Performance Modeling of Adaptive-optics Imaging Systems Using Fast Hankel Transforms

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ABSTRACT

Real-time adaptive-optics is a means for enhancing the resolution of ground based, optical telescopes beyond the limits previously imposed by the turbulent atmosphere. One approach for linear performance modeling of closed-loop adaptive-optics systems involves calculating very large covariance matrices whose components can be represented by sums of Hankel transform based integrals. In this paper we investigate approximate matrix factorizations of discretizations of such integrals. Two different approximate factorizations based upon representations of the underlying Bessel function are given, the first using a series representation due to Ellerbroek and the second an integral representation. The factorizations enable fast methods for both computing and applying the covariance matrices. For example, in the case of an equally spaced grid, it is shown that applying the approximated covariance matrix to a vector can be accomplished using the derived integral-based factorization involving a 2-D fast cosine transform and a 2-D separable fast multipole method. The total work is then $O(N \log N)$ where N is the dimension of the covariance matrix in contrast to the usual $O(N^2)$ matrix-vector multiplication complexity. Error bounds exist for the matrix factorizations. We provide some simple computations to illustrate the ideas developed in the paper.

Keywords: adaptive-optics, atmospheric imaging, phase covariances, fast Hankel transforms, fast multipole methods

1. INTRODUCTION

Atmospheric turbulence has been a limiting factor for imaging since telescopes were invented. Both atmospheric and telescope induced aberrations distort the spherical wavefront of arriving light. Adaptive optics is a scientific and engineering discipline whereby a distorted optical wavefront is compensated to correct for errors induced by the environment, e.g., a turbulent atmosphere, through which it passes. Adaptive optics systems include a wavefront sensor (WFS) to measure the effects of atmospheric turbulence upon telescope performance, a deformable mirror (DM) to actively (in real-time) compensate these effects, and a control algorithm to derive the DM adjustments from WFS measurements. (See the recent books by Roggemann an Welsh[1] and Tyson[2] for extensive details and references.)

We assume the following notation convention throughout this paper. Boldface characters such as $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ denote real vectors in the N-vector space R^N. We use $[\cdot]_{M\times N}$ to denote a $M \times N$ matrix. Alternatively, the notation $A(1:M, 1:N)$, or $A_{1:M,1:N}$, is used to specify that the rows of A are indexed from 1 to M and the columns are indexed from 1 to N.

Various adaptive-optics control systems have been proposed[2] and there is considerable interest in modeling their performances. Necessary inputs for many modeling approaches include the covariance matrices which describe the statistical relationships between the wavefront distortions to be corrected and the WFS measurements driving

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the control algorithm. The covariances that must be computed to evaluate and optimize adaptive-optics system performance are of the form

$$\langle c_1 c_2 \rangle \tag{1}$$

where the angle brackets $\langle \cdots \rangle$ denote ensemble averaging over time. The quantities c_1 and c_2 are two different instances of

$$c_i = \int w_i(\mathbf{r})\phi_i(\mathbf{r}, t_i)d\mathbf{r}, \qquad i = 1, 2 \tag{2}$$

which represent, for example, wavefront sensor measurements of turbulence-induced wavefront distortions.[2,4] The integration variable $\mathbf{r}$ denotes coordinates in the aperture plane of the telescope. The term $\phi_i(\mathbf{r}, t_i)$, usually referred to as the "wavefront", represents the turbulence-induced phase distortion for a wavefront propagating from a point source in or above the atmosphere to the telescope at time t_i. In atmospheric optics,[1] the phase quantifies the deviation of the wavefront from a reference planar wavefront. This deviation is caused by variations in the index of refraction (wave speed) along light ray paths, and is strongly dependent on air temperature. Because of turbulence, the phase varies with time and position in space and is often modeled as a stochastic process.

Ellerbroek,[3,4] has recently studied turbulence outer scale effects in zonal adaptive optics calculations, and provided calculation procedures.* When outer scale effects are considered, the covariances $\langle c_1 c_2 \rangle$ may be represented in the form

$$\begin{aligned} \langle c_1 c_2 \rangle \;=\; & 0.144 \left(\frac{L_0}{r_0}\right)^{\frac{5}{3}} \left[\int_0^\infty C_n^2(h)dh\right]^{-1} \int\int w_1(r)w_2(r')drdr' \int_0^H C_n^2(h)dh \\ & \times \int_0^\infty \frac{x}{(x^2+1)^{11/6}} \left[J_0\left(\frac{2\pi\Delta}{L_0}x\right) J_0\left(\frac{2\pi\delta}{L_0}x\right) - 1\right] dx, \end{aligned} \tag{3}$$

where J_0 is the zero-order Bessel function of the first kind, and the quantities Δ and δ are functions of r, r', and h. See Ellerbroek[4] for notation and further details. In the modeling process, the integral (3) is used to compute the covariance matrix associated with the phase ϕ of the incoming wavefront.†

The covariance matrices required for AO system performance analysis can be very large, and their components must be computed accurately and rapidly. Due to the fact that Δ and δ are functions of r, r', and h, the efficient and accurate evaluation of the inner most integral with respect to x becomes critical to the evaluation of the outer integrals in (3). Denote the *two-parameter Hankel transform*[7] of a function

$$f(x) = \frac{x}{(x^2+1)^{11/6}}$$

by

$$I(a, b) = \int_0^\infty f(x)J_0(ax)J_0(bx)dx \tag{4}$$

where a, b and x are variable nonnegative real numbers. For the inner integral with respect to x in (3),

$$a = \frac{2\pi\Delta}{L_0}, \quad b = \frac{2\pi\delta}{L_0}. \tag{5}$$

The two-parameter Hankel transform of f arises in adaptive-optics under the assumption that the direction of the atmospheric wind velocity is random and uniformly distributed.[4] Note that if instead, the direction of the atmospheric wind velocity is assumed to be known, then $I(a, b)$ becomes a function of a single parameter, $I(a)$, i.e. a single-parameter Hankel transform.

In practice, it is desired to efficiently evaluate (4) for N values of a and b, thus obtaining an $N \times N$ matrix

$$[I(\mathbf{a}, \mathbf{b})]_{N\times N}, \quad \mathbf{a}, \mathbf{b} \in R^N. \tag{6}$$

*Related work on the performance of multiple bandwith AO systems can be found in Ellerbroek et al.[5,6]

†Additional changes in the phase ϕ can occur after the light is collected by the primary mirror, e.g., when adaptive optics are applied. This involves mechanical corrections obtained with a deformable mirror to restore ϕ to planarity.

Matrix-vector products involving the matrix $[I(\mathbf{a},\mathbf{b})]_{N\times N}$ are core operations for the integral computation in (3) after appropriate discretization. Ellerbroek has developed power series formulas analytically expressing each element of $[I(\mathbf{a},\mathbf{b})]_{N\times N}$ using Mellin transforms.[7,8] His power series representation leads to a feasible computational procedure.

Our contributions in this paper are the following. First, we propose *approximate* matrix factorization representations of $[I(\mathbf{a},\mathbf{b})]_{N\times N}$. A matrix factorization formulation aids not only in revealing complexity and approximation accuracy (such as errors introduced by truncation and discretization), but also in identifying operations for which fast algorithms are known. A low rank matrix factorization may allow for efficient computation of matrix-vector products of the type required to evaluate (3). Furthermore, its compact form representation may also allow for storage savings. Second, we provide two approaches to derive such approximate factorizations. These factorizations are based on the representation of the underlying Bessel function J_0 in terms of i) a series representation due to Ellerbroek[4] and ii) an integral form derived in this paper. The series representation covers a class of functions whose Mellin transforms are represented using gamma function ratios. The integral representation approach can be used to a wider class of functions f.

Third, we provide computational procedures as well as efficiency and accuracy analyses. In particular, a factorization is identified for the special case that a and b are nodes on an equally spaced grid, so that computations involving the factorization can be executed efficiently via fast multipole algorithms developed by Greengard and Rokhlin.[9]

The rest of this paper is structured as follows. In §2.1 we introduce an approximate factorization of $[I(\mathbf{a},\mathbf{b})]_{N\times N}$ from the power series representation by Ellerbroek. This factorization aids in identifying key computational aspects of the numerical procedure. Specifically, the approximation obtained is of lower rank in some cases; we describe a hierarchical scheme that leads to fast matrix-vector computations for general cases. In §2.2 we present another factorization by using an integral representation of J_0 and numerical quadratures for the special case that a and b lie on an equally spaced grid. In particular, we introduce and extend the work of Kapur and Rokhlin[10] in using fast cosine transforms and fast multipole algorithms as computational kernels for fast evaluations of Hankel transforms. Among other advantages, the factorizations given in this paper lead to fast algorithms for computing the matrix-vector products involving $[I(\mathbf{a},\mathbf{b})]_{N\times N}$, with arithmetic complexity $O(N\log N)$, rather than $O(N^2)$ as in the usual approach. Some experimental results are presented in §3, and final remarks are given in §4.

2. MATRIX FACTORIZATION BASED REPRESENTATIONS

In this section we present two different approximate factorizations of the matrix (6). Our factorizations are based on i) a series representation of J_0 due to Ellerbroek[4] and ii) an integral representation of J_0.[10]

2.1. Series Expansion Approach

Ellerbroek's[4] scheme involves finding an infinite series for $I(a,b)$ from the inverse Mellin transform of $f(x)$, $J(ax)$, and $J(bx)$, and also clever manipulation and change of variables. The final formula for evaluating each entry $I(a_i,b_j)$ consists of three double series as follows

$$\begin{aligned}
I(a_i,b_j) &= \frac{3}{5}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{u^n}{(n!)^2}\frac{(n+m)!}{(1/6)_{n+m}}\frac{v^m}{(m!)^2} \\
&+\frac{\Gamma(-5/6)}{2\Gamma(11/6)}v^{5/6}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{u^n}{(n!)^2}\frac{(11/6)_{n+m}}{(n+m)!}\frac{v^m}{[(11/6)_m]^2} \\
&+\frac{\Gamma(-5/6)}{2\Gamma(11/6)}v^{5/6}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{u^n(11/6)_n}{n!}\frac{1}{[(n+m)!]^2}(u/v)^m[(-5/6)_m]^2
\end{aligned}$$

where $u=(\frac{a_i}{2})^2$ and $v=(\frac{b_j}{2})^2$. The above formula can be expressed in the following inner product form with a truncation error:

$$\begin{aligned}
I(a_i,b_j) \;=\;& c_1\,\mathrm{vand}_P^T(u_i)\,D_{f,P}^2 H_1 D_{f,P}^2\mathrm{vand}_P(v_j)\,, \\
&+c_2\,\mathrm{vand}_Q^T(u_i)\,H_2 D_{g,Q}^{-2}(11/6)\mathrm{vand}_Q(v_j)\,v_j^{5/6} \\
&+c_2\,\mathrm{vand}_K^T(u_i)\,D_{f,K}T_3 D_{g,K}^2(-5/6)\mathrm{vand}_K(v_j)\,v_j^{5/6}+\mathrm{r}(P,Q,K), \quad a_i\le b_j.
\end{aligned} \tag{7}$$

Here, $r(P,Q,K)$ is the remainder with the respective series truncated at the Pth, Qth and Kth orders in both variables, $c_1 = 3/5$, $c_2 = \dfrac{\Gamma(-5/6)}{2\Gamma(11/6)}$, and Γ is the gamma function.[11] The matrix factors are of the following structures,

- $\mathrm{vand}(z)_L = \begin{pmatrix} 1 & z & \cdots & z^{L-1} \end{pmatrix}^{\mathrm{T}}$ is a Vandermonde vector of length L with node $z \in \{u_i, v_j\}$, where $u_i = (a_i/2)^2$, $v_j = (b_j/2)^2$,
- $D_{f,L} = \mathrm{diag}(k!), \quad k = 0 : L-1$,
- $D_{g,L}(z) = \mathrm{diag}((z)_k), \quad k = 0 : L-1$, where $(z)_k = \dfrac{\Gamma(z+k)}{\Gamma(z)}$ is a Gamma ratio, and
- $H_1(k,l) = \dfrac{(k+l)!}{(1/6)_{k+l}}$, $k,l = 1 : P-1$, $\qquad H_2(k,l) = \dfrac{(11/6)_{k+l}}{(k+l)!}$, $k,l = 1 : Q-1$
- $T_3(k,l) = \dfrac{(11/6)_{k-l}}{(k-l)!}$, $k \geq l$, and $T_3(k,l) = 0, k < l$, $k,l = 1 : K-1$.

Notice that H_1 and H_2 are Hankel matrices and T_3 is a lower triangular Toeplitz matrix.

Denote by $V_L(\mathbf{z}(1:N))$, or simply $V(\mathbf{z})$ when the dimensions are clear from the context, the Vandermonde matrix composed of the Vandermonde vectors of length L with nodes $\mathbf{z}(j)$, i.e., $V(z_j) = \mathrm{vand}_L(z_j), j = 1 : N$. Let $\mathbf{a} = (a_0, \ldots, a_{N-1}), \mathbf{b} = (b_0, \ldots, b_{N-1}) \in R^N$. Then, (6) can be written in the form of an approximate matrix factorization

$$[I(\mathbf{a},\mathbf{b})]_{N\times N} \quad = \quad I_1 + I_2 + I_3 + R_{P,Q,K} \tag{8}$$

where

$$\begin{aligned} I_1 &= c_1 V_Q^T(\mathbf{u})\, D_{f,Q}^2 H_1 D_{f,Q}^2 V_Q(\mathbf{v}), \\ I_2 &= c_2 V_Q^T(\mathbf{u})\, H_2 D_{g,Q}^{-2}(11/6)\, V_Q(\mathbf{v})\, D_N^{5/6}(\mathbf{v}), \\ I_3 &= c_2 V_K^T(\mathbf{u})\, D_{f,K} H_3 D_{g,K}^2(-5/6) V_K(\mathbf{v})\, D_N^{5/6}(\mathbf{v}). \end{aligned}$$

Here, $D_N(\mathbf{z})$ is a diagonal matrix whose elements are the entries of $\mathbf{z}$.

Two remarks are in order. First, if $a_i \leq a_{i+1}$, and $a_{N-1} \leq b_0$, then (8) gives an approximate lower rank representation of $[I(\mathbf{a},\mathbf{b})]_{N\times N}$, the rank is at most $P+Q+K$. The elementwise truncation error in the remainder matrix $R_{P,Q,K}$ is independent of N. Such a lower rank representation of a matrix of order N leads to reductions in both storage for the matrix and operations involving the matrix. For example, only $O(N)$ arithmetic operations are needed when the matrix is applied to a vector. Secondly, $[I(\mathbf{a},\mathbf{b})]_{N\times N}$ is symmetric for the case $\mathbf{b} = \mathbf{a}$, i.e.,

$$\begin{aligned} &\mathrm{triu}([I(\mathbf{a},\mathbf{b})]_{N\times N}) = \mathrm{triu}(I_1) + \mathrm{triu}(I_2) + \mathrm{triu}(I_3) + \mathrm{triu}(R_{P,Q,K}), \\ &\mathrm{tril}([I(\mathbf{a},\mathbf{b})]_{N\times N}) = \mathrm{triu}([I(\mathbf{a},\mathbf{b})]_{N\times N})^{\mathrm{T}} \end{aligned} \tag{9}$$

where, in MATLAB notation, $\mathrm{triu}(A)$ and $\mathrm{tril}(A)$ denote the upper triangular part and the lower triangular part of matrix A, respectively. Notice that (9) is no longer guaranteed to have lower rank, and matrix-vector multiplications, for example, could take $O(N^2)$ arithmetic operations, for this case. However, this computation can still be done efficiently in $O(N \log N)$ operations using multipole techniques. We discuss this approach in §2.3.

2.2. Integral Representation Approach

We now present an alternate factorization of the two-parameter Hankel transform matrix (6). It is based on an integral representation of the Bessel function J_0 and the concepts behind the fast Hankel transform algorithm of Kapur and Rokhlin[10,12] for the single parameter case. We extend the fast Hankel transform algorithm to the two parameter case and provide a matrix representation in factorized form. All the factors are of *fast* type in the sense that they require only $O(N \log N)$ operations when applied to a vector.

A conventional integral represention of $J_0(z)$ is as follows[11]

$$J_0(z) = \frac{1}{\pi}\int_0^\pi \cos(z\cos\theta)d\theta. \tag{10}$$

Replacing $J_0(ax)$ and $J_0(bx)$ in (4) by (10) we have

$$I(a,b) = \frac{1}{\pi^2}\int_0^\infty \int_0^\pi \int_0^\pi f(x)\cos(ax\cos\theta)\cos(bx\cos\phi)d\theta d\phi dx. \tag{11}$$

A change of variables $u = a\cos\theta$ and $v = b\cos\phi$ results in

$$I(a,b) = \frac{1}{\pi^2}\int_{-a}^{a}\int_{-b}^{b} \frac{F(u,v)}{\sqrt{a^2-u^2}\sqrt{b^2-v^2}}dudv, \tag{12}$$

where

$$\begin{aligned} F(u,v) &= \int_0^\beta f(x)\cos(ux)\cos(vx)dx + R_F \\ &= \frac{\beta}{\pi}\int_0^\pi f(x)\cos(ux)\cos(vx)dx + R_F. \end{aligned} \tag{13}$$

The parameter β is a real number such that $|f(x)| < \mu$ for all $x > \beta$ and tolerance μ, and R_F is a negligible truncation quantity.

The following construction is based on the high-order quadrature schemes of Kapur and Rokhlin,[13] which we apply to obtain an accurate factorization of the two-parameter Hankel transform (6). Define the following discretizations of x, u and v,

$$\begin{aligned} x_k &= kh_x, & h_x &= \frac{\beta}{N-1}, & k &= 1:N \\ u_i &= ih, & h &= \frac{\pi}{\beta}, & i &= 1:N \\ v_j &= jh, & h &= \frac{\pi}{\beta}, & j &= 1:N. \end{aligned}$$

A quadrature formulation for (13) at (u_i, v_j) can be expressed as

$$\begin{aligned} Q(F(u_i,v_j)) &= \sum_{k=0}^{N-1} g(x_k)\cos(x_k u_i)\cos(x_k v_j) \\ &= \sum_{k=0}^{N-1} g(x_k)\cos\left(\frac{ik\pi}{N}\right)\cos\left(\frac{jk\pi}{N}\right), \end{aligned}$$

where $g(x_k) = f(x_k)w(x_k)$ for some quadrature weight function w. The particular choices of h_x and h, given above, allow the following inner product representation of $Q(F(u_i,v_j))$,

$$Q(F(u_i,v_j)) = c_{i,:}^T D_N c_{j,:}, \tag{14}$$

where

$$c_{i,:}^T = \left[1, \cos\left(\frac{\pi}{N}i\right), \cdots, \cos\left(\frac{\pi}{N}(N-1)i\right)\right]$$

and $D_N = \mathrm{diag}(g(x_k))$, $k = 0 : N-1$. Thus $F(u_i,v_j)$ can be approximated by $Q(F(u_i,v_j))$ with an approximation error bounded, for some real $\sigma > 0$, as

$$|\eta_{F_{ij}}| < \frac{\sigma}{N^2} \tag{15}$$

for all i and j.

For vectors $\mathbf{u} = (u_1, \ldots, u_N), \mathbf{v} = (v_1, \ldots, v_N) \in R^N$, we can now define the $N \times N$ matrix $[F(\mathbf{u}, \mathbf{v})]_{N\times N}$ in an approximate factorized form

$$[F(\mathbf{u}, \mathbf{v})]_{N\times N} = C^T D_N C + E_{\mathbf{u},\mathbf{v}}, \tag{16}$$

where $E_{\mathbf{u},\mathbf{v}}$ is the approximation error and C is the discrete cosine transform matrix.

It now remains to evaluate (12). Recall the discretization of u and v, and let the elements of $\mathbf{a}$ and $\mathbf{b}$ be written as

$$\begin{aligned} a_i &= ih, \qquad h = \frac{\pi}{\beta}, \quad i = 1 : N \\ b_i &= jh, \qquad h = \frac{\pi}{\beta}, \quad j = 1 : N. \end{aligned}$$

A quadrature formulation of $I(a_i, b_j)$ with a kth order end-point correction results in

$$Q(I(a_i, b_j)) = I_T(a_i, b_j) + I_C(a_i, b_j). \tag{17}$$

The first term $I_T(a_i, b_j)$ is obtained using, for example, the trapezoidal rule,

$$\begin{aligned} I_T(a_i, b_j) &= \sum_{p=-(i-1)}^{i-1} \sum_{q=-(j-1)}^{j-1} \frac{hw_u(p)}{\sqrt{a_i^2 - u_p^2}} F(u_p, v_q) \frac{hw_v(q)}{\sqrt{b_j^2 - v_q^2}} && (18) \\ &= 2\sum_{p=1}^{i-1} \frac{w_u(p)}{\sqrt{i^2 - p^2}} I_T^b(a_i, b_j, u_p) + \frac{w_u(0)}{i} I_T^b(a_i, b_j, u_0). && (19) \end{aligned}$$

where the term $I_T^b(a_i, b_j, u_p)$ is defined as

$$I_T^b(a_i, b_j, u_p) = 2\sum_{q=1}^{j-1} F(u_p, v_q) \frac{w_v(q)}{\sqrt{j^2 - q^2}} + F(u_p, v_0) \frac{w_v(0)}{j}. \tag{20}$$

The second term $I_C(a_i, b_j)$ in (17) corresponds to the k-th order end-point correction[13] to the trapezoidal quadrature rule,

$$\begin{aligned} I_C(a_i, b_j) &= \sum_{p=1}^{k/2} \frac{\nu_p^i}{\sqrt{i^2 - (i-p)^2}} I_C^b(a_i, b_j, u_{i-p}) && (21) \\ &+ \sum_{p=k/2}^{k} \frac{\nu_p^i}{\sqrt{i^2 - (i + (p - k/2))^2}} I_C^b(a_i, b_j, u_{i+(p-k/2)}), && (22) \end{aligned}$$

where the term $I_C^b(a_i, b_j, u_{i-p})$ is the quadrature correction in the direction of b_j,

$$I_C^b(a_i, b_j, u_\ell) = \sum_{q=1}^{k/2} F(u_\ell, v_{j-q}) \frac{\nu_q^j}{\sqrt{j^2 - (j-q)^2}} + \sum_{q=k/2}^{k} F(u_\ell, v_{j+(q-k/2)}) \frac{\nu_q^j}{\sqrt{j^2 - (j + (q - k/2))^2}}. \tag{23}$$

The quadrature correction weights ν_p^i are "pre-computed". See Kapur and Rokhlin[10] for details on how to obtain ν_p^i. The quadrature approximation error depends on the chosen correction order k, which is usually a modest number. For more details on the correction and the error estimate, see Kapur.[13]

Finally, from the elementwise representation (18) we get matrix representation in a symmetric factorization form for $I_T(\mathbf{a}, \mathbf{b})$, $\mathbf{a}, \mathbf{b} \in R^N$,

$$\begin{aligned} [I_T(\mathbf{a}, \mathbf{b})]_{N\times N} &= M\mathrm{diag}(w_\mathbf{u})[F(\mathbf{u}, \mathbf{v})]_{N\times N}\mathrm{diag}(w_\mathbf{v})M^T \\ &= M\mathrm{diag}(w_\mathbf{u})C^{\mathrm{T}} D_N C\mathrm{diag}(w_\mathbf{v})M^T, \end{aligned} \tag{24}$$

where the matrix M is a lower triangular matrix and defined as

$$M_{ij} = \begin{cases} \dfrac{1}{i^2 - j^2}, & j < i \\ 0, & \text{otherwise} \end{cases} . \quad (25)$$

This factorization leads to fast evaluations of the outer integrals in that there are known fast algorithms for matrix-vector multiplcations involving either C or M. A similar factorization can be derived for the quadrature correction part $I_C^b(a_i, b_j, u_{i-p})$.

2.3. Efficient and Accuracute Algorithms

The first factorization (9) of $[I(\mathbf{a}, \mathbf{b})]_{N\times N}$ reveals a low rank representation of the matrix for the case $a_{N-1} \leq b_0$. In such case, the storage for the matrix representation is linear in N and the arithmetic complexity for matrix-vector multiplication is also linear in N. The representation (9) for the case $\mathbf{a} = \mathbf{b}$ offers no advantages in matrix-vector multiplication. One approach to reducing the complexity for this case is to partition the matrix into smaller blocks so that within each block, the condition $a_i \leq b_j$ is met. The low rank factorization of each block is then obtained, saved and applied instead. Using the ideas behind the fast multipole algorithm, the matrix can be partitioned into blocks whose size doubles as the blocks get further away from the diagonal blocks. Thus, the total cost for matrix-vector multiplication can be reduced to $O(N \log N)$ using the hierarchical partitioning and rank-revealing scheme just described.

The fast multipole algorithm can be directly applied to matrix-vector multiplication with matrix M in the second factorization, given in (24), using only $O(N \log N)$ arithmetic operations. Together with the use of fast cosine transform algorithms, the second factorization representation leads naturally to a fast algorithm for matrix-vector multiplications with the matrix $[I(\mathbf{a}, \mathbf{b})]_{N\times N}$. For both approaches, the accuracy of the approximation can be predetermined by choosing the truncation points and the numerical quadrature rules.

3. ILLUSTRATION GRAPHICS

We validate the ideas presented in the previous section using some simple illustrations for the computation of the two-parameter Hankel transform. Recall that this case arises naturally in adaptive-optics modeling when the direction of atmospheric wind velocity is assumed to be random and uniformly distributed.

Figure 1 shows a three-dimensional and a two-dimensional plot of the approximations of $[I(\mathbf{a}, \mathbf{b})]_{N\times N}$ for the assumption that atmospheric wind velocity is random and uniformly distributed. The values of a and b are $0 < a, b \leq 3$ with a stepsize of $\frac{\pi}{\beta}$ for $\beta = 256$.

Figure 2 shows a "rank-map" identification of low rank blocks for the same approximation of $[I(\mathbf{a}, \mathbf{b})]_{N\times N}$.

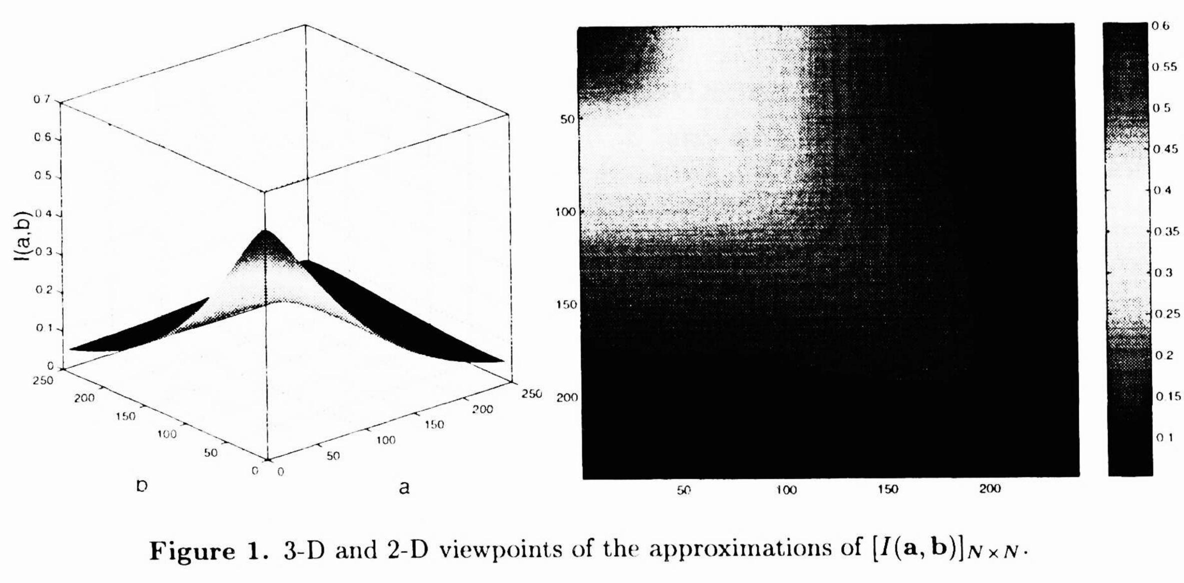

Figure 1. 3-D and 2-D viewpoints of the approximations of $[I(\mathbf{a}, \mathbf{b})]_{N\times N}$.

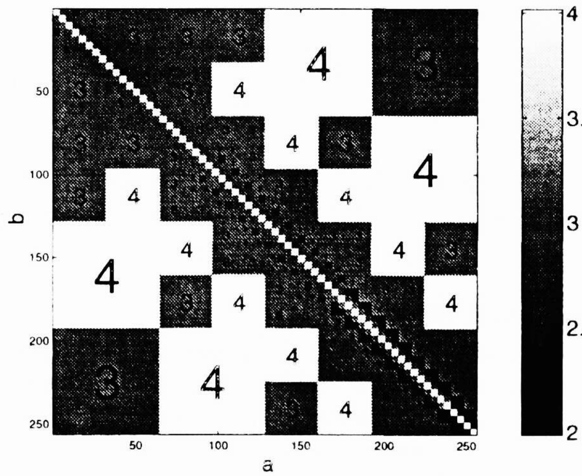

Figure 2. "Rank-map" identification of low rank blocks for the approximation of $[I(\mathbf{a},\mathbf{b})]_{N\times N}$.

4. REMARKS

The series representation approach based on Mellin transforms requires a good amount of algebra for derivation. Extensions of the work by Ellerbroek[4] may be difficult and tedious. Further, computation of the phase covariances $\langle c_1 c_2\rangle$ using his series approach requires hours of computing time.[4] In particular, the functions $w_1(r)$ and $w_2(r')$ in equation (3) for $\langle c_1 c_2\rangle$ lead to matrices upon discretization. Each is the sum of a sparse and a low rank matrix. Hence evaluation of (3) to compute the phase covariances involves several matrix-matrix multiplications with $[I(\mathbf{a},\mathbf{b})]_{N\times N}$. Thus the approximate factorization methods developed in this paper should have clear advantages in performance modeling of adaptive-optics imaging systems.

In addition, the extension of the fast Hankel transform algorithm of Kapur and Rokhlin to the two-parameter case is very efficient, like fast Fourier methods, for the case when a_i and b_j lie on an equally spaced grid. This gives us a very efficient method for approximating the Hankel transform integral

$$\int_0^\infty \frac{x}{(x^2+1)^{11/6}} J_0(ax)J_0(bx)dx$$

by the matrix $[I(\mathbf{a},\mathbf{b})]_{N\times N}$. The restriction on the discretization of a_i and b_j may be dropped without hindering the overall cost of the algorithm. We will investigate this possibility of extending the work of Kapur and Rokhlin to the nonequally spaced discretization case in future work.

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