A $(1+\varepsilon)$-Approximation Algorithm for 2-Line-Center *

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Abstract

We consider the following instance of projective clustering, known as the 2-line-center problem: Given a set S of n points in $\mathbb{R}^2$, cover S by two strips so that the maximum width of a strip is minimized. Algorithms that find the optimal solution for this problem have near-quadratic running time. In this paper we present an algorithm that computes, for any $\varepsilon > 0$, a cover of S by 2 strips of width at most $(1+\varepsilon)w^*$, in $O(n \log n + n/\varepsilon^3 \log(1/\varepsilon))$ time.

1 Introduction

Problem statement and motivation. The 2-line-center problem is defined as follows: Given a set S of n points in $\mathbb{R}^2$, cover S by two strips so that the maximum width of a strip is minimized. This is a special case of *projective clustering.* A projective clustering problem is typically defined as follows. Given a set S of n points in $\mathbb{R}^d$ and two integers $k < n$ and $q \leq d$, find k q-dimensional flats $h_1, \ldots, h_k$ and partition S into k subsets $S_1, \ldots, S_k$ so that

$$\max_{1 \leq i \leq k} \max_{p \in S_i} d(p, h_i)$$

is minimized. The k-line-center problem is the projective clustering problem for $d = 2$ and $q = 1$. That is, we partition S into k clusters and each cluster S_i is projected onto a line (hence the name "k-line-center") so that the maximum distance between a point p and its projection p^* is minimized. Other objective functions have also been proposed [8] for projective clustering. Projective clustering has recently received attention as a tool for creating more efficient nearest neighbor structures, as searching amid high dimensional point sets is becoming increasingly important; see [1] and references therein.

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Previous results. Several algorithms with near-quadratic running time are known for covering a set of n points in the plane by two strips of minimum width; see [9] and references therein. It is an open problem whether a sub-quadratic algorithm exists for this problem. For $k = 1$, projective clustering is the classical *width problem.* The width of a point set can be computed in $\Theta(n \log n)$ time[1] for $d = 2$ [7, 11], and in $O(n^{3/2+\varepsilon})$ expected time for $d = 3$ [3]. Duncan *et al.* [5] gave an algorithm for computing the width approximately in higher dimensions. See also [4].

For the general problem of computing k projective clusters, few theoretical results are known. Meggido and Tamir [12] showed that it is NP-complete to decide whether a set of n points in the plane can be covered by k lines. This immediately implies that projective clustering is NP-Complete even in the planar case. In fact, it also implies that approximating the minimum width within a constant factor is NP-Complete. Agarwal and Procopiuc [2] propose an algorithm with near-linear running time that computes a cover by $O(k \log k)$ strips of width no larger than the width of the optimal cover by k strips. The algorithm extends to covering points by hyper-cylinders in $\mathbb{R}^d$ and to a few special cases of covering points by hyper-strips in $\mathbb{R}^d$. See also [6] for a recent improvement on the running time. Monte Carlo algorithms have been developed for projecting S onto a single subspace [8].

Our result. Let w^* denote the minimum value so that S can be covered by two strips of width at most w^*. We present an algorithm that computes, for any $\varepsilon > 0$, a cover of S by two strips of width at most $(1 + \varepsilon)w^*$, in $O(n \log n + n/\varepsilon^3 \log(1/\varepsilon))$ time.

The paper is organized as follows. In Section 2 we introduce a few definitions and prove a result that is later used in our algorithm. Our approximation algorithm for the 2-line-center problem is described in Section 3; we begin by presenting a 6-approximation algorithm and then use it to derive our $(1 + \varepsilon)$-approximation algorithm.

2 Preliminaries

A *strip* σ in the plane is the region lying between two parallel lines ℓ_1 and ℓ_2. The *width* of σ is the distance between ℓ_1 and ℓ_2, and the *direction* of σ is the direction of ℓ_1 and ℓ_2. A set Σ of two strips is called a *strip cover* of S if each point of S lies in one of the strips of Σ. The *width* of Σ is the maximum width of a strip in Σ. A strip cover Σ is *optimal* if its width is minimum among all strip covers of S.

For any pair of points p, q, let ℓ_{pq} denote the line passing through p and q. If $p = q$ then ℓ_{pq} is the horizontal line through p. For any three, not necessarily distinct, points p, q, r in the plane, we denote by $\sigma(p, q, r)$ the strip having ℓ_{pq} as the median line and of width $2 \cdot d(r, \ell_{pq})$. If $r \in \ell_{pq}$, $\sigma(p, q, r)$ is the same as ℓ_{pq}. We also use the notation $\sigma(p, q; w)$ to denote the strip of width $2w$ whose median line is ℓ_{pq}.

Let $\Sigma^* = \{\sigma_1^*, \sigma_2^*\}$ be an optimal cover of S. For the remainder of this paper, whenever we refer to an optimal cover of S, we mean Σ^* (although S may have other optimal covers as well). We define the *strip subsets* of S to be the (not necessarily disjoint) sets $S_i^* = S \cap \sigma_i^*$.

For a strip σ, we call a pair of points $p, q \in S \cap \sigma$ an *anchor pair* of σ if $d(p, q) \geq$

[1] The base of all logarithms is 2, unless otherwise specified.

diam$(S \cap \sigma)/2$. The following lemma was proved in [2]. We repeat the proof here as it will be useful later on.

Lemma 2.1. *Let $\sigma^* \in \Sigma^*$, and let (p,q) be an anchor pair of σ^*. Then there exists a point $r \in S$ so that $\sigma(p,q,r)$ covers all points of $S \cap \sigma^*$ and $d(r, \ell_{pq}) \le 3w^*$.*

Proof: Let $w \le w^*$ be the width of σ^*, $S^* = S \cap \sigma^*$, and Δ be the diameter of S^*. Define ρ to be the smallest rectangle containing S^* that has two edges lying on the boundaries of σ^* (see Figure 1; ρ is the shaded area). We denote by v_1, v_2, v_3 and v_4 the four vertices of ρ in clockwise order. The width of ρ is w. Let L be the length of ρ. Since the two sides of ρ that are perpendicular to the direction of σ^* must each pass through a point of S^*, $L \le \Delta$. Let σ' be the thinnest strip in direction parallel to the line ℓ_{pq} that contains ρ. The boundaries of σ' are tangent to ρ. Without loss of generality, assume that $\partial\sigma'$ touches ρ at v_2 and v_4. We denote by w' the width of σ', and by $w_1,\ w_2$ the distances from v_1 to the boundaries of σ'. Using the notations of Figure 1, we deduce:

$$w' = w_1 + w_2 \le w + L\sin\alpha \le w + \Delta \cdot \frac{2w}{\Delta} = 3w.$$

We choose $r \in S^*$ to be the point that is farthest away from ℓ_{pq}. Since $r \in \rho$, $d(r, \ell_{pq}) \le 3w$. Moreover, $\sigma(p,q,r) \supset S^*$, and the lemma follows. □

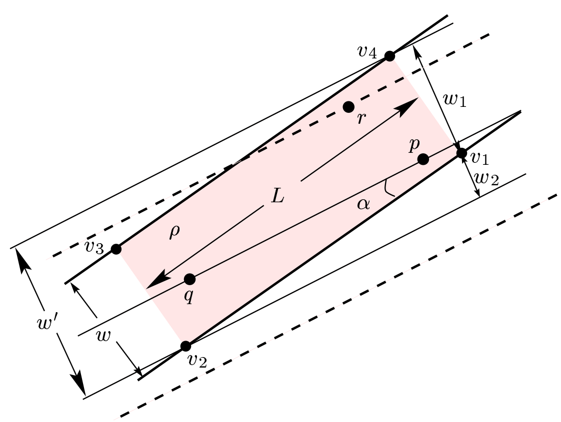

Figure 1: Finding a strip $\sigma(p,q,r)$ (dashed boundaries) that covers $S \cap \sigma^*$.

3 Approximation Algorithm for 2-Line-Center

We describe an algorithm that, given any $\varepsilon > 0$, computes in $O(n\log n + n/\varepsilon^3 \log(1/\varepsilon))$ time two strips of width at most $(1+\varepsilon)w^*$ that cover S. The algorithm works in two phases. The first phase computes a cover Σ of S by two strips of width at most $6w^*$. We then use Σ to compute a new cover of S by two strips of width at most $(1+\varepsilon)w^*$. Each of these steps is detailed below.

3.1 Computing a 6-approximate cover

We first describe an $O(n \log n)$ algorithm for computing a strip cover of width at most $6w^*$, provided that we have an anchor pair (p, q) of a strip in Σ^*. In the next subsection we present an $O(n \log n)$ algorithm for computing a family of at most 11 pairs of points that is guaranteed to contain such an anchor pair.

Without loss of generality, assume that (p, q) is an anchor pair of σ_1^*. By Lemma 2.1 there exists $r \in S$ so that $\text{width}(\sigma(p, q, r)) \leq 6w^*$ and $(S \setminus \sigma(p, q, r)) \subseteq \sigma_2^*$. We will perform a binary search to find such a point r and will use the algorithm by Duncan *et al.* [5] to compute a strip of width at most $2w^*$ that contains $S \setminus \sigma(p, q, r)$. We need the following result to perform the binary search.

For any $w \geq 0$, let $f(w) \leq 2 \cdot \text{width}(S \setminus \sigma(p, q; w))$ be the width of the strip computed by the 2-approximation algorithm by Duncan *et al.*on the set $S \setminus \sigma(p, q; w)$; $f(w)$ is a monotonically decreasing function of w. Set $g(w) = \max\{2w, f(w)\}$. For any given w, $g(w)$ can be computed in $O(n)$ time.

Lemma 3.1. *$g(w)$ is a unimodal function.*

Proof: Let $W = \langle w_i = d(r_i, \ell_{pq})\} \mid r_i \in S \rangle$ be the sequence of distances from points to the line ℓ_{pq}, sorted in a nondecreasing order. The value of $f(w)$ remains the same for all w in an itnerval (w_i, w_{i+1}), and $f(w_n) = 0$. Let w_i be the smallest value in W so that $2w_i \geq f(w_i)$, i.e. $g(w) = f(w_j)$ for $j < i$ and $g(w) = 2w_j$ for $j \geq i$. Then $\langle g(w_1), \ldots, g(w_i) \rangle$ is a monotonically decreasing sequence and $\langle g(w_{i+1}), \ldots, g(w_n) \rangle$ is a monotonically increasing sequence. Hence $g(w)$ is a unimodal function. □

Since $g(\cdot)$ is unimodal and $g(w)$ can be computed in $O(n)$ time for any w, $\min_{w \in W} g(w)$ can be computed in $O(n \log n)$ time by performing a binary search on W. Let $w_i \in W$ be a value for which $g(w)$ is minimized. We return the strip $\sigma(p, q; w_i)$ and the strip computed by the Duncan *et al.*algorithm on $S \setminus \sigma(p, q; w_i)$. We thus obtain the following.

Lemma 3.2. *If (p, q) is an anchor pair then the algorithm described above computes a* 6*-approximation of the optimal cover in $O(n \log n)$ time.*

3.2 Computing an anchor pair

We show how to compute a family $\mathcal{F}$ of at most 11 pairs of points that contains an anchor pair. Our method works as follows (refer to Figure 2):

Compute the *diameter* Δ of S, and let (p, q) be a diametral pair in S. Let $\mathcal{D}_p, \mathcal{D}_q$ be the disks of radius $\Delta/2$, centered at p, respectively q.

Case 1. If $S \setminus (\mathcal{D}_p \cup \mathcal{D}_q) \neq \emptyset$, let $r \in S \setminus (\mathcal{D}_p \cup \mathcal{D}_q)$. Return $\mathcal{F} = \{(p, q), (p, r), (q, r)\}$.

Case 2. Otherwise: Let $P = S \cap \mathcal{D}_p$ and $Q = S \cap \mathcal{D}_q$. We compute the convex hulls $\text{conv}(P)$ and $\text{conv}(Q)$ of P and Q, respectively. Note that these hulls do not intersect. Compute ℓ_1 and ℓ_2, the two lines that are inner common tangents to $\text{conv}(P)$ and $\text{conv}(Q)$. Let $p_1 \in P$ (resp. $p_2 \in P$) and $q_1 \in Q$ (resp. $q_2 \in Q$) be the points lying on ℓ_1 (resp.

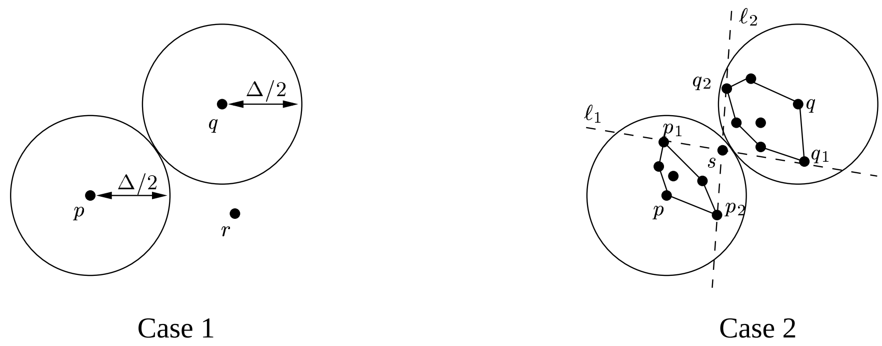

Figure 2: Finding a pair of anchors.

ℓ_2). Let p_3, p_4 be a diametral pair in P, and q_3, q_4 be a diametral pair in Q. Return $\mathcal{F} = \{(p,q), (p_3,p_4), (q_3,q_4)\} \cup \bigcup_{i=1}^{4} (p, q_i) \cup \bigcup_{i=1}^{4} (q, p_i)$.

LEMMA 3.3. *The above algorithm computes in $O(n \log n)$ time a family $\mathcal{F}$ of at most* 11 *pairs of points that contains at least one anchor pair of a strip in Σ^*.*

Figure 3: Analyzing Case 2 in the proof of Lemma 3.3.

Proof: The only non-trivial step of the algorithm is computing the diameters of three sets of at most n points, which can be done in $O(n \log n)$ time (see, e.g., [13]).

Case 1. At least two points among p, q, and r must be in the same strip subset. Since $d(p,q) = \Delta$ and $d(p,r), d(q,r) \geq \Delta/2$, at least one of the pairs in $\mathcal{F}$ is an anchor pair.

Case 2. Suppose on the contrary that no pair of $\mathcal{F}$ is an anchor pair. Let $S_{12}^* = S_1^* \setminus S_2^*$ and $S_{21}^* = S_2^* \setminus S_1^*$. Our assumption implies that S_{12}^* (resp. S_{21}^*) contains either p or q but not both. Without loss of generality, let $p \in S_{12}^*$ and $q \in S_{21}^*$. Since $d(p, q_i), d(q, p_i) \geq \Delta/2$, $1 \leq i \leq 4$, the assumption also implies $p_i \in S_{12}^*$ and $q_i \in S_{21}^*$ for every $1 \leq i \leq 4$. Suppose that $S_{12}^* \cap Q = \emptyset$. Then $S_{12}^* \subset P$, and (p_3, p_4) is an anchor pair, a contradiction. A similar contradiction occurs if we assume $S_{21}^* \cap P = \emptyset$.

Therefore, there exist points $p' \in S_{12}^* \cap Q$ and $q' \in S_{21}^* \cap P$. Let s be the intersection point of ℓ_1 and ℓ_2. Since the strip σ_2^* contains q_1, q_2, and q', it also contains the triangle $\Delta q_1 q_2 s$. Hence, $p' \notin \Delta q_1 q_2 s$. On the other hand, p' lies in the wedge formed by the rays $\overrightarrow{sq_1}$ and $\overrightarrow{sq_2}$, therefore triangle $\Delta p_1 p_2 p'$ intersects the segment $q_1 q_2$ (Figure 3 (a)). Let x be any point in this intersection. Since σ_1^* contains p_1, p_2, and p', it also contains $x \in q_1 q_2$. But q_1 and q_2 do not lie inside σ_1^*, so we deduce that σ_1^* separates q_1 and q_2. By a symmetric argument, we conclude that the strip σ_2^* separates p_1 and p_2. This implies that the interiors of the segments $p_1 p_2$ and $q_1 q_2$ intersect in a point $\xi \in \sigma_1^* \cap \sigma_2^*$ (Figure 3 (b)). Since $p_1, p_2 \in \mathcal{D}_p$ and $q_1, q_2 \in \mathcal{D}_q$, it follows that ξ lies in the interior of both $\mathcal{D}_p$ and $\mathcal{D}_q$. But this is impossible because $\mathcal{D}_p$ and $\mathcal{D}_q$ are tangent to each other. □

We thus conclude the following.

THEOREM 3.4. *For any set S of n points in the plane, we can compute a cover of S by two strips of width at most $6w^*$ in $O(n \log n)$ time.*

3.3 Computing a $(1+\varepsilon)$-approximate cover

Let $\tilde{w} \le 6w^*$ be the width of the cover computed by the previous 6-approximation algorithm. As before, we describe the algorithm for a fixed anchor pair (p, q) of a strip in Σ^*. The overall algorithm then iterates the procedure over all pairs in $\mathcal{F}$.

We apply a transformation to S so that ℓ_{pq} oriented from p to q becomes the $(+x)$-axis. Let R be the rectangle containing p and q and bounded by the following four lines: the two horizontal lines at distance $3\tilde{w}$ from ℓ_{pq}, and the two vertical lines at distance $4d(p,q)$ from the mid point of the segment pq. Intuitively, our approach is as follows. Let $\sigma^* \in \Sigma^*$ be the strip for which (p,q) is an anchor pair. We try to "guess" (within a small error) one of the intersection points of the lower boundary of σ^* with R. We then "guess" the direction of σ^* and the value w^*. For a fixed guess, we draw the corresponding strip σ and compute the thinnest strip σ' that covers the remaining points. If our guess is correct, then σ and σ' have width at most $(1+\varepsilon)w^*$. We prove below that it is sufficient to guess the intersection point, the direction, and the value w^* from three small sets, each of size $O(\varepsilon^{-1})$.

Let $\delta = C\varepsilon$, where C is a constant to be specified later. Draw a grid on the boundary of R, so that there are $\lceil 1/\delta \rceil$ grid points on each of the four sides. Grid points on the same side are equidistant, and the lower left corner of R is a grid point; see Figure 4. Let $\mathcal{Z}$ denote the set of grid points.

Let $\theta \in [0, \pi/2]$ be such that $\sin\theta = \min\{1, \tilde{w}/d(p,q)\}$. Let

$$\Gamma = \{\gamma_i = (i - \lceil 1/\delta \rceil)\delta\theta \mid 0 \le i \le 2\lceil 1/\delta \rceil\}$$

be a set of uniformly placed orientations in the range $[-\theta, \theta]$. Let $\tilde{W} = \{(1 + i\varepsilon/2)\tilde{w}/6 \mid 0 \le i \le 22/\varepsilon\}$ be a set of $O(1/\varepsilon)$ equidistant points in the interval $[\tilde{w}/6, 2\tilde{w}]$.

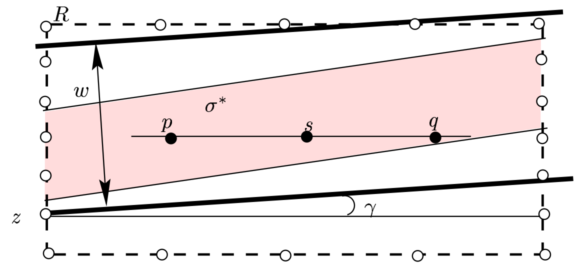

Figure 4: Computing rectangle R (dashed lines), grid points $\mathcal{Z}$ (empty circles), and a strip $\xi(z,\gamma,w)$ (bold solid lines); σ^* is represented shaded.

We approximate the left intersection point of the lower boundary of σ^* by a point in $\mathcal{Z}$ and the direction of σ^* by an angle in Γ. For any $z \in \mathcal{Z}$, $\gamma \in \Gamma$, and $w \in \tilde{W}$, let $\xi(z,\gamma,w)$ be the strip of width w whose lower boundary passes through z and makes angle γ with ℓ_{pq} (see Figure 4). We prove that there exist $z' \in \mathcal{Z}$, $\gamma' \in \Gamma$, and $w' \in \tilde{W}$ such that $w' \leq (1+\varepsilon)w^*$ and $S \cap \sigma^* \subseteq \xi(z',\gamma',w')$. Assuming that we know z' and γ', we compute w' by performing a binary search on $\tilde{W}$. We also use the $(1+\varepsilon)$-approximation algorithm by Duncan *et al.* [5] to compute a strip of width at most $(1+\varepsilon)w^*$ that contains $S \setminus \xi(z',\gamma',w')$. Because we do not know z' and γ', we try all possible pairs of values.

For any $z \in \mathcal{Z}$, $\gamma \in \Gamma$, and $w \geq 0$, let $f_1(z,\gamma,w) \leq (1+\varepsilon)\text{width}(S \setminus \xi(z,\gamma,w))$ be the width of the strip computed by the $(1+\varepsilon)$-approximation algorithm of Duncan *et al.* on the set $S \setminus \xi(z,\gamma,w)$. Let $h(z,\gamma,w) = \max\{w, f_1(z,\gamma,w)\}$. For any given z, γ, and w, $h(z,\gamma,w)$ can be computed in time $O(n/\varepsilon)$. By an argument similar to the one in Section 3.1, if z and γ are fixed then $h(z,\gamma,w)$ is unimodal, and $\min_{w \in \tilde{W}} h(z,\gamma,w)$ can be computed in $O(n/\varepsilon \log(1/\varepsilon))$ time by performing a binary search on $\tilde{W}$. Let $\Xi(z,\gamma)$ be the corresponding pair of strips. We repeat this procedure for all pairs (z,γ) in $Z \times \Gamma$ and report the pair $\Xi(z_0,\gamma_0)$ if

$$\min_{w \in \tilde{W}} h(z_0,\gamma_0,w) \leq \min_{(z,\gamma) \in Z \times \Gamma} \min_{w \in \tilde{W}} h(z,\gamma,w).$$

There are $O(1/\varepsilon^2)$ pairs in $Z \times \Gamma$, and we spend $O(n/\varepsilon \log(1/\varepsilon))$ time on each pair. Hence, the running time of the algorithm is $O(n/\varepsilon^3 \log(1/\varepsilon))$. The proof of correctness follows from the following lemmas.

Lemma 3.5. *Let $\sigma^* \in \Sigma^*$ and let (p,q) be an anchor pair for σ^*. Then $S \cap \sigma^*$ is contained in the rectangle R.*

Proof: We use the same notations from the proof of Lemma 2.1 (see Figure 1). The proof of Lemma 2.1 implies $d(v_2,\ell_{pq}), d(v_4,\ell_{pq}) \leq 3w^* \leq 3\tilde{w}$. Hence, ρ is contained in the strip of width $6\tilde{w}$ having ℓ_{pq} as the median. Let Δ denote the diameter of S^*, and let s denote the midpoint of segment pq. We deduce

$$|sv_1| \leq |v_1v_3| \leq |v_1v_2| + |v_2v_3| \leq 2\Delta \leq 4d(p,q),$$

and similarly $|sv_3| \le 4d(p,q)$. Thus, $v_i \in R$, $1 \le i \le 4$, which implies $S \cap \sigma^* \subset R$. $\square$

LEMMA 3.6. *Let $\sigma^* \in \Sigma^*$ and let (p,q) be an anchor pair for σ^*. Then there exist $z \in \mathcal{Z}$ and $\gamma \in \Gamma$ so that $\xi(z, \gamma, (1+\varepsilon/2)w^*)$ contains $S \cap \sigma^*$.*

Proof: Let $S^* = S \cap \sigma^*$. Let u_1, u_2, u_3, u_4 be the four intersection points of the boundary of σ^* with R, so that the lower boundary of σ^* passes through u_1 and u_2, and so that u_1, u_2, u_3, u_4 are in clockwise order; see Figure 5.

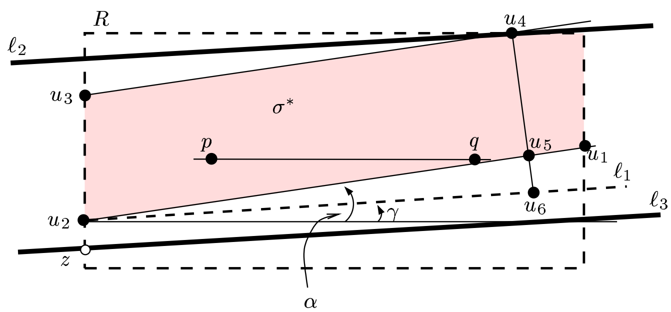

Figure 5: Approximating σ^* (shaded area) by $\xi(z, \gamma, (1+\varepsilon/2)w^*)$ (bold solid lines).

We prove the lemma in two steps. First we prove that there exists $\gamma \in \Gamma$ and a strip σ such that $S^* \subseteq \sigma$, $\text{width}(\sigma) \le (1+\varepsilon/4)w^*$, and the orientation of the lines bounding σ is γ with respect to the x-axis (which is the same as ℓ_{pq}).

Let α be the orientation of σ^* with respect to the x-axis. By Lemma 2.1 and the value of θ, $\alpha \in [-\theta, \theta]$. For simplicity, assume $\alpha \ge 0$ (the other case is similar). Let $\gamma_i \in \Gamma$ so that $\gamma_i \le \alpha < \gamma_{i+1}$. We set $\gamma = \gamma_i$. Let ℓ_1 and ℓ_2 be the two parallel lines whose orientation is γ and that pass through u_2 and u_4, respectively. We define σ to be the strip bounded by ℓ_1 and ℓ_2. By Lemma 3.5, $S^* \subseteq R$, which implies $S^* \subseteq \sigma$. We prove that $\text{width}(\sigma) \le (1+\varepsilon/4)w^*$. Let u_5 be the projection of point u_4 on the lower boundary of σ^*, and let u_6 be the intersection point between ℓ_1 and the line through u_4 and u_5. Then

$$\begin{aligned} d(\ell_1, \ell_2) &\le |u_4u_6| = w^* + |u_5u_6| = w^* + |u_2u_5| \tan(\alpha - \gamma) \\ &\le w^* + |u_2u_4| \tan(\delta\theta) \le w^* + \text{diam}(R) \tan(\delta\theta) \\ &\le w^* + (6\tilde{w} + 8d(p,q)) \tan(\delta\theta). \end{aligned}$$

We consider two cases. If $1/2 \le \tilde{w}/d(p,q)$, then $d(p,q) \le 2\tilde{w}$. Assuming $\delta \le 2/3$ and using the inequality $\sin x \le x$, for $x \ge 0$, we deduce

$$(6\tilde{w} + 8d(p,q)) \tan(\delta\theta) \le 22\tilde{w} \frac{\delta\theta}{\cos(\delta\pi/2)} \le 22\tilde{w} \frac{\delta\pi/2}{\cos(\pi/3)} \le 132\pi\delta w^*.$$

Otherwise, $2\tilde{w} < d(p,q)$, which implies $\theta < \pi/6$. Using the fact that $\tan(\delta x) \le \delta \tan x$, for

$0 \leq x < \pi/2$ and $0 < \delta < 1$, we deduce

$$\begin{aligned}(6\tilde{w} + 8d(p,q))\tan(\delta\theta) &\leq 11d(p,q)\delta\tan\theta \leq 11d(p,q)\delta\frac{\sin\theta}{\cos(\pi/6)} \\ &\leq (22/\sqrt{3})d(p,q)\delta\frac{\tilde{w}}{d(p,q)} \leq (132/\sqrt{3})\delta w^*.\end{aligned}$$

Hence, choosing $\delta \leq \min\{2/3, \varepsilon/(528\pi)\}$ we obtain width$(\sigma) \leq \varepsilon w^*/4$.

We now prove that there exists $z \in \mathcal{Z}$ so that $S^* \subseteq \xi(z, \gamma, (1+\varepsilon/2)w^*)$. Let $z_j, z_{j+1} \in Z$ be two consecutive grid points so that u_2 lies between z_j and z_{j+1}. Choose $z \in \{z_j, z_{j+1}\}$ to be the point that lies below the lower boundary of σ^*. Let ℓ_3 be the line parallel to ℓ_1 passing through z, and let σ' be the strip bounded by ℓ_3 and ℓ_2. If z and u_2 lie on a vertical boundary of R (as in Figure 5) then $d(\ell_3, \ell_1) \leq |zu_2| \leq 6\delta\tilde{w} < (\varepsilon/4)w^*$. Otherwise, z and u_2 lie on a horizontal side of R and

$$d(\ell_3, \ell_1) = |zu_2|\sin\gamma \leq |zu_2|\sin\alpha \leq 8\delta d(p,q)\frac{w^*}{d(p,q)} \leq 8\delta w^* < (\varepsilon/4)w^*.$$

□

We are now ready to prove the main result of this subsection.

Lemma 3.7. *If (p,q) is an anchor pair of a strip in an optimal strip cover, then the above algorithm computes a $(1+\varepsilon)$-approximation of the optimal cover in time $O(n/\varepsilon^3 \log(1/\varepsilon))$.*

Proof: Let $\sigma^* \in \Sigma^*$ be the strip for which (p,q) is an anchor pair. By Lemma 3.6, there exist $z \in \mathcal{Z}$ and $\gamma \in \Gamma$ such that $\xi(z, \gamma, (1+\varepsilon/2)w^*)$ contains $S \cap \sigma^*$. Let w_k be the smallest element in $\tilde{W}$ so that $(1+\varepsilon/2)w^* \leq w_k$. Then $w_k \leq (1+\varepsilon/2)w^* + \varepsilon\tilde{w}/12 \leq (1+\varepsilon)w^*$. Obviously, $\xi(z, \gamma, w_k)$ contains $S \cap \sigma^*$. Moreover, $S \setminus (S \cap \sigma^*)$ can be covered by a strip of width w^*. Therefore, the above procedure returns a strip cover of width at most $(1+\varepsilon)w^*$. □

As mentioned in the beginning, we repeat the above procedure for all pairs in $\mathcal{F}$, which can be computed in $O(n \log n)$ time. In addition, we compute the value $\tilde{w}$ (used in the above procedure) in $O(n \log n)$ time. We conclude with the following.

Theorem 3.8. *For any set S of n points in the plane, we can compute a cover of S by two strips of width at most $(1+\varepsilon)w^*$ in $O(n \log n + n/\varepsilon^3 \log(1/\varepsilon))$ time.*

Remark 3.9. The constant hidden by the big-Oh notation in the analysis of the running time is quite large. A much smaller constant can be obtained with some additional work. For example, using the technique by Kirkpatrick and Snoeyink [10], our 6-approximation algorithm can be modified to compute two strips of width at most $\tilde{w}_1 \leq 3w^*$ in the same time bounds. Hence, we can replace $\tilde{w}$ by $\tilde{w}_1$ in the $(1+\varepsilon)$-approximation algorithm. Also, a more careful analysis shows that it is sufficient to choose a larger value for δ, further reducing the constant in the running time. For simplicity, we did not attempt to minimize this constant.

4 Conclusions

We have presented a simple, efficient $(1+\varepsilon)$-approximation algorithm for computing a 2-line-center. An obvious open question is whether the running time can be improved to $O(n + 1/\varepsilon^{O(1)})$. The next step is to extend this approach to the k-line-center problem, for fixed k, and to higher dimensions. We would also like to extend our approach to covering the points by hyper-strips. It is unclear whether we can extend the definition of anchor pairs of planar strips, to anchor tuples of hyper-strips, in a manner that allows us to efficiently compute a small set of candidate tuples.

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