathbf{e} \| & = 1
\end{align*}
\]

What is the integrality gap of this SDP? Can you round it to a ratio of \( 0.5 + \epsilon \) for some \( \epsilon > 0 \)? For related work on semidefinite programming, please refer [2, 3, 4].
References


