Eliciting information for decision making from individual and multiple experts

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Abstract

We consider a setting in which a principal faces a decision and asks an external expert for a recommendation and a probabilistic prediction about what outcomes might occur if the recommendation were implemented. The principal then follows the recommendation and observes an outcome. Finally, the principal pays the expert based on prediction and outcome according to some decision scoring rule. In this paper, we ask the question: What does the class of proper decision scoring rules look like, i.e., what scoring rules incentivize the expert to honestly reveal the action he believes to be best for the principal and the prediction for that action?

We first show that in addition to an honest recommendation, proper scoring rules can only incentivize the expert to reveal the expected utility of taking the recommended action. The principal cannot strictly incentivize honest reports on other aspects of the recommendation’s distribution without setting poor incentives on the recommendation itself. We then characterize proper decision scoring rules as ones which give or sell the expert shares in the principal’s project. Each share pays, e.g., $1 per unit of utility obtained by the principal. Owning these shares makes the expert want to maximize the principal’s utility by giving the best-possible recommendation. Furthermore, if shares are offered at a continuum of prices, this makes the expert reveal the value of a share and therefore the expected utility of the principal under following the recommendation.

We proceed to extend our analysis to eliciting recommendations and predictions from multiple experts. With a few modifications, the above characterization for the single-expert case carries over. Among other implications, this characterization suggests that no expert should be able to “short-sell” shares in the principal’s project and thereby profit if the project goes poorly. Inspired by our results, we finally give a detail-free auction-based mechanism for eliciting information for decision making.

Keywords: algorithmic game theory, mechanism design, prediction markets, decision markets, principal-agent problems, principal-expert problems, proper scoring rules
Contents

1 Introduction

1.1 General notation ............................................ 5

2 Eliciting from a single expert

2.1 Setting ....................................................... 6
2.2 Only means matter .......................................... 9
2.3 Characterization ............................................ 12
2.4 Interpretation and alternative statements ................. 16
2.4.1 Selling shares at different prices ....................... 16
2.4.2 A characterization of differentiable scoring rules .... 17
2.4.3 Characterization in terms of convex functions and subgra-
dients ......................................................... 18
2.5 Which decision scoring rule to pick ....................... 18
2.5.1 The agency cost of eliciting the mean ................. 18
2.5.2 Strategic principals ................................... 20
2.6 Related work ............................................... 20
2.6.1 Othman and Sandholm [2010] ......................... 20
2.6.2 Chen et al. ............................................... 20
2.6.3 Eliciting under ex post revelation of the world state . 21
2.6.4 Direct elicitation of properties ....................... 22
2.6.5 Principal-agent problems ............................... 23

3 Eliciting from multiple experts

3.1 Setting ....................................................... 24
3.2 Characterizing right-action proper multi-expert scoring rules for
decision making ............................................... 27
3.3 Notable impossibilities ................................... 29
3.3.1 Rewarding big updates ............................... 30
3.3.2 Ungated participation ................................ 31
3.4 Related work ............................................... 31
3.5 Decision auctions – a proposal for “realistic” mechanisms for
eliciting decision-relevant information .................... 32

4 Conclusion ..................................................... 35
1 Introduction

Consider a firm that is about to make a major strategic decision. It wishes to maximize the expected value of the firm. It hires an expert to consult on the decision. The expert is strictly better informed than the firm, but it is commonly understood that the outcome conditional on the chosen option is uncertain even for the expert. The firm can commit to a compensation package for the expert; compensation can be conditional on both the expert’s predictions and on what happens (e.g., in terms of the value of the firm) after a decision is made. (The compensation cannot depend on what would have happened if another option had been chosen.) The firm cannot or does not want to commit to an arbitrary mapping from expert reports to options: once the report is made, the firm will always choose the option that maximizes expected value, conditional on that report. What compensation schemes will incentivize the expert to report truthfully? One straightforward solution is to give the expert a fixed share of the firm at the outset. Are there other schemes that also reward the accuracy of the prediction? What compensation schemes can be used if the firm can consult multiple experts?

Our approach to formalizing and answering these questions is inspired by existing work on eliciting only honest predictions about an event that the firm or principal cannot influence. In the single-expert case, such elicitation mechanisms are known as proper scoring rules (Brier, 1950; Good, 1952; McCarthy, 1956; Savage, 1971; Gneiting and Raftery, 2007). Formally, a scoring rule for prediction $s$ takes as input a probability distribution $\hat{P}$ reported by the expert, as well as the actual outcome $\omega$, and assigns a score or reward $s(\hat{P}, \omega)$. A scoring rule $s$ is proper if the expert maximizes his expected score by reporting as $\hat{P}$ his true beliefs about how likely different outcomes are. The set of proper scoring rules has been characterized in existing work (e.g. Gneiting and Raftery, 2007, Section 2).

In Section 2, we derive a similar characterization of what we call proper decision scoring rules – scoring rules that incentivize the expert to honestly report the best available action and a prediction. We show that proper decision scoring rules cannot give the expert strict incentives to report any properties of the distribution for the recommendation other than its expected utility (Section 2.2). Intuitively, rewarding the expert for getting anything else about the distribution right will make him recommend actions whose outcome is easy to predict as opposed to actions with high expected utility. Hence, the expert’s reward can depend only on the reported expected profit for the recommended option, and the realized profit. We then give a complete characterization of proper decision scoring rules (Section 2.3). The mechanisms can be interpreted as offering the expert to buy at different prices shares in the eventually realized profits (Section 2.4.1). The price schedule does not depend on the option chosen. Thus, given the chosen option, the expert is incentivized to buy shares up to the point where the price of a share exceeds the expected utility, thereby revealing

\footnote{We use “he” for experts and “she” for the principal, i.e. the firm or person setting up the mechanism.}
the expected utility. Moreover, once the expert has some share in the principal’s utility, he will be (strictly) incentivized to recommend an optimal option.

Our work on the multi-expert case is also inspired by work on eliciting mere predictions. This work has focused on a particular type of elicitation mechanism called prediction market (e.g., Hanson, 2003; Pennock and Sami, 2007). Prediction markets can be constructed from single-expert scoring rules as a building block. For example, in a market scoring rule, agents successively update the probability estimate, and an agent that updated the estimate from $\hat{P}_t$ to $\hat{P}_{t+1}$ is eventually rewarded $s(\hat{P}_{t+1}, \omega) - s(\hat{P}_t, \omega)$. But prediction markets can also be implemented in a way that resembles real-world securities markets. In particular, one could let the experts trade so-called Arrow-Debreu securities that each pay out a fixed amount—say, $\$1$—if a given event happens, and $\$0$ otherwise. Then, at any point, the price at which this security trades can be seen as the current market consensus of the probability that the event takes place.

Our goal is again to lay the foundations for a theory of eliciting recommendations and predictions from multiple experts that is in some sense analogous to existing work on prediction markets. Indeed, existing work has proposed decision markets (or conditional prediction markets) as a variation of prediction markets that can advise a decision maker (e.g., Hanson, 1999; Hanson, 2002; Berg and Rietz, 2003; Hanson, 2006; Hanson, 2013, Section IV). As we will see, these do not set good incentives in their most commonly described form. Roughly, the problem is that experts will misreport in order to make the principal take an action that they can predict well (relative to the market) (cf. Othman and Sandholm, 2010, Section 3). Chen, Kash, Ruberry, et al. (2014, Section 4) show that choosing suboptimal actions with some small probability can solve the incentive problems while still ensuring that we take the best option most of the time. But we consider the goal of deterministically (rather than with high probability) taking an optimal action.

To find out what elicitation mechanisms for decision making with multiple experts have to look like, we provide a characterization of such mechanisms (Section 3). We can build on the single-expert characterization by designing each expert’s scoring function on the assumption that everyone else reports honestly (i.e., non-strategically). If all scoring rules are designed in this way, honest reporting is a Nash equilibrium. The resulting characterization is structurally the same as that for the single-expert case, but significantly richer, because the multi-expert setting allows for a number of new possibilities. For example, one may (but is not required to) reward experts for accurately predicting other experts’ reports. Moreover, an expert’s recommended option and prediction of the profits to be expected from that option are, in general, conditioned on other experts’ reports.

The characterization implies several impossibilities (3.3). In prediction markets, traders who change the market probabilities significantly and toward the truth—and therefore contribute “a significant amount of accurate information”
in some sense – reap higher rewards. Our characterization shows that, other things being equal, a trader gets the same reward for honestly reporting a specific mean, regardless of whether the report helps in identifying the optimal action. Another desirable property of prediction markets is that we can allow anyone to participate, potentially anonymously. Those who do not have additional relevant information cannot consistently profit from participating. In proper mechanisms for decision making, on the other hand, being scored is profitable even to those who simply agree with the market estimates.

Finally, we briefly outline a mechanism which might be seen as the analogue of a prediction market with Arrow-Debreu securities for decision making (Section 3.5). Roughly, we let the principal auction off shares in her project and then have the shareholders decide what action to take. Importantly, we do not allow short-selling of shares in the principal’s project.

1.1 General notation

We here introduce some notation that is used throughout this paper. Since much of this is self-explanatory or sufficiently widely used, the reader may skip this section and refer to it later.

For any set $\Omega$, define

$$\Delta(\Omega) := \{ P : \Omega \to [0, 1] \mid \sum_{\omega \in \Omega} P(\omega) = 1 \}$$

(1)

to be the set of probability distributions over $\Omega$. For probabilities $p_1, \ldots, p_n \in [0, 1]$ with $\sum_{i=1}^{n} p_i = 1$, we write $p_1 \ast \omega_1 + \ldots + p_n \ast \omega_n$ for the random variable that assumes value $\omega_i$ with probability $p_i$ for $i = 1, \ldots, n$.

For any probability distribution $P \in \Delta(\Omega)$ define the support of $P$

$$\text{supp}(P) := \{ \omega \in \Omega \mid P(\omega) > 0 \}$$

(2)

to be the set of outcomes with positive probability. For any real-valued function $u : \Omega \to \mathbb{R}$, we use

$$\mathbb{E}_{O \sim P}[u(O)] := \sum_{\omega \in \Omega} P(\omega)u(\omega)$$

(3)

to denote the expected value of $u(O)$ where $O$ is distributed according to $P$. When it is clear that $O$ is distributed according to $P$, we may drop the subscript $O \sim P$ for brevity. When it is clear that $O$ is the random variable that the expectation is over, but there is ambiguity as to how $O$ is distributed, we give only $P$ in the subscript. We will also use $\mathbb{E}_{O \sim P}[u(O) \mid e] := \mathbb{E}_{O \sim P(\cdot \mid e)}[u(O)]$ for conditional expectations.

In the second part of this paper (Section 3), we will furthermore use the following bit of notation from game theory. If $X = \times_{i=1}^{n} X_i$ (for sets $X_1, \ldots, X_n$), then $X_{-j} := \times_{\ell \in \{1, \ldots, i-1, j+1, \ldots, n\}} X_i$. We use bold font to denote vectors like $x \in X$, and extend the above notation to $x_{-j} := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. We will also use $x_{-j}$ to denote elements of $X_{-j}$ directly.
2 Eliciting from a single expert

2.1 Setting

We consider a setting in which a principal faces a choice from some finite set of options or actions $A$. After taking any such action, the principal observes an outcome from some finite set $\Omega$, to which she assigns a utility according to some utility function $u: \Omega \to \mathbb{R}$.

In addition to the principal, there is an expert who holds beliefs described by some probability distribution $P(\cdot | \cdot) \in \Delta(\Omega)^A$, which specifies for each action $a \in A$ and outcome $\omega \in \Omega$ the probability $P(O = \omega | a)$ that outcome $\omega$ is obtained if action $a$ were to be taken by the principal.

The principal may ask the expert to recommend some action $\hat{a}$ – with the intention of getting the expert to report one that maximizes $E_P[u(O | a)]$ – and make a prediction $\hat{P} \in \Delta(\Omega)$ about what outcome action $\hat{a}$ will give rise to. The principal then always follows that recommendation. Note that it makes little sense to ask the expert to report his beliefs $P(\cdot | a)_{a \neq \hat{a}}$ about what would happen if the principal were to take some action other than the recommended one. After all, such predictions are never tested. Hence, the expert could have no incentive to report them honestly. For a formal statement and proof of this point, see Othman and Sandholm (2010, Theorems 1 and 4) or Chen, Kash, Ruberry, et al. (2014, Theorem 4.1).

Others have considered principals who take suboptimal actions with some (small) probability (Chen, Kash, Ruberry, et al., 2014; cf. Zermeño, 2011, Zermeño, 2012). This can help incentivize the expert to report honestly. For instance, Chen, Kash, Ruberry, et al. (2014) show that taking an action $a$ with a small positive probability is enough to make the expert honestly report his belief $P(\cdot | a)$. The principal may therefore learn from the expert about the effects of all actions, not just the one she ends up taking. However, randomizing also has a number of disadvantages. First and obviously, the principal prefers taking the best action all of the time over taking the best action almost all of the time. Second, the principal, who will finally make the decision, may be unable to credibly commit to taking, say, the worst available action if the dice demand

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3 Throughout this paper, we assume that the expert could hold any belief $P \in \Delta(\Omega)^A$. That is, we imagine that the principal does not know anything about what kinds of beliefs the expert may have. An alternative setting would be one in which the expert has exclusive access to some private piece of information $e \in H$. Each such piece of evidence gives rise to a posterior $P(\cdot | \cdot, e) \in \Delta(\Omega)^A$ and the principal knows both the possible pieces of evidence and each of the posteriors. For some posteriors, this gives rise to ways of scoring that are not available here. For instance, $O$ might inform the principal directly about which $e$ the expert observed (cf. Boutilier, 2012, Section 3.1; Carroll, 2019). Of course, it would be desirable to do the kind of work we do in this paper for any such given evidence structure. But such characterizations for this more general case seem hard to obtain. Furthermore, in practice, the principal may not know the evidence structure very well and might therefore favor the more “generic” scoring rules we discuss in this paper.

4 One reason to want to learn about this might be intrinsic curiosity. The principal may also be interested in how large the counterfactual benefit is of asking an expert – “how better off am I now than I would have been had I taken the option that I thought was best before asking the expert? Was this benefit worth the payment made to the expert via the scoring rule?”
it. Thus, there may be trust issues regarding whether the principal will really go through with a promise of choosing a suboptimal option some of the time (cf. Chen, Kash, Ruberry, et al., 2014, Section 5). Third, if the probability of suboptimal options being chosen is small, then rewards or punishments based on the outcomes of these events must be scaled up inversely proportionately to generate proper incentives (Chen, Kash, Ruberry, et al., 2014, Theorem 4.2). While in theory this poses no problem, in practice there are limits to rewards and punishments, due to budgets, limited liability, and other constraints. Operating within these constraints will thus require suboptimal options to be chosen significantly more often. Fourth, taking each action with positive probability is only an option when the set of options is finite. Fifth and finally, in the real world an action may produce different effects depending on the probability with which it is taken. For instance, if the decision mechanism is transparent, actions may be more effective when they are chosen with low probability as a result of randomization over suboptimal options, because they will then catch adversaries and competitors by surprise.

In our setting, the principal has no way of verifying the information she receives. We also assume that the expert has no intrinsic interests in the principal’s endeavors. To nevertheless incentivize the expert to report his private information honestly, the principal may therefore use a decision scoring rule (DSR) \( s : \Delta(\Omega) \times \Omega \rightarrow \mathbb{R} \), which maps a report and an outcome observed after taking \( \hat{a} \) onto a reward for the expert. As we have noted earlier, we do not let the score depend on predictions about what would happen if an action other than \( \hat{a} \) were to be taken. Furthermore, we do not let the score depend on what action is recommended – other than through the outcome obtained after implementing \( \hat{a} \). It is easy to see why that would set poor incentives for some beliefs. We do not give a formal proof here to avoid the introduction of alternative formalisms. However, such a proof could easily be conducted as part of the proof of Lemma 1. Incidentally, because the DSR does not take as input the recommended action, all of the present work generalizes to settings in which the expert could affect the outcome directly (not only through the recommendation). In particular, a proper DSR (as characterized in this paper) not only incentivizes the expert to make good recommendations but also to (at least before – but as we will see also after – submitting the prediction \( \hat{P} \)) affect the outcome favorably whenever he can (cf. the notion of “principal alignment” in prediction markets, as discussed by Shi, Conitzer, and Guo, 2009).

Ideally, the principal sets up \( s \) such that the expert is incentivized to recommend an action \( \hat{a} \) from \( \text{opt}(P) = \arg\max_{a \in A} \mathbb{E}_{O \sim P} [u(O) \mid a] \), the set of optimal actions, and further to report about \( P(\cdot \mid \hat{a}) \) honestly. The most basic form of this requirement is (non-strict) properness: among the reports that give the expert the highest expected score should always be one that consists of an optimal action and an honest prediction. Formally, we can define this as follows.

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5Of course, if the expert wants the principal to succeed, things become easier for the principal. However, the expert’s preferences may also conflict with the principal’s. For a discussion of eliciting decision-relevant information from experts under such conflicts of interest, see Boutilier (2012).
Definition 1. We say that a DSR \( s \) is proper if for all beliefs \( P(\cdot | \cdot) \in \Delta(\Omega)^A \) and all possible recommendations \( \hat{a} \in A \) and predictions \( \hat{P} \in \Delta(\Omega) \) it is

\[
E_{O \sim P} \left[ s(\hat{P}, O) \mid \hat{a} \right] \leq \max_{a^* \in \text{opt}(P)} E_{O \sim P} \left[ s(P(\cdot | a^*), O) \mid a^* \right].
\] (4)

It is clear that we can limit our attention to mechanisms that are proper. However, this non-strict form of properness alone is not very useful. While it implies that the expert has no bad incentives, it does not require that the expert has any good incentives. For example, any constant \( s \) is (non-strictly) proper. We might therefore be much more interested in strictly proper DSRs, i.e. ones where inequality (4) is strict unless \( \hat{a} \) is optimal and \( \hat{P} = P(\cdot | \hat{a}) \) is honest. As we will see (Lemma 2), no DSR is strictly proper in this sense. We will therefore define partially strict versions of properness.

Definition 2. We say that \( s \) is right-action proper if it is proper and for all beliefs \( P(\cdot | \cdot) \in \Delta(\Omega)^A \) and all possible recommendations \( \hat{a} \in A \) and predictions \( \hat{P} \in \Delta(\Omega) \),

\[
E_{P} \left[ s(\hat{P}, O) \mid \hat{a} \right] = \max_{a^* \in \text{opt}(P)} E_{P} \left[ s(P(\cdot | a^*), O) \mid a^* \right]
\] (5)

implies \( \hat{a} \in \text{opt}(P) \). We call \( s \) strictly proper w.r.t. the mean if eq. (5) implies

\[
E_{O \sim P(\cdot | \hat{a})} [u(O)] = E_{O \sim \hat{P}} [u(O)].
\] (6)

Right-action properness should be a main goal. Fortunately, such scoring rules do indeed exist. One class of examples of such rules is especially easy to identify. For any \( a,b \in \mathbb{R} \) with \( a > 0 \), we can use

\[
s(e, \omega) = au(\omega) + b.
\] (7)

Essentially, this corresponds to giving the expert some share of the principal’s profit (Chen, Kash, Ruberry, et al., 2014, Section 5). It is much less obvious what the entire class of proper and right-action proper DSRs looks like, and what other aspects of the report \( \hat{P} \) they can set strict incentives on. As we will see, the only further form of strictness that proper DSRs can achieve is strict properness w.r.t. the mean. This is why we define no other form of strict properness here.

If we drop the demand of being right-action proper or proper at all, then the characterization of scoring rules for prediction (see, e.g., Gneiting and Raftery, 2007, Section 2) tells us which DSRs are “strictly proper w.r.t. the reported probability distribution” (Chen, Kash, Ruberry, et al., 2014, Sections 3-4). It is

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6Of course, there may not be a problem with using a mechanism that strictly incentivizes some “harmless” form of misrepresentation – in particular misrepresentation that is guaranteed not to affect our decision. But we could modify any such mechanism to obtain a proper one by “doing the misrepresentation for the player”. This is essentially a single-player version of the revelation principle (Shoham and Leyton-Brown, 2009, Section 10.2.2; Börgers, 2015, Section 3.2.2, 4.2.2, 10.6).
illustrative of the challenge of right-action properness to work through an example of such a scoring rule for prediction and why it fails to be right-action proper. Consider the quadratic scoring rule (originally proposed by and sometimes named after Brier, 1950)

\[ s(\hat{P}, \omega) = 2\hat{P}(\omega) - \sum_{\omega' \in \Omega} \hat{P}(\omega')^2. \] (8)

This rule is well-known to elicit honest predictions for a given probability distribution. Hence, even if we allow the expert to choose the principal’s action and thereby the random variable he is being scored on, it will elicit honest predictions for that random variable. However, \( s \) is not right-action proper. It incentivizes the expert to recommend an action \( a \) that makes the outcome easier to predict. For instance, imagine that \( \Omega = \{\omega_1, ..., \omega_m\} \), that the optimal action \( a^\ast \) leads to the uniform distribution \( O_{a^\ast} = \frac{1}{m} * \omega_1 + ... + \frac{1}{m} * \omega_m \), while \( a' \) leads to \( O_{a'} = 1 * \omega_i \) deterministically. Then the expert will (assuming \( m > 1 \)) always prefer recommending the suboptimal \( a' \), since

\[ \mathbb{E}[s(P_{a'}, O_{a'})] = 1 > \frac{2}{m} = \frac{2}{m} - \sum_{\omega' \in \Omega} \frac{1}{m^2} = \mathbb{E}[s(P_{a^\ast}, O_{a^\ast})]. \] (9)

Because the decision making setting is so generic and because decision makers can often profit from getting advice from disinterested outside entities, many real-world situations can be viewed as principal-expert decision problems. Throughout this paper, we will often use the example of a firm paying disinterested experts (e.g., consultants). We use this example in great part because it yields illustrative interpretations of our results. However, we can also consider very different types of problems as principal-expert decision problems. For instance, how should societies use social rewards (e.g., fame and status) to incentivize individuals to reveal decision-relevant information?

2.2 Only means matter

We have argued (and it has been formally proven in earlier work) that proper DSRs cannot strictly incentivize the expert to honestly report on what would happen if a non-recommended action were taken. In this section, we derive a less obvious limitation on what aspects of the reported information the principal can set strict incentives on. In particular, we will show that if a DSR is to be proper, it can only strictly incentivize the expert to be honest about the the report and the reported expected utility of the optimal (recommended) action. That is, proper scoring rules can be strictly proper w.r.t. the mean and recommendation but nothing else.

To prove that result, we need a simple lemma. From the definition of properness it follows that if one action is better than another, the expert must prefer recommending the better action, even if, say, the worse action makes the outcome more predictable. But perhaps if two actions’ expected utilities are the same, a proper DSR could induce the expert to prefer recommending one of the two (say, the one with an easier-to-predict distribution over outcomes)? It turns
out that this is not the case. That is, we show that under honest reporting the expected scores for two different recommendations are the same whenever the expected utilities of the two recommendations are the same.

**Lemma 1.** Let $s$ be a proper DSR and $P_1, P_2 \in \Delta(\Omega)$. Then if
\[
\min_{\omega \in \Omega} u(\omega) < E_{O \sim P_1}[u(O)] = E_{O \sim P_2}[u(O)] < \max_{\omega \in \Omega} u(\omega)
\]
it must be
\[
E_{O \sim P_1}[s(P_1, O)] = E_{O \sim P_2}[s(P_2, O)].
\]

**Proof.** Let $\omega_L/\omega_H = \arg\min/\max_{\omega \in \Omega} u(\omega)$ (with ties being broken arbitrarily) and $L = u(\omega_L)$, $H = u(\omega_H)$. Then for $p \in (0,1)$ let $R_p = p * \omega_H + (1 - p) * \omega_L$ and $Q_p$ be the distribution of that random variable. Note first that because $s$ is proper $E[s(Q_p, R_p)]$ is non-decreasing in $p$. We claim that $E[s(Q_p, R_p)]$ is continuous in $p$. From this, we will directly derive the claim of the lemma.

We conduct a proof by contradiction. Assume there is a discontinuity at $p = \tilde{p}$, i.e. that there is a $\delta > 0$ such that for all $\epsilon > 0$
\[
E[s(Q_{\tilde{p} - \epsilon}, R_{\tilde{p} - \epsilon})] > \delta + E[s(Q_{\tilde{p} + \epsilon}, R_{\tilde{p} - \epsilon})].
\]
Now imagine that the expert believes that under the optimal action, $O$ is distributed according to $R_{\tilde{p} - \epsilon}$: Then for small enough $\epsilon$, the expert prefers submitting $Q_{\tilde{p}}$ over submitting $Q_{\tilde{p} - \epsilon}$:
\[
E[s(Q_{\tilde{p}}, R_{\tilde{p} - \epsilon})] = (\tilde{p} - \epsilon)s(Q_{\tilde{p}}, \omega_H) + (1 - (\tilde{p} - \epsilon))s(Q_{\tilde{p}}, \omega_L)
\]
\[
\rightarrow (\tilde{p} - \epsilon)s(Q_{\tilde{p}}, \omega_H) + (1 - \tilde{p})s(Q_{\tilde{p}}, \omega_L)
\]
\[
= E[s(Q_{\tilde{p}}, R_{\tilde{p}})]
\]
\[
> \delta + E[s(Q_{\tilde{p} - \epsilon}, R_{\tilde{p} - \epsilon})] \quad \text{for all } \epsilon > 0
\]
Hence, for any proper DSR $s$, the term $E[s(Q_p, O_p)]$ must be continuous in $p$.

We now use this fact to show that $E_{O \sim P}[s(P, O)]$ must be the same for all $P$ with the same non-degenerate mean $\mu$ (i.e., $L < \mu < H$). For $s$ to be proper, it must be $E_{O \sim P}[s(P, O)] \geq E[s(Q_p, R_p)]$ for all $p$ with $\mu > pH + (1 - p)L$ and $E[s(P, R_P)] \leq E[s(Q_p, R_p)]$ for all $p$ with $\mu < pH + (1 - p)L$. But because $E[s(Q_p, R_p)]$ is continuous, this implies $E_{O \sim P}[s(P, O)] = E[s(Q_p, R_p)]$ for the $p$ s.t. $\mu = pH + (1 - p)L$. So for any $P$ with mean $\mu$, this uniquely fixes the expected score under honestly reporting $P$ to the same $E[s(Q_p, R_p)]$. \hfill \Box

It is worth noting that the proof is based on the lack of “space” in the set $\mathbb{R}$ of possible scores. We could imagine experts who maximize a lexicographic score.\footnote{Note that lexicographic utility functions describe the preferences of agents who may violate the continuity axiom but are otherwise vNM-rational (see, e.g., Fishburn, 1971; Blume, Brandenburger, and Dekel, 1989). For some philosophical discussion on the (descriptive and normative) plausibility of lexical preferences, see, e.g., Hájek (2012).} Then our result only shows that the lexically highest value of the scores
– under honest reporting – of two equally good recommendations must be the same. But the lexically lower values could be given according to some scoring rule for prediction (such as the quadratic scoring rule) and thus make the expert prefer one of two recommendations with equal expected utility for the expert.

We have now shown that the expected utility of an action uniquely determines the expected reward the expert gets for recommending that action and reporting his prediction about the outcome given that action honestly. Next, we show that – perhaps more surprisingly – on the reporting side, in some sense the mean is the only piece of information the principal can elicit from the expert. That is, as long as the expert reports the expected utility of the recommendation honestly, he can almost arbitrarily mis-predict the outcome to a right-action proper DSR.

**Lemma 2.** Let $s$ be a proper DSR and $P, \hat{P} \in \Delta(\Omega)$. Then if

$$\min_{\omega \in \Omega} u(\omega) < \mathbb{E}_P [u(O)] = \mathbb{E}_{\hat{P}} [u(O)] < \max_{\omega \in \Omega} u(\omega)$$

(17)

and $\text{supp}(P) \subseteq \text{supp}(\hat{P})$, it must be

$$\mathbb{E}_P [s(P, O)] = \mathbb{E}_{\hat{P}} [s(\hat{P}, O)].$$

(18)

**Proof.** If $\mathbb{E}_P [u(O)] = \mu = \mathbb{E}_{\hat{P}} [u(O)]$ and $\text{supp}(P) \subseteq \text{supp}(\hat{P})$, there is a $P'$ and a $p \in (0, 1]$ s.t. $\hat{P} = pP + (1 - p)P'$ and $\mathbb{E}_{P'} [u(O)] = \mu$. Then

$$\mathbb{E}_{P'} [s(P, O)] = p \mathbb{E}_P [s(\hat{P}, O)] + (1 - p) \mathbb{E}_{P'} [s(P', O)].$$

(19)

Because the term at the beginning is the same as the term in the end, the $\leq$-inequalities in the middle must be equalities. Therefore, because $p > 0$, it must be

$$\mathbb{E}_P [s(P, O)] = \mathbb{E}_{\hat{P}} [s(\hat{P}, O)].$$

(20)

Lemma 2 implies that proper DSRs cannot be strictly proper w.r.t. anything but the mean (and the recommended action). Thus, we will henceforth only consider DSRs $s(\hat{\mu}, \omega)$, which take only the reported mean as input. Note that not all proper scoring rules can be expressed as a scoring rule that depends only on the mean. For one, we could punish the expert if the support of the reported probability distribution does not contain the observed outcome. Of course, unless $\min/\max_{\omega \in \Omega} u(\omega)$ will occur with certainty, the expert has no
reason not to report full support. Furthermore, we could let the submitted probability distribution determine the scoring rule in ways that do not affect the expected score since none of these dependencies on details of the submitted probability distribution seem helpful, we will ignore them.

Next we argue that \( s(\hat{\mu}, \omega) \) should only depend on \( \hat{\mu} \) and \( u(\omega) \). So let \( \omega, \omega' \in \Omega \) with \( \omega \neq \omega' \) and \( u(\omega) = u(\omega') \). Clearly, if \( \mu = \min/\max_{\omega \in \Omega} u(\hat{\omega}) \) and \( u(\omega) \neq \mu \) then the score can be different for \( \omega \) and \( \omega' \), because in that case both are predicted to occur with probability 0. We ignore this degenerate case, e.g. because in this case it is not very useful to treat \( \omega \) and \( \omega' \) differently.

We may also not want our scoring rule to make use of information about what \( \min/\max_{\omega \in \Omega} u(\hat{\omega}) \) is. So assume that \( \mu \) is not an extreme value. Then there is \( p > 0 \) and random variable \( X \) over \( \Omega \) s.t. the two random variables \( Y = p \ast \omega + (1 - p) \ast X \) and \( Y' = p \ast \omega' + (1 - p) \ast X \) both have an expected utility of \( \hat{\mu} \). It is

\[
ps(\hat{\mu}, \omega) + (1 - p)\mathbb{E}[s(\hat{\mu}, X)] = \mathbb{E}[s(\hat{\mu}, Y)] \tag{25}
\]

\[
\mathbb{E}[s(\hat{\mu}, Y')] = ps(\hat{\mu}, \omega') + (1 - p)\mathbb{E}[s(\hat{\mu}, X)]. \tag{26}
\]

Hence, \( s(\hat{\mu}, \omega) = s(\hat{\mu}, \omega') \). So from now on, we we will only consider scoring rules \( s: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) that map a reported mean \( \hat{\mu} \) and the obtained utility \( y \) onto a score \( s(\hat{\mu}, y) \).

### 2.3 Characterization

Now that we have shown that we can limit our attention to scoring rules \( s \) that map a reported expected utility and an observed utility onto a score, we can finally characterize the classes of proper decision scoring rules. The change in inputs to \( s \) also allows us to consider scoring rules independently of any utility function and outcome set, which in turn lets us ignore the degenerate cases of the reported mean \( \mu \) being the lowest-possible and highest-possible utility. With this, we can characterize proper DSRs as follows. (Structurally, the characterization resembles a few existing results on proper elicitation. We discuss these in Section 2.6, after giving some alternative forms of the result in Section 2.4.)

**Theorem 3.** A DSR \( s: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is proper iff

\[
s(\hat{\mu}, y) = f(\hat{\mu})(y - \hat{\mu}) + \int_{0}^{\hat{\mu}} f(x)dx + C \tag{29}
\]

for some non-negative, non-decreasing \( f \) and constant \( C \in \mathbb{R} \). Furthermore, \( s \) is right-action proper if \( f > 0 \) and strictly proper w.r.t. the mean iff \( f \) is strictly increasing.

\(^8\)For example, in the characterization given in Theorem 3, we can let the expert modify \( f \) at a set of points with measure zero as long as \( f \) remains positive and non-decreasing.
A more fine-grained version of strictness w.r.t. the mean can be proven: The expert is indifferent between submitting means $\hat{\mu}_1$ and $\hat{\mu}_2$ iff $f(\hat{\mu}_1) = f(\hat{\mu}_2)$. The proof of such a more fine-grained version can be conducted in just the same way as the proof of full strict properness. We omit it here for simplicity.

**Proof.** “⇒” Let $s$ be a right-action proper DSR. We now show that $s$ is of the form given in the theorem.

First, we show that $s(\hat{\mu}, y)$ is affine in $y$, i.e. that

$$s(\hat{\mu}, y) = f(\hat{\mu})y - g(\hat{\mu})$$

for some functions $f$ and $g$ with the claimed restrictions on the sign of values of $f$. Let $\hat{\mu} \in \mathbb{R}$ be a reported mean, $X$ be a random variable over $\mathbb{R}$ with mean $\mu$. Consider $Y = p * X + (1 - p) * X'$ and $Y' = p * x + (1 - p) * X'$, where $p, X'$ are s.t. both $Y, Y'$ have mean $\hat{\mu}$. Then it is

$$p \mathbb{E}[s(\hat{\mu}, X)] + (1 - p) \mathbb{E}[s(\hat{\mu}, X')]$$

$$= \mathbb{E}[s(\hat{\mu}, Y)]$$

$$= \mathbb{E}[s(\hat{\mu}, Y')]$$

$$= ps(\hat{\mu}, x) + (1 - p) \mathbb{E}[s(\hat{\mu}, X')].$$

Hence, even if $\hat{\mu} \neq \mu$ is not the mean of $X$, it is $\mathbb{E}[s(\hat{\mu}, X)] = s(\hat{\mu}, \mu)$ for all $\hat{\mu}, X$. This is exactly the characterization of $s(\hat{\mu}, \cdot)$ being affine and therefore of the form in eq. 30. Further, notice that for $s$ to be proper, $f$ has to be non-negative and for $f$ to be right-action proper it has to be strictly positive.

It is left to show that $f$ must be non-decreasing and that

$$g(\mu) = f(\mu)\mu - \int_0^\mu f(x)dx - C$$

for some $C \in \mathbb{R}$. For both of these, we will need a relationship between the rates at which $f$ and $g$ change. For $s(\hat{\mu}, \mu)$ to be maximal at $\hat{\mu} = \mu$, it has to be for all $d > 0$

$$s(\mu + d, \mu) \leq s(\mu, \mu),$$

which – using eq. 30 – we can rewrite as

$$g(\mu + d) - g(\mu) \geq \mu(f(\mu + d) - f(\mu)).$$

Similarly, it has to be for $d > 0$,

$$s(\mu, \mu + d) \leq s(\mu + d, \mu + d),$$

which we can rewrite as

$$g(\mu + d) - g(\mu) \leq (\mu + d)(f(\mu + d) - f(\mu)).$$

Note that all of these inequalities must be strict if $s$ is to be strictly proper (w.r.t. the mean).
We now show that $f$ is non-decreasing. From ineq. 37 and 39 it follows that for all positive $d$

$$\mu(f(\mu + d) - f(\mu)) \leq (\mu + d)(f(\mu + d) - f(\mu)), \quad (40)$$

which implies that $f(\mu + d) - f(\mu) \geq 0$ for all $d > 0$. If $s$ is to be strictly proper (w.r.t. the mean), then this inequality is strict.

Finally, it is left to show that $g$ is structured as suggested in the theorem. By telescoping, it is for any $n \in \mathbb{N}_{>0}$ and any $\hat{\mu} \in \mathbb{R}$

$$g(\hat{\mu}) = g(0) + \sum_{i=1}^{n} \left( \frac{i\hat{\mu}}{n} \left( f \left( \frac{i\hat{\mu}}{n} \right) - f \left( \frac{(i-1)\hat{\mu}}{n} \right) \right) \right). \quad (41)$$

Since relative to any $f$, $g$ can only be unique up to a constant, we will write $C$ instead of $g(0)$. From equations 37 and 39 it follows that

$$\sum_{i=1}^{n} \frac{(i-1)\hat{\mu}}{n} \left( f \left( \frac{i\hat{\mu}}{n} \right) - f \left( \frac{(i-1)\hat{\mu}}{n} \right) \right) \leq g(\hat{\mu}) - C \leq \sum_{i=1}^{n} \frac{i\hat{\mu}}{n} \left( f \left( \frac{i\hat{\mu}}{n} \right) - f \left( \frac{(i-1)\hat{\mu}}{n} \right) \right) \quad (42)$$

for all $n \in \mathbb{N}_{>0}$.

We would now like to find $g$ by taking the limit w.r.t. $n \to \infty$ of the two series. To do so, we interpret the two sums as the (right and left) Riemann sums of some function. In fact, we could immediately interpret them as Riemann sums of the function $f^{-1}$ for the partition $(f \left( \frac{ih}{n} \right))_{i=1,\ldots,n}$. This works out but leads to a number of technical issues that are cumbersome to deal with: if $f$ is discontinuous, then $(f \left( \frac{ih}{n} \right))_{i=1,\ldots,n}$ might not get arbitrarily fine, and $f^{-1}$ could be empty somewhere between $f(0)$ and $f(\hat{\mu})$; and if $f$ is constant on some interval, then $f^{-1}$ contains more than one element. We can avoid these issues by first re-writing the above sum. This re-writing corresponds directly to a well-known, very intuitive formula for the integral of the inverse (e.g. Key, 1994 Theorem 1). (Again, we could interpret the sums of lines 42 and 44 as Riemann sums for $f^{-1}$ and then apply that formula for integrals, but this is more cumbersome.)

It is

$$\sum_{i=1}^{n} \frac{i\hat{\mu}}{n} \left( f \left( \frac{i\hat{\mu}}{n} \right) - f \left( \frac{(i-1)\hat{\mu}}{n} \right) \right) \quad (45)$$

\begin{align*}
&= \sum_{i=1}^{n} \frac{i\hat{\mu}}{n} f \left( \frac{i\hat{\mu}}{n} \right) - \frac{(i-1)\hat{\mu}}{n} f \left( \frac{(i-1)\hat{\mu}}{n} \right) - \sum_{i=1}^{n} \frac{\hat{\mu}}{n} f \left( \frac{(i-1)\hat{\mu}}{n} \right) \quad (46) \\
&= \hat{\mu} f(\hat{\mu}) - \sum_{i=1}^{n} \frac{\hat{\mu}}{n} f \left( \frac{(i-1)\hat{\mu}}{n} \right). \quad (47)
\end{align*}
The last step is due to telescoping of the left-hand sum. Analogously,

\[
\sum_{i=1}^{n} \frac{(i-1)\hat{\mu}}{n} \left( f \left( \frac{i\hat{\mu}}{n} \right) - f \left( \frac{(i-1)\hat{\mu}}{n} \right) \right) = \frac{n}{n} \sum_{i=1}^{n} i\hat{\mu} f \left( \frac{i\hat{\mu}}{n} \right) - \frac{n}{n} \sum_{i=1}^{n} \frac{\hat{\mu}}{n} f \left( \frac{(i-1)\hat{\mu}}{n} \right) - \sum_{i=1}^{n} \frac{n}{n} f \left( \frac{i\hat{\mu}}{n} \right) \]

(48)

\[
= \hat{\mu} f(\hat{\mu}) - \frac{\hat{\mu}}{n} \sum_{i=1}^{n} f \left( \frac{i\hat{\mu}}{n} \right). 
\]

(50)

First note that the subtrahends are the left and right Riemann sums of \( f \) on \([0, \hat{\mu}]\). Because \( f \) is non-decreasing on \( \mathbb{R} \), it is integrable (e.g. Rudin, 1976, Theorem 6.9). That is, both the left and right Riemann sum converge to the integral:

\[
\sum_{i=1}^{n} \frac{\hat{\mu}}{n} f \left( \frac{i\hat{\mu}}{n} \right) \rightarrow \int_{0}^{\hat{\mu}} f(x) \, dx \quad \text{as} \quad n \rightarrow \infty
\]

(51)

So for \( n \rightarrow \infty \), the lower and upper bound on \( g(\hat{\mu}) \) converge to the same value. Hence, \( g(\hat{\mu}) \) must be that value, i.e.

\[
g(\hat{\mu}) = C + \hat{\mu} f(\hat{\mu}) - \int_{0}^{\hat{\mu}} f(x) \, dx.
\]

(52)

From this, eq. 29 follows as claimed.

“⇐” The other direction is easy to verify.

As an example, we can construct the simplest possible right-action proper DSR \( s(\hat{\mu}, y) = ay + C \) for \( a > 0 \) (see eq. 7) from using the function \( f = a \). Again, this \( s \) is not strictly proper w.r.t. the reported mean. But it is easy to see why it is right-action proper, as the expert can influence his expected payoff only by recommending an action that yields the highest utility \( y \) in expectation. Imagine the principal manages a company with the utility being the profit generated by the company. Then for, say, \( a = 1/5 \) scoring according to eq. 7 is like promising the expert 20% of the profits of the company. Of course, the expert will now share the principal’s goal of maximizing the company’s profit.

The second-simplest example (which up to a factor of 1/2 is also given by Chen and Kash, 2011, end of Section 4) arises from \( f(\hat{\mu}) = \hat{\mu} \), which gives

\[
s(\hat{\mu}, y) = \hat{\mu} y - \frac{1}{2} \hat{\mu}^2.
\]

(53)

Because \( s \) is linear in \( y \), it is again easy to see that the expert wants to recommend the best action. But it is not immediately obvious why \( s \) is also strictly proper w.r.t. the mean of the optimal action; though it is easy to verify by taking the derivative w.r.t. \( \hat{\mu} \) and so forth.
In the next section, we will give some alternative interpretations of Theorem 3, one of which (Section 2.4.1) in particular will make it easier to immediately “see” why — if $f$ is strictly increasing — the scoring rules of Theorem 3 are strictly proper w.r.t. the mean of the optimal action. In Section 2.5, we then give some criteria to decide between different scoring rules.

### 2.4 Interpretation and alternative statements

#### 2.4.1 Selling shares at different prices

We can interpret the proper scoring rules of theorem 3 as ones where the principal “sells” $f(\hat{\mu})$ “shares” for an overall price of $f(\hat{\mu}) \hat{\mu} - \int_0^{\hat{\mu}} f(x)dx$. Since the given price term is not very intuitive, we will re-write it a bit. For technical convenience, assume that $f$ is strictly increasing and continuous.

By Theorem 1 of Key (1994) (a well-known and very intuitive — see figure 1 — formula for the integral of the inverse), it is

$$ f(\hat{\mu}) \hat{\mu} - \int_0^{\hat{\mu}} f(x)dx = \int_{f(0)}^{f(\hat{\mu})} f^{-1}(z)dz. \quad (54) $$

So

$$ s(\hat{\mu}, y) = f(\hat{\mu})y - \int_0^{f(\hat{\mu})} f^{-1}(z)dz + C' \quad (55) $$

for some constant $C' \in \mathbb{R}$. Now imagine that instead of reporting a mean $\hat{\mu}$, the expert chooses the quantity $q = f(\hat{\mu})$ of shares to buy and then pays $\int_0^{q} f^{-1}(z)dz$. Then we can interpret $f^{-1}(z)$ as the price of the $z$-th share. Note that if $f$ is strictly increasing, $f^{-1}$ is, too. Of course, a utility-maximizing expert stops buying shares when the cost of the marginal share is exactly the value of a single share, i.e. when $f^{-1}(q)$ (and therefore $f^{-1}(f(\hat{\mu})) = \hat{\mu}$) is the expected utility of the expert.

So right-action proper scoring rules are like sets of offers to the expert to buy shares at increasing prices. One might also say that being scored according to $s$ is like owning a set of stock options with different prices. If the expert accept/rejects to buy a share for some price $p \in \mathbb{R}$, the principal can infer that the expert believes the share to be worth at least/at most $p$. This in turn (by the size of the share), allows the principal to infer what the expert believes the value of the company to be (upon implementation of the expert’s advice). The larger (or perhaps the more fine-grained) the sets of prices are, the more precisely the principal can infer the expert’s beliefs about the company. In particular, if $f$ is continuous (as we have assumed for this section for technical reasons), then the expert will reveal precisely what he believes the principal’s expected utility to be.

---

9As noted in the proof of Theorem 3, if $f$ is potentially not strictly increasing and not continuous, this analysis still works but only with a few cumbersome complications.
In the proof of Theorem 3, we first show that \( s(\hat{\mu}, y) = f(\hat{\mu})y - g(\hat{\mu}) \) and then infer how \( f \) and \( g \) relate to each other for \( s \) to be maximal in \( \hat{\mu} \). Hence \( f \) and \( g \) are differentiable in \( \hat{\mu} \). This gives us the following result for differentiable scoring rules. For those used to thinking about optimization in terms of derivatives, the result gives an alternative perspective on why these types of scoring functions work. The theorem in this form makes it easier to compare our result to that of Othman and Sandholm (2007) (discussed in Section 2.3.2). As an example, we can rewrite the scoring rule resulting from \( f(\hat{\mu}) = \hat{\mu} \) (see eq. 53) as

\[
\begin{align*}
\hat{s}(\mu, y) &= \mu y - \int_0^\mu z \, dz \\
&= \mu y - \frac{\mu^2}{2}.
\end{align*}
\]

This gives us a way to easily see why it is strictly proper w.r.t. the mean.

Figure 1 illustrates how \( f^{-1} \) serves as a pricing function. As noted earlier, it also shows why eq. 54 is true. Further, it illustrates why, up to a constant, the expert’s profit under honest reporting is \( \int_0^\mu f(x) \, dx \) – rotating the graph in figure 1 counter-clockwise by 90° shows that this is the area marked “expert’s profit”. It is worth noting that scoring rules for eliciting mere predictions (see Gneiting and Raftery, 2007, for an overview and introduction) can be interpreted in a similar way. Roughly, to elicit the probability of some outcome \( \omega \), we can offer the expert Arrow-Debreu securities on \( \omega \) – assets which pay some fixed amount if \( \omega \) occurs and are worthless otherwise (cf. Savage, 1951; Gneiting and Raftery, 2007).
Section 2.6.1 of our paper); as well as to some results on direct elicitation of properties (see Section 2.6.4 and in particular footnote 12).

**Corollary 4.** A differentiable DSR \( s \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is proper iff there are differentiable \( f, g \) s.t.
\[
s(\hat{\mu}, y) = f(\hat{\mu})y - g(\hat{\mu})
\] (57)
with \( g'(\hat{\mu}) = \hat{\mu}f'(\hat{\mu}) \) for all \( \hat{\mu} \in \mathbb{R} \); \( f' \geq 0 \); and \( f \geq 0 \). Furthermore, \( s \) is right-action proper if \( f > 0 \) and strictly proper w.r.t. to the reported mean of the optimal action iff \( f' > 0 \).

**2.4.3 Characterization in terms of convex functions and subgradients**

Existing work on elicitation often uses the terminology of convex functions and their subgradients (see Section 2.6.2). Indeed, our result can be put in these terms as well.

**Corollary 5.** A DSR \( s \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is right-action proper iff there is a convex, non-decreasing \( h \) with a subgradient \( h' \) s.t.
\[
s(\hat{\mu}, y) = h'(\hat{\mu})(y - \hat{\mu}) + h(\hat{\mu}).
\] (58)
Furthermore, \( s \) is strictly proper w.r.t. to the reported mean of the optimal action iff (in addition to the above) \( h \) is strictly convex.

**Proof.** Follows directly from Theorem 3 and the equivalence of convex functions and integrals over subderivatives, see e.g. Theorem 24.2 and Corollary 24.2.1 of Rockafellar (1970).

**2.5 Which decision scoring rule to pick**

**2.5.1 The agency cost of eliciting the mean**

By using a non-constant \( f \), we can incentivize the expert to honestly report information about the principal’s expected utility conditional on following his recommendation. This seems to be a desirable property of a DSR since the principal might want to know what to expect. Unfortunately, rewarding the expert for reporting \( \mu \) honestly comes at a cost: The expert is rewarded for knowing as precisely as possible what the mean of his recommendation is. Hence, the expert will sometimes prefer to have information that is less relevant for decision making. In particular, if the expert has the choice between receiving as his private piece of information \( X_1 \) which merely improves his estimate of the mean (relative to the prior), and receiving \( X_2 \) which identifies what the best action is, he will sometimes prefer the decision-irrelevant \( X_1 \). In the following we sketch the argument for this.

Imagine that given his current beliefs, the expert would recommend some action \( a \) which gives rise to an outcome variable \( O_a \) with \( u(O_a) = pU_1 + (1 - p)U_2 \) where \( \mu_1 := \mathbb{E}[U_1] < \mathbb{E}[U_2] =: \mu_2 \) and \( \mu := p\mu_1 + (1 - p)\mu_2 \). Consider a
proper DSR $s$ resulting from some $f$ which is increasing somewhere on $(\mu_1, \mu_2)$. Imagine further that the expert can choose which piece of information he receives at the start of the game. His first option is to receive $X_1$ which tells him whether $u(O_1) = U_1$ or $u(O_2) = U_2$ but does not tell him anything about whether some other action is better than $a$ (e.g. because absent further evidence $\mu_1$ is still higher than the expected utility of any action other than $a$). Then the expert values knowing that piece of information (relative to having to report based on the prior) at

$$p\mathbb{E}[s(\mu_1, U_1)] + (1 - p)\mathbb{E}[s(\mu_2, U_2)] - \mathbb{E}[s(\mu, U)]$$

(59)

$$= p\int_{\mu_1}^{\mu_2} f(x)dx + (1 - p)\int_{\mu_2}^{\mu_1} f(x)dx - \int_{\mu_1}^{\mu_2} f(x)dx$$

(60)

$$> (1 - p)\int_{\mu_2}^{\mu_1} f(x)dx - p\int_{\mu_1}^{\mu_2} f(x)dx$$

(61)

$$= (1 - p)(\mu_2 - \mu)f(\mu) - p(\mu_1 - \mu)f(\mu)$$

(62)

The last step is because $\mu = p\mu_1 + (1 - p)\mu_2$ implies (by mere arithmetic) that $(1 - p)(\mu_2 - \mu) = p(\mu_2 - \mu_1)$. This shows that if $f$ is increasing somewhere on $(\mu_1, \mu_2)$, the expert assigns some positive value to knowing $X_1$ as opposed to reporting based on his prior.

Now imagine further that the expert knows that one of the alternative actions $a_1, \ldots, a_n$ gives rise to a distribution $U^*$ over utilities with $\mu^* := \mathbb{E}[U^*] = \mu + \epsilon$ for some known positive $\epsilon$. However, he does not yet know for which of these actions this is the case. That is why he would – given his current beliefs – recommend $a$. Let $X_2$ be the random variable which tells him which action is optimal. Then the value of acquiring $X_2$ is

$$\mathbb{E}[s(\mu + \epsilon, U^*)] - \mathbb{E}[s(\mu, U)]$$

(64)

$$= \left(\int_{\mu}^{\mu+\epsilon} f(x)dx + C\right) - \left(\int_{\mu_1}^{\mu} f(x)dx + C\right)$$

(65)

$$= \int_{\mu}^{\mu+\epsilon} f(x)dx$$

(66)

$$\leq \epsilon f(\mu + \epsilon).$$

(67)

This value approaches 0 as $\epsilon$ approaches 0. Hence, if $\epsilon$ is small enough, the expert would prefer to have $X_1$ rather than $X_2$ as his private piece of information, even though $X_2$ is action-guiding and $X_1$ is not.

We have shown that linear scoring rules are the only DSRs which incentivize the expert to focus acquisition on decision-relevant information. Very similar arguments can be found in the economics literature on the principal–agent problem (Stoughton, 1993; Diamond, 1998; Carroll, 2015; Chassang, 2013; Carroll, 2019; Dütting, Roughgarden, and Talgam-Cohen, 2019; Oesterheld and Conitzer, 2019).
2.5.2 Strategic principals

Imagine that both the expert’s utility and the expert’s score are given in some transferable form of utility such as money. Imagine also that the principal chooses her action strategically, i.e. that she can (e.g., secretly) take an action other than the optimal one. (In practice, it may not be fully observable how much effort the principal invests in her project, or whether she self-sabotages.) Then once the expert has submitted a mean of $\hat{\mu}$, the principal will want to choose according to

$$\arg \max_{a \in A} (1 - f(\hat{\mu})) E [u(O) | a].$$

(68)

Therefore, the principal’s new preferences over actions are aligned with her original preferences iff $f(\hat{\mu}) < 1$. This motivates the use of scoring rules with $f < 1$. For instance, if we know from the start that the mean is greater than 1, we might use $f(x) = 1 - 1/x$.

2.6 Related work

2.6.1 Othman and Sandholm (2010)

As far as we can tell, Othman and Sandholm (2010) are the first to consider the problem of designing scoring rules for decision making. They study a simplified case in which the set of outcomes $\Omega$ has only two elements, one with a utility of 1, the other with a utility of 0. Note that the two-outcome-case is unusual because the mean of a binary random variable fully determines its distribution. In Section 2.3.2, they give a characterization of differentiable scoring rules with good incentives, which is a special case of our Corollary 1.

2.6.2 Chen et al.

Chen, Kash, Ruberry, et al. (2014) also characterize scoring rules for decision making (cf. Frongillo and Kash, n.d., Section E.1). Their setting is more general than ours in that they allow arbitrary decision rules. That is, they allow principals

\[ \text{We briefly show that the two characterizations are equivalent. They characterize differentiable scoring rules } s \text{ as ones where A) } s(p, 1) > s(p, 0) \text{ and B) } \frac{s'(p, 1)}{s'(p, 0)} = \frac{p - 1}{p} \text{ for all } p. \text{ A is equivalent to being able to write } s(p, 1) = f(p) - g(p) \text{ and } s(p, 0) = -g(p) \text{ for some differentiable } g \text{ and positive, differentiable } f. \text{ With that we can re-write B:}
\]

\[ \frac{s'(p, 1)}{s'(p, 0)} = \frac{p - 1}{p} \iff \frac{f'(p) - g'(p)}{g'(p)} = \frac{p - 1}{p} \iff g'(p) = pf'(p). \]

(70)

(71)

which is exactly the relationship between $f', g'$ stated in our Corollary 1 except that Othman and Sandholm seem to overlook the necessity of $s'(p, 1) > 0$ (which is equivalent to $f' \geq 0$ in our framework). Note that Othman and Sandholm use different names for scoring rules. In particular, they use “$f(p)’” for $s(p, 1)$ and “$g(p)’” for $s(p, 0)$.
who do not choose the best action, but, for instance, randomize over the best few options with probabilities depending on the expert’s prediction. Randomizing over all options in particular allows any proper scoring rule for mere prediction to be used to construct strictly proper scoring rules for decision making (Chen, Kash, Ruberry, et al., 2014, Section 4).

They also characterize a much larger class of scoring rules: they merely require that the expert honestly reports the probability distribution that the recommendation gives rise to and allow scoring rules which strictly incentivize misreporting what the best action is. For principals who (like those in our setting) choose the best action according to the expert’s report, their result is especially easy to derive and understand. To elicit an honest report about the probability distribution resulting from taking the expert’s recommended action, the principal can use any (strictly) proper scoring rule for mere prediction (as defined and characterized by, e.g., Gneiting and Raftery, 2007, Section 2). As an example, consider the quadratic (Brier) scoring rule discussed at the end of Section 2.1. Interest in this larger class of scoring rules is harder to motivate – usually the principal will primarily want to ensure that the best action is taken. Eliciting honest and accurate predictions about the best action will only be a secondary priority.

Chen, Kash, Ruberry, et al. (2014, Section 5) do also consider the goal of characterizing preferences over lotteries for which making the best recommendation can be incentivized. But they do not give a characterization of (right-action) proper scoring rules for utility-maximizing principals or of what information can be extracted along with the best action.\footnote{In earlier, work they mention in passing (Chen and Kash, 2011, Section 6) that a particular scoring rule – the one described in eq. 53 – strictly incentivizes the expert to not only recommend the best action but also the distribution resulting from taking that action. But this is not the case, as the scoring rule of eq. 53 depends only on the reported mean. More generally, we have shown in Lemma 2 that no right-action proper scoring rule can strictly incentivize honesty about anything other than the mean of the recommended action.}

That said, Chen et al.’s main result or Gneiting and Raftery’s characterization (or even the more general characterization of Frongillo and Kash, n.d.) could have replaced part of our proof. Notice that Corollary 5 is structurally very similar to these results. The only difference is that we impose additional constraints on the constituent functions, for instance, that $f$ only depends on the mean. Of course, our proof worked by first showing that only the reported mean matters and then deriving the structure of $s$. But alternatively we could have used Gneiting and Raftery’s characterization to first derive the structure without using the results from Section 2.2 and only then derived and incorporated the additional requirements necessary to ensure that $s$ elicits honest recommendations (and not only honest reporting of the distribution for the recommended action).

### 2.6.3 Eliciting under ex post revelation of the world state

Some authors have studied a setting like ours with the modification that the expert’s private information makes only predictions about the state of the world.
and that the state of the world is revealed after the action is taken (Boutilier, 2012; Carroll, 2019). This allows the principal to elicit any relevant information from the expert by using a regular scoring rule for prediction to score the expert on his predictions about the state. The main motivation for studying this setting – rather than the generic setting of making predictions without basing decisions on these predictions – is that the existence of a principal with preferences makes it clear how to prioritize between different goals of the mechanisms, e.g. between sizes of payments and quality of the elicited recommendations. Zermeño (2011, 2012) studies an interesting variation of this setup where only some actions reveal the state; it is shown that to set incentives, it can be prudent for the principal to sometimes take a suboptimal action to find out the true state.

2.6.4 Direct elicitation of properties

Typically, when designing scoring rules for prediction (without the recommendation component) the goal is to elicit entire probability distributions over outcomes. But a recent line of work has explored the direct elicitation of particular properties of the distribution without eliciting the entire distribution (e.g. N. Lambert, Pennock, and Shoham, 2008; Gneiting, 2011; Abernethy and Frongillo, 2012; Bellini and Bigi, 2015). Of course, in principle, one could elicit entire distributions and would thereby elicit all properties. But eliciting, say, a single-valued point forecast may be required “for reasons of decision making, market mechanisms, reporting requirements, communications, or tradition, among others” (Gneiting, 2011, Section 1). Lemma 2 gives another reason to study scoring rules for eliciting just the expected utility, albeit with the additional requirements that the expected score under honest reporting must be the same for two variables with equal mean (Lemma 1) and that the expected score under honest reporting must be increasing in the true mean of the random variable. The literature on property elicitation therefore provides yet another
way of deriving our main result.  

### 2.6.5 Principal-agent problems

One lens through which we can view our work is through that of information elicitation and mechanism design. Another is that of the principal-agent problem in economics (for overviews refer to Rees, 1985; Eisenhardt, 1989; Sappington, 1991; Laffont and Martimort, 2002). Generally, this literature considers a principal with some kind of project who pays the agent for some partially observable “work” or contribution to the principal’s project. The agent’s goals may be unaligned with those of the principal. In particular, the agent is generally assumed to want to invest as little effort as possible; needless to say, he also wants to maximize the payment made to him.

The principal-expert problem of this paper can be seen as a particular type of principal-agent problem in which the agent is an “expert” whose work consists in acquiring and revealing information. Indeed, this type of problem has been studied in the literature on the principal-agent problem (e.g. R. A. Lambert, 1986; Demski and Sappington, 1987; Stoughton, 1993; Core and Qian, 2002; Barron and Waddell, 2003; Feess and Walzl, 2004; Gromb and Martimort, 2007; Malcomson, 2009; Zermeño, 2011; Zermeño, 2012; Chassang, 2013; Carroll, 2019; Oesterheld and Conitzer, 2019). However, that literature differs substantially in its objectives from the present work. In accordance with the principal–agent framework, it focused on incentivizing costly information acquisition – whereas we have assumed (with the exception of Section 2.5.1) that the expert already possesses the information he reports. The problem posed in these papers is then which scoring rule (or contract) maximizes the principal’s payoff from her project minus payments made to the expert. The frameworks of these papers are often

---

For instance, consider Theorem 3 of N. Lambert, Pennock, and Shoham (2008), which characterizes continuously differentiable scoring rules for any elicitable property. Up to normalization, we can use as a “signature” of the expected utility, \( v(t, \omega) = u(\omega) - t \). So the class of scoring rules for the mean according to Theorem 3 of N. Lambert, Pennock, and Shoham (2008) is the class of functions \( s \), which can be written as

\[
s(\mu, y) = s_0(y) + \int_{\mu_0}^{\mu} \lambda(t)(y-t)dt,
\]

where \( \mu_0 \) is the lowest reportable mean and \( \lambda \) is some non-negative, continuous weight function. In general, \( s_0 \) can be an arbitrary function. But for \( s \) to be a proper scoring rule for decision making, \( s_0 \) has to be positive affine in \( y \).

Now take our characterization of good differentiable scoring rules for decision making in corollary 4. Here, it must be \( g'(\mu) = \mu f'(\mu) \). Integrating both sides yields

\[
g(\mu) = C + \int_{\mu_0}^{\mu} tf'(t)dt
\]

for some constant \( C \in \mathbb{R} \). Corollary 4 thus becomes

\[
s(\mu, y) = f(\mu)y - \int_{\mu_0}^{\mu} tf'(t)dt - C = \int_{\mu_0}^{\mu} f'(t)(y-t)dt + C'y - C,
\]

for some constant \( C' \geq 0 \). So using Theorem 3 of Lambert, Pennock, and Shoham gives the same result as ours.
more detailed (e.g., by incorporating risk aversion and limited liability) and less general (e.g., by assuming specific types of distributions or by not letting the expert submit a prediction). As noted earlier, some of these papers come to the same conclusion as Section 2.5.1, they, too, show (albeit by different means) that linear scoring rules (or contracts) are best-suited for aligning the expert with the principal (Stoughton, 1993; Diamond, 1998; Carroll, 2015; Chassang, 2013; Carroll, 2019; Dütting, Roughgarden, and Talgam-Cohen, 2019; Oesterheld and Conitzer, 2019).

3 Eliciting from multiple experts

3.1 Setting

Again, we consider a principal who selects from a set of options $A$. After she has taken an action, an outcome from $\Omega$ is obtained. The principal would like to select the action that maximizes the expectation of the value of some utility function $u: \Omega \rightarrow \mathbb{R}$. This time the principal consults $n$ different experts. Again, she asks for information, then takes the best action given the information submitted and finally rewards the experts based on the submitted information and the outcome obtained.

When it comes to the format and reporting of information, however, switching to the multi-expert setting poses a few additional challenges compared to the single-expert setting. The first (and easier to defuse) issue is that we need to adapt our definition of properness. What report maximizes an expert’s expected score depends, in general, on what the other experts report. In this paper, we go with one of the standard answers to this question from the mechanism design literature: Bayes-Nash equilibrium (or Bayes-Nash incentive compatibility). That is, we will call the scoring rule $s_i$ for expert $i$ proper if given that everyone else reports honestly, $i$ also weakly prefers reporting honestly.

The second challenge is trickier and a heavier burden to carry throughout this section: of what types are the beliefs and reports of the experts? Clearly, we cannot simply let each expert’s beliefs be some probability distribution $\Delta(\Omega)^A$ for there would be no principled way of aggregating the experts’ beliefs. Again, we make use of a standard solution from the literature: the common prior model.

We assume that each expert $i = 1, \ldots, n$ has a access to a private piece of information from some set $H_i$. The experts and principal share a common prior $Q$ over $H := \times_i H_i$ and a common posterior $P$, which for any action $a \in A$ and evidence vector $e \in H$ specifies a probability distribution $P(\cdot | a, e)$ over outcomes given that $e$ is observed and action $a$ is taken. As a report, each expert $i$ – after observing $e_i \in H_i$ – submits $\hat{e}_i \in H_i$ and the principal chooses an action

\[13\] Note that – although we will not prove this formally – the stronger goal of making honest reporting a dominant strategy (i.e., of making our mechanism dominant-strategies incentive-compatible) is unattainable. Roughly, if the other experts misreport, an expert may be able to improve his payoff by misreporting in a way that improves the quality of the recommendation and prediction resulting from the overall evidence.
that is best given the reported evidence, i.e. an action from

\[ \text{opt}_P(\hat{e}) := \arg \max_{a \in A} \mathbb{E}_P[u(O) \mid \hat{e}, a]. \]  \hspace{1cm} (75) \]

It would now be natural to consider as scoring rules functions from \( H \times \Omega \) to \( \mathbb{R} \), i.e. functions that take as input everyone’s report and the outcome obtained after taking \( \text{opt}_P(\hat{e}) \). But the set of allowed scoring rules for player \( i \) will then in general depend a lot on the structure of \( H, Q \) and \( P \) (cf. footnote 3). Instead, we will again work toward generic DSRs, i.e. ones that work without assuming anything about the beliefs that the experts could have.

To make our scoring rules generic, we have to return – partially – to scoring probability distributions. In particular, for the purposes of scoring expert \( i \), we imagine that expert \( i \) can recommend/believe in

- any probability distribution \( \hat{Q}_i \in \Delta(H_{-i}) \) over what evidence everyone else submits,
- any policy \( \hat{\alpha} : H_{-i} \to A \) mapping the others’ evidence onto a recommendation, and
- any probability distribution \( \hat{P} \in \Delta(\Omega)^{H_{-i}} \) that describes what outcomes to expect for any report of the other experts if expert \( i \)’s recommendation is implemented.

At the same time we imagine that all experts other than \( i \) still observe and submit (honestly) from \( H_{-i} \). For properness we then require that even with this much broader set of possible beliefs and reports, expert \( i \) always prefers honest reporting assuming that everyone else submits their \( H_{-i} \) honestly.

**Definition 3.** A multi-expert DSR is a collection \( (s_i)_{i=1, \ldots, n} \) of functions

\[ s_i : \Delta(H_{-i}) \times \Delta(\Omega)^{H_{-i}} \times \Delta(H) \times \Delta(\Omega)^{H \times A} \times H_{-i} \times \Omega \to \mathbb{R}, \]  \hspace{1cm} (76) \]

each of which maps

- expert \( i \)’s reported distribution over \( H_{-i} \),
- expert \( i \)’s reported distribution over \( \Omega \) under following expert \( i \)’s conditional recommendation,
- the prior \( Q \) over \( H \),
- the posterior \( P(\cdot \mid \cdot, \cdot) \) which specifies a distribution over outcomes for each evidence vector and action,
- the submitted evidence of the other experts, and

\footnote{This non-generic multi-expert setting reduces cleanly to the non-generic single-expert setting described in footnote 3. Since we have no characterization for the non-generic single-expert setting, though, this helps us little here.}
We say that $s_i$ is proper if for any priors $P, Q$ and true beliefs $Q_i \in \Delta(H_{-i})$, $P_i \in \Delta(\Omega)^{A \times H_{-i}}$, all possible reports $\hat{Q}_i \in \Delta(H_{-i})$, $\hat{P}_i \in \Delta(\Omega)^{H_{-i}}$ with recommended policy $\hat{\alpha}$

\[
\xi_{Q_i,P_i}(\hat{\alpha}, \hat{P}_i, \hat{Q}_i) \leq \max_{\alpha^* \in \text{opt}(P_i)} \xi_{Q_i,P_i}(\alpha^*, P_i(\cdot | \alpha^*), Q_i)
\]

(77)

where

\[
\xi_{Q_i,P_i}(\hat{\alpha}, \hat{P}_i, \hat{Q}_i) = E_{\hat{\alpha} \sim Q_i, \hat{P}_i \sim P(\cdot | \hat{\alpha}(e_{-i})), e_{-i} \sim \hat{E}_{-i}} \left[ s_i(\hat{Q}_i, \hat{P}_i, P, Q, e_{-i}, O) \right]
\]

(78)

is expert $i$’s expected score of reporting $\hat{\alpha}, \hat{P}_i, \hat{Q}_i$ under the belief $Q_i, P_i$ and opt$(P_i)$ is the set of optimum deterministic policies $H_{-i} \rightarrow A$ given the belief $P_i$. We say that $s_i$ is right-action proper if for all $P, Q, P_i, Q_i, \hat{Q}_i, \hat{P}_i$ as above it is

\[
\xi_{Q_i,P_i}(\hat{\alpha}, \hat{P}_i, \hat{Q}_i) = \max_{\alpha^* \in \text{opt}(P_i)} \xi_{Q_i,P_i}(\alpha^*, P_i(\cdot | \alpha^*), Q_i)
\]

(79)

only when $\hat{\alpha} \in \text{opt}(P_i)$. We call $s_i$ strictly proper w.r.t. the means if eq. 79 implies

\[
E_{\hat{P}_i} [u(O) | e_{-i}] = E_{\hat{P}_i} [u(O) | e_{-i}, \hat{\alpha}(e_{-i})].
\]

(80)

There are a few things to point out. First, notice that the score of $i$ depends only on the distribution given the recommended action. Predictions about what would happen if a suboptimal action were to be taken are not scored. And the score cannot depend on which action is recommended (other than through the outcome that the recommendation gives rise to). All of this can be justified in the same way as it is justified for the single-expert case.

The above definitions are somewhat schizophrenic. The score of expert $i$ can depend on the other experts’ reports $e_{-i} \in H_{-i}$ themselves. Moreover, DSRs can depend on the prior probability distributions $P$ and $Q$. This stands in contradiction to the goal of genericism! After all, it allows the scoring rule to know exactly what probability distributions the expert can submit. However, allowing dependences of this sort is necessary for many things we may like generic decision scoring rules to do in practice. For example, we may like to scale $i$’s reward up or down depending on the value of expert $i$’s information to the principal (cf. the scoring rule we propose below in eq. 90). We could apply some restrictions on how $s_i$ can depend on the priors $P, Q$; for instance, we may require that $s_i$ only depends on conditionals like $Q(\cdot | e_j)$ and not on the (unconditioned) prior. However, the general characterization needs no such restrictions.

Rather than the definition of a multi-expert DSR itself, the genericism in definition 3 lies primarily in the definitions of properness. For properness, we assume that the scoring rule (or the principal using that scoring rule) does not know (despite the information submitted by the other experts and the priors $P, Q$) what the possible reports $H_i$ of expert $i$ are or what beliefs these give rise
to. We require that $s_i$ sets good incentives even if the expert had beliefs that given $P, Q, e_{-i}$ he cannot have.

Multi-expert DSRs are (again) a type of direct-revelation mechanism for information elicitation. But in general, there are many other complex games one could set up to get the experts to reveal their information. For instance, much of the existing literature has been concerned with “decision markets”, in which experts repeatedly report probabilistic estimates or even trade some kind of securities (see Section 3.4). As is often the case in mechanism design, it turns out that every generic proper mechanism can be transformed into a generic (direct-revelation) multi-expert DSR with the same payoffs. Thus, the class of direct-revelation mechanisms in some sense characterizes the set of all proper mechanisms for eliciting information for decision making. Anyone familiar with revelation principles for other settings from the literature (e.g. see Shoham and Leyton-Brown, 2009 Section 10.2.2; Börgers, 2015 Section 3.2.2, 4.2.2, 10.6) will be able to derive a revelation principle for the current setting. At the same time, the revelation principle in our setting ends up somewhat cumbersome to state. For these reasons, we here omit a formal statement of the revelation principle.

3.2 Characterizing right-action proper multi-expert scoring rules for decision making

Definition 3 (in particular the underlying solution concept of Bayes-Nash equilibrium) make properness a property of single-player games – for judging whether $s_i$ is proper, we need only consider the case in which everyone else submits non-strategically. What’s more, this single-player game resembles the elicitation situations studied in Section 2: the expert makes a recommendation which the principal implements; the expert further submits a prediction; finally the expert is scored based on prediction and outcome. Unfortunately, we nevertheless cannot directly apply Theorem 3 to obtain a characterization of multi-expert DSRs. This is because the corresponding single-expert elicitation situations have some additional structure that we assumed not to exist in our characterization: we know that part of the outcome – namely the evidence submitted by the other experts $E_{-i}$ – is not affected by what action the principal chooses. Contrary to the general case, incentivizing an expert to reveal honest predictions about $E_{-i}$ along with good recommendations is easy and can be done using regular scoring rules for prediction (as characterized by, e.g., Gneiting and Raftery, 2007 Section 2).

To apply our existing results, consider for any DSR $s_i$ and any honest distribution $Q_i$ over $H_{-i}$ the function

$$
\tilde{s}_i(\hat{P}_i, Q_i, P, e_{-i}, \omega) = s_i(\hat{P}_i, Q_i, Q, P, e_{-i}, \omega).
$$

Here, $\hat{P}_i$ is any (reported) probability distribution over $\Omega$. Intuitively, $\tilde{s}_i$ is the scoring function resulting from $s_i$ if $Q_i$ was somehow already common knowledge or if the principal had some way of forcing expert $i$ to report $Q_i$ honestly. It is

15This is a standard point in mechanism design, see, e.g., Narahari (2014 Section 16.1).
left to incentivize $i$ to give honest recommendations and predictions conditional on each $e_{-i} \in H_{-i}$. We can view this as there being a separate decision for each $e_{-i} \in H_{-i}$. The agent gives a recommendation and a prediction for each of those decisions, but each of them is played only with some probability (which for now we assume to be known). It can be shown (e.g. by transferring the arguments of Section 2.2) that for $\hat{s}_i$ to be right-action proper, $\hat{s}_i$ can be written as a function of only the reported means $E_{Q \sim P_i} [u(O) \mid e_{-i}]$ for each $e_{-i}$ and the obtained utility. Write $\hat{\mu}_i$ for the collection of such reported means and $\hat{\mu}_e_{-i}$ for the mean for any particular $e_{-i} \in H_{-i}$. Then we can write $\hat{s}_i$ as $\hat{s}_i(\hat{\mu}_i, Q, P, e_{-i}, y)$.

For simplicity we will henceforth only consider functions $\hat{s}_i$ where $\hat{s}_i(\hat{\mu}_i, Q, P, e_{-i}, y)$ depends only on $\hat{\mu}_{e_{-i}}$ and none of the other parts of $\hat{\mu}_i$.

Theorem 3 then implies that for $\hat{s}$ to be proper it has to be

\[
\hat{s}_i(\hat{\mu}_i, Q, P, e_{-i}, y) = f_{Q,P,e_{-i}}(\hat{\mu}_e_{-i})(y - \hat{\mu}_{e_{-i}}) + h_{Q,P,e_{-i}}(\hat{\mu}_{e_{-i}}),
\]

where $f_{Q,P,e_{-i}}$ is non-negative and non-decreasing for every $e_{-i}, Q, P$ and

\[
h_{Q,P,e_{-i}}(\mu) = \int_0^\mu f_{Q,P,e_{-i}}(x)dx + C_{Q,P,e_{-i}}.
\]

Further, $s$ is right-action proper iff the $f_{Q,P,e_{-i}}$ is strictly positive and strictly proper w.r.t. the means iff $f_{Q,P,e_{-i}}$ are strictly increasing.

For any honestly reported and therefore also any dishonestly reported $Q_i$ over $H_{-i}$ we can therefore write $s_i$ as

\[
s_i(\hat{\mu}_i, Q_i, P, Q, e_{-i}, y) = f_{Q_i,P,e_{-i}}(\hat{\mu}_e_{-i})(y - \hat{\mu}_{e_{-i}}) + h_{Q_i,P,e_{-i}}(\hat{\mu}_{e_{-i}}),
\]

with the usual constraints on $f_{Q_i,P,e_{-i}}$ and $h_{Q_i,P,e_{-i}}$.

It is left to add to eq. 87 the requirement that honest reporting of $Q_{E_{-i}}$ is optimal. First notice that regardless of what $Q_i$ the expert reports, he is incentivized to report the means $\hat{\mu}_i$ honestly (potentially up to some interval)

\[\text{More generally, we could have } \hat{s}_i \text{ depend on the reported expected utility across multiple } e_{-i}. \text{ For instance, we could have a rule like}
\]

\[
\hat{s}_i(\mu_i, y) = f(\mu_i)(y - \mu_i) - g(\mu_i),
\]

where

\[
\mu_i := \sum_{e_{-i} \in H_{-i}} Q(e_{-i} \mid e_i)\hat{\mu}_{e_{-i}}
\]

is the overall mean implied by $\hat{\mu}_i$. Similarly, we could pick a partition of $H_{-i}$ into subsets $S$ across which we use the reported mean within $S$

\[
\mu_S := \frac{1}{Q(E_{-i} \in S \mid e_i)} \sum_{e_{-i} \in S} Q(e_{-i} \mid e_i)\hat{\mu}_{e_{-i}}
\]

for scoring the expert when $e_{-i} \in S$. This way, we can incentivize $\hat{\mu}_S$ to be the mean for the across $e_{-i} \in S$ without incentivizing the individual $\hat{\mu}_{e_{-i}}$ to be accurate.
s.t. when deciding which \( \hat{Q}_i \) to submit, he maximizes

\[
\mathbb{E}_{E_i \sim Q_i} \left[ s_i(\mu_i, \hat{Q}_i, P, Q, e_{-i}, y) \right]
= \sum_{e_{-i} \in H_{-i}} Q_i(e_{-i}) \left( \int_{0}^{\mu_{e_{-i}}} f_{Q_i, Q, P, e_{-i}, y}(x) dx + C_{\hat{Q}_i, Q, P, e_{-i}} \right),
\]

(88)

where \( Q_i \) is expert \( i \)'s honest probability distribution over \( H_{-i} \). Hence, \( s_i \) is (strictly) proper w.r.t. \( Q_{E_{-i}} \) iff for any \( \mu_i \)

\[
\int_{0}^{\mu_{e_{-i}}} f_{Q_i, Q, P, e_{-i}, y}(x) dx + C_{\hat{Q}_i, Q, P, e_{-i}}
\]

(89)

is a (strictly) proper scoring rule for prediction (as defined and characterized by Gneiting and Raftery, 2007).

We want to briefly give (without formal discussion or justification) an example of an appealing scoring rule for the multi-expert case to illustrate our characterization:

\[
s_i(\mu_i, \hat{Q}_i, P, Q, e_{-i}, y) = \alpha \text{VoI}_{P, Q}(E_i | e_{-i}) y - \beta,
\]

(90)

where \( \alpha, \beta \in \mathbb{R}, \alpha > 0 \) and \( \text{VoI}_{P, Q}(E_i | e_{-i}) \) is some measure of the value of information for the principal of finding out \( E_i \) given that \( e_{-i} \) is already known. Roughly, the idea here is to give each expert a number of shares that is proportional in how valuable that expert’s information is, given all the information we receive from other experts. Of course, this scoring rule is dependent on \( Q \) and \( P \), but only in their posterior forms. In practice, we may imagine that the reports of all experts but \( i \) together state what \( i \) is likely to report and how much that could change the principal’s beliefs. Some variations of this scoring rule are plausible, cf. Conitzer’s (2009, Section 4) discussion of similar scoring rules for prediction.

### 3.3 Notable impossibilities

By the standard revelation principle argument of mechanism design, we can use the results of the previous section to draw conclusions about what kind of characteristics any generic proper mechanism for eliciting information for decision making from multiple experts must have. For instance, it shows that we cannot incentivize experts to – along with an honest recommendation – reveal anything other than the expected utility of taking the recommended action. In this section, we consider a few other properties that we might want a mechanism to have but that as a consequence of our results we cannot obtain. Many of these properties are inspired by properties of prediction markets. This is in part because existing work in eliciting information for decision making from multiple experts has focused on designing prediction-market-like mechanisms, essentially “decision markets” (Hanson, 1999; Hanson, 2002; Berg and Rietz, 2003; Hanson, 2013, Section IV; Othman and Sandholm, 2010, Section 3; Chen, Kash, Ruberry, et al., 2014).
It is worth noting that we can get around all of the impossibility results of this section by removing some of our underlying assumptions or weakening our goals. For instance, they disappear if we allow randomization over which option we take (Chen, Kash, Ruberry, et al., 2011; Chen, Kash, Ruberry, et al., 2014, Section 4) or if we look for non-generic mechanisms for specific decision problems in which the principal has a detailed model of experts’ information structure. The impossibilities may therefore serve as motivation to study elicitation for decision making on different assumptions or desiderata. Genericism and deterministic choice of the best action are desirable properties. But the limitations they imply may outweigh their benefits.

3.3.1 Rewarding big updates

We might like to reward experts in proportion to how much their report updates the principal’s beliefs. This is one of many desirable properties of prediction markets: experts (or “traders”) are rewarded based on how far they can move the “market probabilities” (the probabilities resulting from the combined evidence of all traders) toward the truth. For example, an expert who at any point simply agrees with the market probabilities can earn no money (in expectation). An expert who updates the probabilities from, say, $0.5 \cdot \omega_1 + 0.5 \cdot \omega_2$ to $0.1 \cdot \omega_1 + 0.9 \cdot \omega_2$ makes a large profit in expectation (assuming $0.1 \cdot \omega_1 + 0.9 \cdot \omega_2$ represents his true beliefs over the outcome of the random event). Rewarding experts for their impact on the final distribution has many advantages. For instance, it sets a natural incentive to acquire relevant information. Therefore, we might want an elicitation mechanism for decision making – perhaps a kind of “decision market” (see Section 3.4) – that similarly rewards experts for submitting evidence that yields large (justified) changes in the principal’s beliefs.

However, from our results it follows immediately that a number of types of changes cannot be rewarded at all. An expert’s score cannot depend on how much the trader’s report moves the distributions for suboptimal actions. Experts also cannot be rewarded for changing the recommended action. Generally, if two pieces of information $e_1 \in H_1, e_2 \in H_2$ have the same implications for the expected utility given the best action, the expert makes the same money in expectation from honestly reporting $e_1$ and honestly reporting $e_2$. This is the case even if $e_1$ affects what the best action is and implies wild changes to the distributions of all actions; while $e_2$ may not change the principal’s beliefs at all.

How can the principal make sure that despite these impossibilities, experts with more useful information receive higher scores? The only way out, it seems, is to reward experts based on the \textit{ex ante} value of their information in the style of eq. 90. That is, pay expert $i$ (in shares or constant reward) in proportion to how much the principal would be willing to pay to learn $E_i$. One could also

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17That said, our proofs do not seem to rely on the usual pathological edge cases. For example, a typical problematic case for information aggregation is that of fully (or even partly) complementary information (Chen and Waggoner, 2017; Geanakoplos and Polemarchakis, 1982, Proposition 3). So we may be willing to forego guarantees of information aggregation in these cases. But none of our proofs rely on cases of fully complementary information.
use the willingness to pay given that one already knows $E_{-i}$ or given that one will know $E_{-i}$. This way, obtaining $E_i$ is incentivized to the extent that $E_i$ is useful to the expert. The practical mechanism proposed below in Section 3.5 implements this strategy.

3.3.2 Ungated participation

Another desirable property of prediction markets is that we can offer anyone to participate without having to worry that we lose money to the vast number of potential participants who do not have any relevant information. Any participant who does not know more than the market can make no money at all. This also means that – whatever opinions the participants other than $i$ may have about $i$’s information – $i$ can make money if he disagrees with the market.

In generically right-action proper mechanisms as we have characterized them, participation will generally be a privilege even for those who – relative to the other experts – do not have any additional (decision- or otherwise relevant) information. (That is, assuming that the expected reward of honest reporting is never negative.) Hence, we cannot allow large numbers of ignorant experts to participate in generically right-action proper mechanisms.

That said, if we slightly weaken some of our goal of generic right-action properness, we might get around this limitation. Generic right-action properness demands that regardless of what the other experts submit, expert $i$ strictly prefers to report honestly. But we might allow that if all other experts believe that expert $i$’s information is of no use to decision making, we are indeed allowed to treat expert $i$ differently. There are a few options. For one, we could just use the type of scoring rules we have considered so far but allow $f_{\hat{Q},Q,P,e_{-i}}$, and thereby the expected score under honest reporting to be zero on some submitted means. In the strongest form, we could just let $f_{\hat{Q},Q,P,e_{-i}} = 0$ and ignore $i$’s submitted evidence (or belief) if given $e_{-i}$ player $i$’s evidence is uninformative. Arguably, this is just an alternative way of imposing a constraint on participation, as an expert who disagrees with the aggregated beliefs of the others still cannot profit from this disagreement. Alternatively, we could let $f_{\hat{Q},Q,P,e_{-i}}$ only be positive and our choice only be dependent on expert $i$’s recommendation if $i$ reports that some action has a higher utility than what is promised for the other experts’ current recommendation. This way, if an expert feels that he has a better plan than the other experts, he is incentivized to reveal that plan. If he indeed has nothing to add to the beliefs of the other experts, he cannot make any profit, as desired. The “realistic” mechanism described in Section 3.5 implements this partial version of right-action properness.

3.4 Related work

As far as we are aware, most work on eliciting decision-relevant information from multiple agents has focused on designing prediction-market-like mechanisms or “decision markets” (as opposed to considering the class of direct-revelation mechanisms discussed in this paper) (e.g. Hanson, 1999 Hanson, 2002 Berg and
Rietz, 2003; Hanson, 2006; Hanson, 2013 Section IV). Othman and Sandholm (2010 Section 3) are the first to point out incentive problems with this model. Our impossibility results can be seen as an extension of their result (though we have limited attention to mechanisms which guarantee full information aggregation, which may not be a primary goal for the design of decision markets). Inspired by Othman and Sandholm’s proof that decision markets sometimes set poor incentives, Teschner, Rothschild, and Gimpel (2017) conduct an empirical study in which human subjects took the roles of the experts (or “traders”) to show that strategic reporting may not be a problem in decision markets in practice. Chen et al. (2011; 2014) show that by randomizing over all options (while potentially giving all but \(\epsilon\) probability mass to the best action) decision markets can, in some sense, be made to be analogous to prediction markets.

3.5 Decision auctions – a proposal for “realistic” mechanisms for eliciting decision-relevant information

Inspired by a revelation principle, we have characterized the class of generic right-action proper direct-revelation mechanisms. But direct-revelation mechanisms are generally impractical. The expert’s knowledge may be tacit and therefore not easily transferable to the principal in the form of a “report”. Even if the reports are submitted, the principal may not be able to aggregate them using the postulated common prior. Cf. Wilson’s doctrine which asks that the mechanism itself does not make use of the imagined common prior (Wilson, 1985 Section 6; cf. Shoham and Leyton-Brown, 2009 Section 11.1.8). In line with this doctrine, prediction and decision markets as they are usually discussed do not involve experts revealing their evidence directly (though consider Conitzer, 2009). Besides Wilson’s doctrine, another plausible demand for realism that prediction markets satisfy is resemblance with real-world security markets. In this section, we would like to briefly sketch a mechanism for eliciting for decision making that satisfies both of these requirements. To differentiate it from existing prediction-market-like decision markets, we call it a decision auction. It goes as follows.

The principal runs a series of (e.g. second-price sealed-bid) auctions. In each round, the item on sale is a share in the principal’s project, which means receiving some dollar amount per unit of utility achieved by the principal. We may allow shares to be re-sold according to any procedure. In particular, the shareholders should be able to give away shares for free to specific experts in order to incentivize them to reveal decision-relevant information. But the auction must be the only way in which such shares can be created – remember the no-short-selling conclusion drawn in Section 3.3.

The series of auctions ends at some point. For instance, the principal may only sell some fixed number of shares (cf. Section 2.5.2) or wait until prices

\[\text{More generally, the items on sale could be assets which allow the expert to be scored according the scoring rules we characterized in Theorem 3. We can view this as selling stock options rather than selling shares themselves. See Section 2.4 on background for these interpretations of Theorem 3.}\]
converge (across time and experts/bidders). At this point, the shareholders – all those who have bought shares – convene to share their information and collectively decide which action to take. To prevent any bargaining (e.g. via threats not to reveal relevant information), one may forbid the shareholders from making transactions amongst each other at this point. Note that the shareholders’ incentives are now aligned with each other and with maximizing the principal’s expected utility. So in theory it does not matter much how the shareholders reveal their information to each other or which of the many voting rules (see, e.g., Shoham and Leyton-Brown, 2009, ch. 9 for an overview) they use to decide which action to take.

This mechanism satisfies Wilson’s doctrine: the principal need not know the common prior. While the mechanism may involve direct revelation of information at the stage where shareholders make a decision, “the burden of coping with the complexity of the common knowledge features is assumed by the traders [or experts] in the construction of their strategies” (Wilson, 1985, Section 6). Like prediction markets, decision auctions resemble a real-world market institution: joint stock companies or corporations. There are some differences, however. For instance, the stock market allows short-selling, which our model has to forbid lest a trader with decision-relevant information ends up holding a short position. Moreover, it is somewhat unclear to which extent real-world corporations are controlled by shareholders.

Let us ask whether decision auctions fully aggregate available information. First, it is not so clear whether the iterated auction of shares aggregates information about the value of the shares (and therefore the true expected utility of the principal). For general discussions of information aggregation in common value auctions, see Pesendorfer and Swinkels (1997) and Kremer (2002). Of course, information aggregation on this level is not our main goal, anyway. (If the principal auctions off options rather than shares, experts report means later on; also, we care more about taking the best available action than about knowing the expected utility of our decision.) However, it may be beneficial to the way the auction works if the auction prices come to represent everyone’s believed

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19In practice (at least in the US), most corporations are controlled by the CEO and board of directors; while the shareholders can theoretically elect the board of directors, they generally have little practical power and exercise their control rights sparsely, partly on the assumption that the directors have more expertise (Black, 1990, Velasco, 2006, Section I.A.2, II.B). Black (1992) argues that shareholder control of a company is beneficial and that legislation should be changed to make it easier for shareholders to collectively control their jointly-owned corporation. For the most part, the intended role of the shareholders would be that of meta-managers who ensure that, for instance, the CEO’s incentives are aligned with that of the company (Black, 1992, Section III.B). The same could be true in our model. The available actions may be electing a different board of directors or the like. That said, the assumption that the shareholders are competent at implementing whatever measures are necessary to incentivize the experts is left implicit by, e.g., Black (1992, Section III.B).

Of course, depending on the exact contract determining their compensation, we could model employees as a kind of “shareholders” as well, as they, too, contribute information in decision-making processes and are sometimes rewarded as if they had a share in the company. Some employees may even literally own part of the company. However, instead of paying for their “shares” with money, they pay for them with work (Velasco, 2006, Section IV.A).
value of a share.

Second, the most relevant type of information aggregation is still done “manually” in that we have the shareholders “discuss” and vote on what action to take as opposed to having them express their beliefs about which action is best via bets. While shareholders have a strong monetary incentive to identify and vote for the best option, within a single decision better-informed shareholders \textit{ceteris paribus} do not make more money than those who are uninformed. As noted in Section \ref{subsec:inform}, our characterization seems to show that there is no way around this – two otherwise identical experts cannot be rewarded differently based on how much impact they have on which action is selected. Unfortunately, group deliberation is often unreliable even if group members have a stake in the decision (Sunstein, \citeyear{sunstein2006}), ch. 2).

Finally, let us ask whether all decision-relevant information will be revealed. Of course, there will generally be experts with decision-relevant information who do not acquire any shares (e.g., because they know little about what a share is worth and therefore bid low in the auction). However, decision auctions ensure (somewhat trivially) that any expert ends up with shares if he is believed to have decision-relevant information (cf. Section \ref{subsec:inform}): To maximize the value of their shares, shareholders will give away shares for free to experts who have decision-relevant information but who do not acquire shares themselves. In our setting, neither acquiring nor reporting information is associated with a cost. Hence, the shareholders can give away individual or even fractional shares (which pay \(\epsilon\) per unit of utility achieved). In practice, shareholders would potentially need to give away larger numbers of shares to offset acquisition and reporting costs.

Besides this arguably trivial procedure, there are other ways in which an expert \(i\) with decision-relevant information will have an advantage in obtaining shares (even if his information provides no evidence about the principal’s expected utility). For instance, imagine there were some equilibrium in which \(i\) ends up without shares. Then one would expect the price at which the auction sells shares to reflect this lack of information. But then if \(i\) has a sufficiently good understanding of the situation, he has an opportunity to outbid other experts in the auction, because he knows the shares will be worth more than what they currently sell at if he becomes a shareholder.

Further research is needed to figure out whether and if so in which form decision auctions are a viable mechanism for eliciting information for decision making. Experimental work has shown that prediction markets perform well at aggregating information relative to other mechanisms (see, e.g., Wolfers and Zitzewitz, \citeyear{wolfers2004}; Sunstein, \citeyear{sunstein2006} ch. 4, Section “Practice and Evidence”); and similar work has already been done for decision \textit{markets} (Teschner, Rothschild, and Gimpel, \citeyear{teschner2017}). The same type of experimental study could be done on decision auctions. On the more theoretical side, we may hope to obtain results about information aggregation under some assumptions about the information structure analogous to results for prediction (and other information) markets (like those of Feigenbaum et al., \citeyear{feigenbaum2005} Chen, Dimitrov, et al., \citeyear{chen2010} Iyer, Johari, and Johari, \citeyear{johari2010} Ostrovsky, \citeyear{ostrovsky2012}).
4 Conclusion

We have characterized mechanisms for eliciting decision-relevant information from both individual and multiple experts, thus – we hope – laying the foundations for theories analogous to those of proper scoring rules and prediction markets. We have shown that to strictly incentivize optimal recommendations the principal can only set very limited incentives on the experts’ predictions: only the report on the expected utility of taking the recommended action can be strictly incentivized. Moreover, the right-action proper scoring rules allow for a simple, intuitive interpretation. In the single-expert case, the principal offers the expert at different prices shares in the principal’s project. From the price above which the expert rejects the offers, it can be inferred what the expert believes the expected value of a share and therefore the expected utility of the principal’s project to be. Further, since the expert owns a share in the project, he wants to give the best-possible recommendation. A similar result holds for a setting with multiple experts. By the usual revelation principle, our characterization tells us how any right-action proper mechanism for eliciting decision-relevant information must look like. Many desirable properties cannot be obtained. For instance, short-selling of the principal’s project – as is possible in prediction markets – cannot be allowed in a right-action proper mechanisms. The most natural object of further research therefore is the search for a natural, practical equivalent of a (Arrow-Debreu securities or market scoring rule) prediction market in the decision making case. We propose the decision auction as one candidate. We also outline what kind of work should be done to evaluate its practicality: detailed theoretical analysis and empirical work, such as lab experiments with human subjects as experts and comparisons with real-world market mechanisms. But it is also worth thinking of other mechanisms.

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