Maintaining Approximate Extent Measures of Moving Points*

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Abstract

We present approximation algorithms for maintaining various descriptors of the extent of moving points in $\mathbb{R}^d$. We first describe a data structure for maintaining the smallest orthogonal rectangle containing the point set. We then use this data structure to maintain the approximate diameter, smallest enclosing disk, width, and smallest area or perimeter bounding rectangle of a set of moving points in $\mathbb{R}^2$ so that the number of events is only a constant. This contrasts with $\Omega(n^2)$ events that data structures for the maintenance of those exact properties have to handle.

1 Introduction

With the rapid advances in positioning systems, e.g., GPS, ad-hoc networks, and wireless communication, it is becoming increasingly feasible to track and record the changing position of continuously moving objects. These developments have raised a wide range of challenging geometric problems involving moving objects, including efficient data structures for answering proximity queries, for clustering, and for maintaining connectivity information. Many of these problems ask for maintaining a descriptor of the *extent* of a point set. For example, in R-trees, one of the most widely used spatial data structures in practice, each node is associated with a subset of points and the smallest orthogonal rectangle containing this subset. If the points are moving, then one has to maintain the smallest enclosing rectangle as the points move continuously. Other applications of maintaining an extent include collision detection, clustering, animation, and physical simulation.

1.1 Problem statement

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points moving in $\mathbb{R}^d$. For a given time t, let $p_i(t) = (x_i^1(t), \ldots, x_i^d(t))$ denote the position of p_i at time t. We will use $P(t)$ denote the set P at time t. We say that the motion of P has *degree* k if every $x_i^j(t)$ is a polynomial of degree at most k. We call a motion of degree 1 *linear*. In this case each point of P moves along a straight line with fixed speed. We say that the motion of P is *algebraic* if it is of degree d for some constant $d \geq 0$. For most of the discussion in this paper, we will assume that the motion of P is linear, i.e., $p_i(t) = \mathbf{a}_i + \mathbf{b}_i t$, where $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{R}^d$. The values $\mathbf{a}_i, \mathbf{b}_i$ may change over time. We assume that the objects are responsible for updating the values of $\mathbf{a}_i, \mathbf{b}_i$.

In this paper we develop efficient approaches for maintaining various descriptors of the extent of P, including the smallest enclosing orthogonal rectangle, diameter, width, smallest enclosing disk, and the smallest enclosing rectangle of arbitrary orientation. These extent measures indicate how spread out the point set P is. As the points move continuously, the extent measure of interest changes continuously as well, though its combinatorial realization changes only at certain discrete times. For example, the smallest orthogonal rectangle containing P can be represented by a sequence of $2d$ points, each lying on one of the facets of the rectangle. As the points move, the rectangle also changes continuously. At certain discrete times, the points lying on the boundary of the rectangle change, and we have to update the sequence of points defining the rectangle. Similarly, in the case of diameter, the pair of points defining the diameter changes at certain discrete times. Our approach is to focus on these discrete changes (or *events*) and track through time the combinatorial description of the extent measure of interest.

It turns out that maintaining the exact description of extent measures is quite expensive. For example, a result of Agarwal *et al.* [2] shows that the diameter of a point set under linear motion in the plane can change

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quadratic number of times. We therefore investigate the problem of maintaining extent measures approximately. More precisely, suppose we want to maintain the smallest enclosing orthogonal rectangle. Let $B(t)$ denote the smallest orthogonal rectangle containing $P(t)$. For a given rectangle R and a parameter $\lambda > 0$, we will use λR to denote the rectangle R scaled by the factor λ with respect to the center of R. For a given parameter $\varepsilon > 0$, let $B^{\varepsilon}(t)$ denote a rectangle such that $B(t) \subseteq B^{\varepsilon}(t) \subseteq (1+\varepsilon)B(t)$, where $(1+\varepsilon)B(t)$ denotes the rectangle resulting from scaling $B(t)$ by a factor of $(1+\varepsilon)$ and centering it at the center of $B(t)$. We call $B^{\varepsilon}(t)$ an *(outer) ε-approximation* of $B(t)$. The intuition is that one has to change the combinatorial description of $B^{\varepsilon}(t)$ considerably fewer times than that of $B(t)$. We can define similar ε-approximations for other extent measures, including diameter, width, and smallest enclosing ball.

1.2 Previous work

Motivated by various applications, there has been a flurry of activity in computational geometry, databases, and networking on problems dealing with moving objects. In the computational geometry community, earlier work on moving points focused on bounding the number of changes in various geometric structures (e.g., convex hull, Delaunay triangulation) as the points move [5, 14]. In their paper, Basch *et al.* [7] introduced the notion of *kinetic data structures*. Their work has lead to several interesting results related to moving objects, including results on kinetic convex-hull, binary space partition trees, collision detection, and closest pair; see [1, 11] and references therein. The main idea in the kinetic framework is that as the points move and their *configuration* changes, *kinetic updates* are performed on the data structure when certain events occur. Although the points move continuously, the combinatorial structure changes only at certain discrete times at which certain events occur, and therefore one does not have to update the data structure continuously. In contrast to fixed-time-step methods, in which the fastest moving object determines the update time step for the entire data structure, a kinetic data structure is based on events, which have a natural interpretation in terms of the underlying structure.

In the context of maintaining an extent measure of a point set P, a result by Basch *et al.* [7] gives a kinetic data structure for maintaining the smallest orthogonal rectangle $B(t)$ containing $P(t)$. It processes $O(n \log n)$ events if the motion is linear, and each event requires $O(\log n)$ time to update the combinatorial description of $B(t)$. A point can be inserted or deleted in $O(\log^2 n)$ time. Saltenis *et al.* [13] describe a heuristic to maintain a small rectangle enclosing a set of points moving in the plane. Agarwal *et al.* [2] proposed a kinetic data structure for maintaining the diameter, width, and a smallest enclosing rectangle (of arbitrary orientation) of a point set in the plane. Their structure processes $O(n^{2+\varepsilon})$ events for algebraic motion of points, and each event requires $O(\log n)$ time. No efficient data structure is known for maintaining the diameter or width of a point set in higher dimensions. The best known data structure for maintaining the smallest enclosing disk of a point set in the plane is the same as the one that maintains the farthest point Voronoi diagram. This processes $O(n^3)$ events for linear motion.

1.3 Our results.

Most of the work on kinetic data-structures had focused on maintaining exact geometric structure which forces them to process many events. We develop efficient algorithms for maintaining various extent measures approximately. The most salient feature of our data-structures is that the number of events processed depends only on the approximation factor and not on the input size. In the following, let P be a set of n points in $\mathbb{R}^d$, and let $\varepsilon > 0$ be a parameter. This paper contains the following main results.

Extent of points in $1D$. Let P be a set of n points in $\mathbb{R}$. We present a data structure for maintaining $B^{\varepsilon}(t)$, each endpoint of B^{ε} follows a piecewise-linear trajectory. The combinatorial structure of B^{ε} is updated $O(\sqrt{1/\varepsilon})$ times, and at each such event the extent can be updated in $O(\log n)$ time after $O(n \log n)$ preprocessing. Note that the number of combinatorial changes depends only on ε, and not on the number of points. A point can be inserted into or deleted from the structure in $O(\log^2 n)$ time.

Actually, we define the notion of extent for a set $\mathcal{H}$ of hyperplanes. We show that there exists a small set $\mathcal{J}$ of hyperplanes whose extent approximates the extent of $\mathcal{H}$ and whose size is independent of $|\mathcal{H}|$. Maintaining B^{ε} is a special case of maintaining the extent of hyperplanes.

Smallest enclosing rectangle under linear motion. Next, we consider the problem of maintaining the smallest enclosing orthogonal rectangle of a point set P in $\mathbb{R}^d$. The problem of computing the smallest rectangle is decomposable. That is, for each $j = 1, \ldots, d$, we can project the points in P onto the x_j-axis, compute the extent of the projected points (the smallest interval containing the points), and combine the result to get the bounding box. If I_j is the extent of the points projected on the x_j-axis, then the smallest rectangle containing P is $I_1 \times \cdots \times I_d$. Thus our one-dimensional result implies that we can maintain B^{ε} whose combinatorial structure changes $O(\sqrt{1/\varepsilon})$ times, and at each such event the rectangle can be updated in $O(\log n)$ time after $O(n \log n)$ preprocessing. A point can be inserted into or deleted from in $O(\log^2 n)$ time.

We also describe a data structure for maintaining a rectangle $\beta^\varepsilon(t)$ such that $(1-\varepsilon)B(t) \subseteq \beta^\varepsilon(t) \subseteq B(t)$ and such that $\beta^\varepsilon(t)$ is defined by a sequence of $2d$ points $(p_1, \ldots p_{2d})$ of P in the sense that $\beta^\varepsilon(t) = \prod_{i=1}^{d} [x^i_{2i-1}(t), x^i_{2i}(t)]$. The data structure has the same performance bounds as the previous one.

As mentioned above, this algorithm can be used as a building block for R-trees (or other hierarchical data structures) on moving points in $\mathbb{R}^d$. Namely, at each node v of the tree, we maintain an ε-approximation of the smallest rectangle B^ε_v enclosing the point subset associated with v. Since the representation of the kinetic bounding rectangle is small, we can store it at the node. In order to answer a range query — report all points that lie inside a rectangle at time t — we check at each node v whether $R \subseteq B^\varepsilon_v(t)$. If so, we report all points in $S_v(t)$. If $R \cap B^\varepsilon(t) \neq \emptyset$, we recursively visit the children of v.

Smallest enclosing rectangle under algebraic motion. For degree k motion, we present a data structure for maintaining $B^\varepsilon(t)$, which processes $O(\log(1/\varepsilon)/\varepsilon)$ events. The rectangle can be updated in $(\log(n)/\varepsilon)^{O(1)}$ time at each event.

Diameter. For linear motion, we can maintain a pair of points (p, q) such that $d(p, q) \geq (1-\varepsilon)\,\mathrm{diam}(P)$. The data structure processes $O((1/\varepsilon^{d/2})\log(1/\varepsilon)\alpha(1/\varepsilon))$ events. The total time spent in updating these events is $O((n/\varepsilon^{(d-1)/2})\log n)$. A similar approach can also maintain the smallest enclosing ball of a point set. In the plane, the number of events is roughly $O(1/\varepsilon^{5/2})$.

Width. For a set P of n points in $\mathbb{R}^2$, in which each point is moving linearly, and a parameter $\varepsilon > 0$, we can maintain a strip of width at most $(2+\varepsilon)\,\mathrm{width}(P)$ whose combinatorial structure changes only $O(1/\varepsilon^2)$ times. However, we currently do not have an efficient procedure to update the width at each such instance. The naïve procedure takes $O(n)$ time to update the strip at each event. A similar procedure works for maintaining an approximate minimum area rectangle (of arbitrary orientation) of moving points in the plane.

The paper is organized as follows. In Section 2, we define the extent for hyperplanes, review some techniques for handling the problem at hand, and prove bounds on maintaining the approximate extent of hyperplanes and the smallest enclosing rectangle. In Section 3, we present data-structures for maintaining the smallest enclosing rectangle. In Section 4, we extend these data-structures to maintaining the approximate diameter, smallest enclosing ball, and width. We then describe in Section 5 how to maintain B^ε for algebraic motion. We use the linearization technique to map the trajectories of points to linear functions in a higher dimension, compute an approximate extent in this parametric space, and then map the approximation back to the original plane. Because of lack of space, we omit the descriptions for maintaining width and min-area rectangle of a moving point-set.

2 Approximating the Extent

Let P be a set of n points in $\mathbb{R}$, each moving with fixed speed. That is $p_i(t) = a_i + b_i t$, where $a_i, b_i \in \mathbb{R}$. The *extent* $B(t)$ of $P(t)$ is the smallest interval containing $P(t)$. For a parameter ε, an *outer ε-extent* of $P(t)$ is an interval $B^\varepsilon(t) \subseteq (1+\varepsilon)B(t)$. (Here, and in the following, $(1+\varepsilon)I(t)$ denotes the set resulting by scaling $I(t)$ by a factor of $(1+\varepsilon)$, and centering it in the middle of $I(t)$.) Similarly, an *inner ε-extent* of $P(t)$ is an interval $\beta^\varepsilon(t) \supseteq (1-\varepsilon)B(t)$.

It will be convenient to work in a parametric xt-plane in which a moving point $p(t) \in \mathbb{R}$ at time t is mapped to the point $(t, p(t))$. For $1 \leq i \leq n$, we map the point $p_i \in P$ to the line $\ell_i = \bigcup_t (t, p_i(t))$, for $i = 1, \ldots, n$. Let $L = \{\ell_1, \ldots, \ell_n\}$ be the resulting set of lines, and let $\mathcal{A}(L)$ be their arrangement. Clearly, the extent $B(t_0)$ of $P(t_0)$ is the vertical interval $I(t_0)$ in the arrangement $\mathcal{A}(L)$ connecting the upper and lower envelopes of L at $t = t_0$. See Figure 1(i). The combinatorial structure of $I(t)$ changes at the vertices of the two envelopes of L, and all the different combinatorial structures of $I(t)$ can be computed in $O(n \log n)$ time by computing the upper and lower envelopes of L.

We want to maintain a vertical interval $I^\varepsilon(t)$ so that $I(t) \subseteq I^\varepsilon(t) \subseteq (1+\varepsilon)I(t)$ for all t, so that the endpoints of $I^\varepsilon(t)$ follow piecewise-linear trajectories, and so that the number of combinatorial changes in $I^\varepsilon(t)$ is small. Geometricly, this has the following interpretation: We want to simplify the upper and lower envelopes of $\mathcal{A}(L)$ by convex and concave polygonal chains, respectively, so that the simplified upper (resp. lower) envelope lies above (resp. below) the original upper (resp. lower) envelope and so that for any t, the vertical segment connecting the simplified envelopes is contained in $(1+\varepsilon)I(t)$. See Figure 1 (ii). Actually, we generalize the notion of extent to arrangements of hyperplanes and prove a result on approximating the extent of hyperplanes.

2.1 Outer and inner approximations of extent

Let $h : x_d = a_1 x_1 + \cdots a_{d-1} x_{d-1} + a_d$ be a hyperplane in $\mathbb{R}^d$. We will regard h as the graph of a linear function

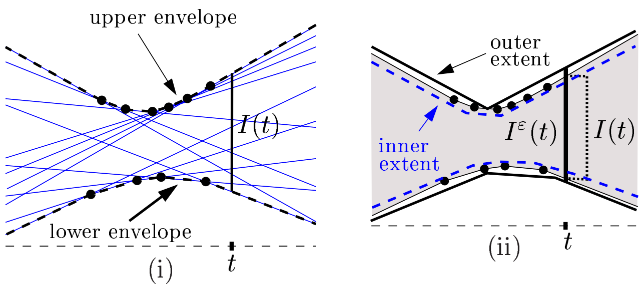

Figure 1: (i) The extent of the moving points, is no more than the vertical segment connecting the lower envelope to the upper envelope. The black dots mark where the movement description of $I(t)$ changes. (ii) The approximate extent.

$h(x_1, \ldots, x_{d-1}) = a_1x_1 + \cdots + a_{d-1}x_{d-1} + a_d$ and will not distinguish between the function and its graph. A point $p = (p_1, \ldots, p_d)$ lies *above* (resp. *below*) h if $p_d > a_1p_1 + \cdots + a_{d-1}p_{d-1} + a_d$ (resp. $p_d < a_1p_1 + \cdots + a_{d-1}p_{d-1} + a_d$). For two parallel hyperplanes $h : x_d = a_d + \sum_{i=1}^{d-1} a_ix_i$ and $h' : x_d = a'_d + \sum_{i=1}^{d-1} a_ix_i$, the vertical distance between h and h' is defined as $\delta(h, h') = |a_d - a'_d|$. We will use $d(h, h')$ to denote the normal distance between h and h'. Note that $d(h, h') = \delta(h, h')/\sqrt{1 + \sum_{i=1}^{d-1} a_i^2}$.

Let $\mathcal{H} = \{h_1, \ldots, h_n\}$ be a set of hyperplanes in $\mathbb{R}^d$. The *lower envelope* of $\mathcal{H}$ is the graph of the linear function $\mathcal{L}_\mathcal{H} : \mathbb{R}^{d-1} \to \mathbb{R}$

$$\mathcal{L}_\mathcal{H}(\mathbf{x}) = \min_{h \in \mathcal{H}} h(\mathbf{x}).$$

Similarly, the *upper envelope* of $\mathcal{H}$ is the graph of the function

$$\mathcal{U}_\mathcal{H}(\mathbf{x}) = \max_{h \in \mathcal{H}} h(\mathbf{x}).$$

The *extent* $I_\mathcal{H} : \mathbb{R}^{d-1} \to \mathbb{R}$ of $\mathcal{H}$ is defined as

$$I_\mathcal{H}(\mathbf{x}) = \mathcal{U}_\mathcal{H}(\mathbf{x}) - \mathcal{L}_\mathcal{H}(\mathbf{x}).$$

With a slight abuse of notation, we will also use $\mathcal{L}_\mathcal{H}(\mathbf{x})$ and $\mathcal{U}_\mathcal{H}(\mathbf{x})$ to denote the points $(\mathbf{x}, \mathcal{L}_\mathcal{H}(\mathbf{x}))$ and $(\mathbf{x}, \mathcal{U}_\mathcal{H}(\mathbf{x}))$, respectively, in $\mathbb{R}^d$. Let $\varepsilon > 0$ a parameter. A set $\mathcal{J}$ of hyperplanes is an *outer ε-approximation* of $\mathcal{H}$ if the following two conditions hold for every point $\mathbf{x} \in \mathbb{R}^{d-1}$:

(i) $\mathcal{L}_\mathcal{H}(\mathbf{x}) - \frac{\varepsilon}{2} I_\mathcal{H}(\mathbf{x}) \le \mathcal{L}_\mathcal{J}(\mathbf{x}) \le \mathcal{L}_\mathcal{H}(\mathbf{x})$, and

(ii) $\mathcal{U}_\mathcal{H}(\mathbf{x}) \le \mathcal{U}_\mathcal{J}(\mathbf{x}) \le \mathcal{U}_\mathcal{H}(\mathbf{x}) + \frac{\varepsilon}{2} I_\mathcal{H}(\mathbf{x})$.

This implies that $I_\mathcal{H}(\mathbf{x}) \le I_\mathcal{J}(\mathbf{x}) \le (1 + \varepsilon) I_\mathcal{H}(\mathbf{x})$.

Similarly, a set $\mathcal{K}$ of hyperplanes in an *inner ε-approximation* of $\mathcal{H}$ if the following two conditions hold for every point $\mathbf{x} \in \mathbb{R}^{d-1}$:

(i) $\mathcal{L}_\mathcal{H}(\mathbf{x}) \le \mathcal{L}_\mathcal{K}(\mathbf{x}) \le \mathcal{L}_\mathcal{H}(\mathbf{x}) + \frac{\varepsilon}{2} I_\mathcal{H}(\mathbf{x})$, and

(ii) $\mathcal{U}_\mathcal{H}(\mathbf{x}) - \frac{\varepsilon}{2} I_\mathcal{H}(\mathbf{x}) \le \mathcal{U}_\mathcal{K}(\mathbf{x}) \le \mathcal{U}_\mathcal{H}(\mathbf{x})$.

Therefore $(1 - \varepsilon) I_\mathcal{H}(\mathbf{x}) \le I_\mathcal{K}(\mathbf{x}) \le I_\mathcal{H}(\mathbf{x})$.

2.2 Duality and extent

The *dual* of a point $b = (b_1, \ldots, b_d)$ is a hyperplane $b^* : x_d = b_1x_1 + \cdots b_{d-1}x_{d-1} - b_d$, and the dual of a hyperplane $h : x_d = a_1x_1 + a_2x_2 + \cdots + a_{d-1}x_{d-1} + a_d$ is the point $h^* = (a_1, \ldots, a_{d-1}, -a_d)$. Under this definition of duality, $b^{**} = b$ and the point b lies above (resp. below, on) the hyperplane h if and only if the point h^* lies above (resp. below, on) the hyperplane b^*. The vertical distance between b and h is the same as that between b^* and h^*, and the vertical distance $\delta(h, h')$ between two parallel hyperplanes h and h' is the same as the length of the vertical segment $h^*h'^*$. It can be checked that the hyperplane dual to the point $\mathcal{L}_\mathcal{H}(\mathbf{x})$ (resp. $\mathcal{U}_\mathcal{H}(\mathbf{x})$) is normal to the vector $(\mathbf{x}, -1) \in \mathbb{R}^d$ and supports $\mathrm{conv}(\mathcal{H}^*)$ so that $\mathcal{H}^*$ lies below (resp. above) the hyperplane and so that $I_\mathcal{H}(\mathbf{x})$ is the vertical distance between these two parallel supporting planes.

2.3 Approximating the extent via duality

In this section we use duality to show the existence of inner and outer ε-approximations of $\mathcal{H}$ of small size.

Let S be a set of points in $\mathbb{R}^d$. For a point $\mathbf{x} \in \mathbb{R}^{d-1}$, let $\lambda_S(\mathbf{x})$ (resp. $\rho_S(\mathbf{x})$) be the supporting hyperplane of $\mathrm{conv}(S)$ in direction $(\mathbf{x}, -1)$ so that S lies below (resp. above) it. Set $W_S(\mathbf{x}) = d(\lambda_S(\mathbf{x}), \rho_S(\mathbf{x}))$. In view of the above discussion, $\lambda_{\mathcal{H}^*}(\mathbf{x}), \rho_{\mathcal{H}^*}(\mathbf{x})$ are the hyperplanes dual to the points $\mathcal{L}_\mathcal{H}(\mathbf{x})$ and $\mathcal{U}_\mathcal{H}(\mathbf{x})$, respectively, and $I_\mathcal{H}(\mathbf{x}) = \delta(\lambda_{\mathcal{H}^*}(\mathbf{x}), \rho_{\mathcal{H}^*}(\mathbf{x}))$. For a parameter $\varepsilon > 0$, a set R of points in $\mathbb{R}^d$ is called an *outer ε-approximation* of S if $\mathrm{conv}(S) \subseteq \mathrm{conv}(R)$ and for all $\mathbf{x} \in \mathbb{R}^{d-1}$

$$d(\lambda_R(\mathbf{x}), \lambda_S(\mathbf{x})),\ d(\rho_R(\mathbf{x}), \rho_S(\mathbf{x})) \le (\varepsilon/2) W_S(\mathbf{x}).$$

An *inner ε-approximation* of S is defined similarly.

LEMMA 2.1. *Let $\mathcal{H}$ be a set of hyperplanes in $\mathbb{R}^d$, and let $\varepsilon > 0$ be a parameter. A set $\mathcal{J}$ of hyperplanes is an outer (resp. inner) ε-approximation of $\mathcal{H}$ if and only if $\mathcal{J}^*$ is an outer (resp. inner) ε-approximation of $\mathcal{H}^*$.*

Proof. We prove the lemma for outer approximations. Suppose $\mathcal{J}$ is an outer ε-approximation of $\mathcal{H}$. Then

$$\begin{aligned} & d(\rho_{\mathcal{H}^*}(\mathbf{x}), \rho_{\mathcal{J}^*}(\mathbf{x})) \\ &= \frac{\delta(\rho_{\mathcal{H}^*}(\mathbf{x}), \rho_{\mathcal{J}^*}(\mathbf{x}))}{\|(\mathbf{x}, 1)\|} \\ &= \frac{\mathcal{L}_\mathcal{H}(\mathbf{x}) - \mathcal{L}_\mathcal{J}(\mathbf{x})}{\|(\mathbf{x}, 1)\|} \le \frac{(\varepsilon/2) I_\mathcal{H}(\mathbf{x})}{\|(\mathbf{x}, 1)\|} \\ &= \frac{\varepsilon}{2} \frac{\delta(\rho_{\mathcal{H}^*}(\mathbf{x}), \lambda_{\mathcal{H}^*}(\mathbf{x}))}{\|(\mathbf{x}, 1)\|} = \frac{\varepsilon}{2} W_{\mathcal{H}^*}(\mathbf{x}). \end{aligned}$$

Similarly, we can prove that $d(\lambda_{\mathcal{H}^*}(\mathbf{x}), \lambda_{\mathcal{J}^*}(\mathbf{x})) \leq \varepsilon W_{\mathcal{H}^*}(\mathbf{x})/2$. Hence, $\mathcal{J}^*$ is an outer ε-approximation of $\mathcal{H}^*$. A similar argument shows that if $\mathcal{J}^*$ is an outer ε-approximation of $\mathcal{H}^*$, then $\mathcal{J}$ is an outer ε-approximation of $\mathcal{H}$. □

By constructing a d-dimensional grid of appropriate size in the dual space, choosing the vertices of a subset of "extremal" grid cells that contain the points of $\mathcal{H}^*$, and returning the hyperplanes dual to these vertices, one can prove the following lemma. We omit the proof from this version.

LEMMA 2.2. *Given a set $\mathcal{H}$ of n points in $\mathbb{R}^d$ and a parameter $\varepsilon > 0$, one can compute in $O(n)$ time a set $\mathcal{J}$ of $O(1/\varepsilon^{d-1})$ hyperplanes that is an outer ε-approximation of $\mathcal{H}$. A similar claim holds for an inner ε-approximation of $\mathcal{H}$.*

The size of the approximation of Lemma 2.2 can be improved, as the following theorem testifies.

THEOREM 2.1. *Given a set $\mathcal{H}$ of n hyperplanes in $\mathbb{R}^d$ and a parameter $\varepsilon > 0$, there exists a set $\mathcal{J}$ of $O(1/\varepsilon^{(d-1)/2})$ hyperplanes that is an outer (resp. inner) ε-approximation of $\mathcal{H}$.*

Proof. We first compute in linear time, using the algorithm of Barequet and Har-Peled [6], a box B containing $\mathcal{H}^*$ whose volume is at most $2d!\,\mathrm{Vol}(\mathcal{CH}(\mathcal{H}))$. Let T be the affine transformation that maps B to the unit-cube, and set $S = T(\mathcal{H}^*)$. By Lemma 2.1, if U is an outer ε-approximation of S, then the set of hyperplanes dual to the points in $T^{-1}(U)$ is an outer ε-approximation of $\mathcal{H}$.

Let $C = \mathcal{CH}(S)$. The polytope C is "fat" (i.e., its volume is at least $1/(2d!)$), and its diameter is $\leq \sqrt{d}$. By [8], there exists a point-set V of size $O(1/\varepsilon^{(d-1)/2})$, so that $C \subseteq \mathcal{CH}(V)$, and the Hausdorff distance between C and $\mathcal{CH}(V)$ is at most $\varepsilon c_d/(2\sqrt{d})$. In particular, V is an outer ε-approximation of S. A similar argument works for the inner ε-approximation. □

REMARK 2.1. (i) The algorithm of [8] works by computing an $\sqrt{\varepsilon}$-dense set V' of points on the boundary of $C = \mathcal{CH}(S)$ (the details are somewhat similar to the method of [10]), and then slightly lifting each point in V' from the boundary of C to get V. See [8] for the details. Somewhat surprisingly, we are not aware of any efficient (i.e., linear) algorithm to compute this approximation for small sets (i.e., $n = O(1/\varepsilon^{d-1})$), for $d > 3$.

(ii) If we skip the lifting part of the algorithm of [8], then the resulting set of points $V' \subseteq \partial\mathcal{CH}(S)$, and the Hausdorff distance between $\mathcal{CH}(V')$ and $\mathcal{CH}(S)$ is smaller than $\varepsilon c_d/(2\sqrt{d})$. In particular, in the dual, this corresponds to an inner ε-approximation of S^*.

(iii) The lower-bound construction in [8] implies that the bounds presented above in Theorem 2.1 are tight.

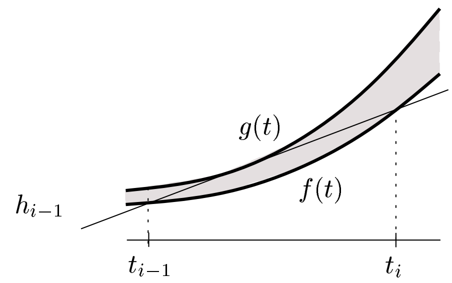

Figure 2: Maintaining an outer ε-extent.

An immediate corollary of Theorem 2.1 is the following result on maintaining the smallest enclosing rectangle of a moving point set.

COROLLARY 2.1. *Given a set $P(t)$ of n linearly moving points in $\mathbb{R}^d$ and a parameter $\varepsilon > 0$, one can compute an ε-approximation $B^\varepsilon(t)$ of the smallest rectangle enclosing $P(t)$ such that the number of events is $O(\sqrt{1/\varepsilon})$.*

3 Maintaining the Extent

In this section, we describe how to maintain outer and inner ε-extents of a point set in $\mathbb{R}$. One can, of course, use the construction described in the previous section to maintain an outer ε-extent that changes $O(1/\sqrt{\varepsilon})$ times, but we describe a different algorithm that computes an outer ε-extent whose combinatorial structure changes at most $\mathrm{Opt}(\varepsilon) + 2$ times, where $\mathrm{Opt}(\varepsilon)$ is the minimum size of an ε-extent. Again, we work in the parametric xt-plane.

3.1 Maintaining an outer ε-extent

Let L be the set of lines in the xt-plane as defined in the beginning of Section 2. Unlike the algorithm in the previous section, we do not compute an outer ε-approximation of L. Instead we compute the trajectories of the endpoints of the vertical segment I^ε. We describe the algorithm for maintaining the trajectory of the upper endpoint of I^ε; the lower endpoint can be maintained in a similar manner.

Let $f(t) = \mathcal{U}_L(t)$ be the linear function defining the upper envelope of L, and let

$$g(t) = f(t) + \varepsilon I_L(t)/2 = (1 + \varepsilon/2)f(t) - \varepsilon\mathcal{L}_L(t)/2.$$

Since $f(t)$ is a convex function and $\mathcal{L}_L(t)$ is a concave function, the graph of $g(t)$ is a convex polygonal chain.

We first compute in $O(n \log n)$ time the graphs of f and g. We then compute an almost minimum-link path in the "corridor" lying between $f(t)$ and $g(t)$. This path is the trajectory of the upper endpoint of ε-extent I^ε. Inductively, we maintain the invariant that the upper endpoint of I^ε moves along a line tangent to $g(t)$ and that it lies in the corridor. Initially, we choose a ray passing through the leftmost vertex of g and parallel to the leftmost edge of f. Suppose currently, the upper endpoint of $I^\varepsilon(t)$ is following a ray h_{i-1}. We compute the time t_i at which h_{i-1} intersects the upper envelope f (i.e., the time at which it tries to leave the corridor). We then compute in $O(\log n)$ time the rightward directed ray h_i emanating from $f(t_i)$ and tangent to g. The details are straightforward, and we omit them. If the minimum-link polygonal chain that lies between f and g consists of k vertices, then the above algorithm computes a convex polygonal chain with at most $k+1$ vertices [4]. We summarize:

Lemma 3.1. *One can compute in $O(n \log n)$ time a convex polygonal chain C lying between f and g with at most $k+1$ vertices, where k is the number of vertices in the minimum-link chain lying between f and g.*

We run the same algorithm for computing the trajectory of the lower endpoint of I^ε. For a given $\varepsilon > 0$, let $\mathrm{Opt}(\varepsilon)$ denote the minimum number of times the combinatorial structure of any ε-extent of P has to change, i.e., the number of times the (linear) trajectory of one of its endpoints changes. The number of events processed by our algorithm is bounded by $\mathrm{Opt}(\varepsilon) + 2$ (the algorithm might compute one extra link for each of the two endpoints). We thus have the following result.

Theorem 3.1. *Given a set P of n points in $\mathbb{R}^d$ under linear motion and a parameter $\varepsilon > 0$, one can maintain an outer ε-approximation $B^\varepsilon(t)$ of the smallest rectangle $B(t)$ containing $P(t)$ in a total of $O(n \log n)$ time, whose combinatorial structure changes at most* $\mathrm{Opt}(\varepsilon) + 2d = O(\sqrt{1/\varepsilon})$ *times.*

3.2 Maintaining an inner extent

The problem with the algorithm described above is that since $B(t) \subseteq B^\varepsilon(t)$, the endpoints of $B^\varepsilon(t)$ are not two of the input points. In some applications, we would like to maintain an interval $\beta(t) \supseteq (1-\varepsilon)B(t)$ whose endpoints are two of the input points; these endpoints act as a witness of the extent of P. Again, we focus on the one dimensional case. We call an inner ε-extent $\beta(t)$ of $P(t)$ *strong* if the endpoints of β are two of the input points. The algorithm described in Section 2.3 for computing the inner extent also does not compute a stronger inner ε-extent. We therefore describe a new algorithm.

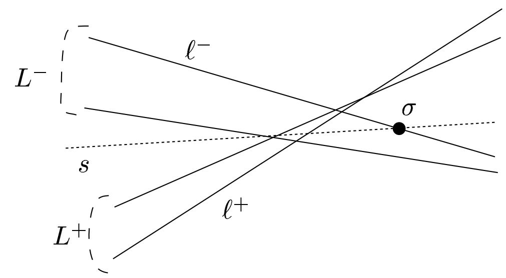

Figure 3: Illustration of the proof of Lemma 3.2.

Lemma 3.2. *let P be a set of points in $\mathbb{R}$. Given an ε-outer extent $B^\varepsilon(t)$ of $P(t)$, whose combinatorial structure changes k times, one can compute a strong 2ε-inner extent of $P(t)$ with at most $4k+4$ events, where $\varepsilon \leq 1$.*

Proof. As in the previous subsections, we work in the parametric xt-plane. Let $I(t)$ (resp. $I^\varepsilon(t)$) denote the vertical segment in the xt-plane corresponding to the extent (resp. ε-outer extent) of $P(t)$. Define $I'(t) = (1-\varepsilon)I^\varepsilon(t)$. It is easily seen that $(1-2\varepsilon)I(t) \subseteq I'(t) \subseteq I(t)$. Note, that each event point of $I(t)$ becomes an event point for both the upper and lower envelope of $I'(t)$, and thus $I'(t)$ complexity is twice the complexity of $I(t)$. In particular, the number of segments in the upper/lower envelopes of $I'(t)$ is $\leq k+1$. The endpoints of $I'(t)$ follow polygonal chains whose vertices lie at the values of t at which the combinatorial structure of $I^\varepsilon(t)$ changes. We construct a vertical interval $i^\varepsilon(t)$ whose endpoints lie on the lines of L and $I'(t) \subseteq i^\varepsilon(t) \subseteq I(t)$.

We will choose (described below) a subset $R^+ \subset L$ of at most $2k+2$ lines so that its upper envelope lies above $I'(t)$ for all t. We use the upper envelope of lines of R^+ as the trajectory of the upper endpoint of $i^\varepsilon(t)$. Let s be an edge of the polygonal chain followed by the upper endpoint of $I'(t)$. If s lies entirely below a line $\ell \in L$, we add ℓ to R^+. Otherwise, let L^- (resp. L^+) be the subset of lines in L whose slopes are less (resp. greater) than that of s, and let $\ell^- \in L^-$ be the line whose intersection with s has the maximum t-coordinate. Set $\sigma = \ell^- \cap s$. Since σ lies below the upper envelope of L, a line ℓ^+ of L lies above σ. One can verify that the upper envelope of $\{\ell^-, \ell^+\}$ lies above s, and we add them both to R^+. See Figure 3.

Similarly, we can find another subset $R^- \subseteq L$ of at most $2k+2$ lines so that their lower envelope lies below $I'(t)$ for all values of t. Hence, we can maintain an 2ε-inner extent of P with at most $4k+4$ events. □

Omitting all the algorithmic details, which are similar to the previous algorithm, we obtain the following.

LEMMA 3.3. *We can maintain a strong inner ε-extent of $P(t) \subseteq \mathbb{R}$ in $O(n \log n)$ time, whose combinatorial structure changes at most $O(1/\sqrt{\varepsilon})$ times.*

Proof. We maintain an $\varepsilon/2$-outer extent of $P(t)$ using the algorithm of Theorem 3.1. Let $J(t)$ be the corresponding vertical segment in the parametric xt-plane. We describe how to compute the trajectory of the upper endpoint of $i^\varepsilon(t)$. Whenever the combinatorial structure of $J(t)$ changes, we compute the new edges e^+, e^- along which the upper and lower endpoints of $(1-\varepsilon)J(t)$ move. We then compute the two lines ℓ_1, ℓ_2 of L whose upper envelope lies above e^+. We omit the details of this procedure from this abstract. As long as the upper endpoint of $(1-\varepsilon)J(t)$ moves along e^+, the upper endpoint of $i^\varepsilon(t)$ moves along the upper envelope of ℓ_1, ℓ_2. We repeat the same procedure for e^-. The lemma now follows. □

THEOREM 3.2. *Given a set P of n points in $\mathbb{R}^d$ under linear motion and a parameter $\varepsilon > 0$, one can maintain, in a total of $O(n \log n)$ time, an inner ε-approximation $\beta^\varepsilon(t)$ of $B(t)$ whose facets contain input points and whose combinatorial structure changes at most $O(1/\sqrt{\varepsilon})$ times.*

3.3 Maintaining the extent dynamically

Recall that our algorithm for maintaining the extent performs the following two queries: (i) compute the intersection point of a line with the upper envelope f, and (ii) given a point p below g, find the ray emanating from p in the rightward direction and tangent to g. We can use the dynamic data structure by Overmars and van Leeuwen [12], which inserts or deletes a line in $O(\log^2 n)$ time and answers a query of type (i) or (ii) in $O(\log n)$ time. However, we have an additional difficulty because the convex chain g depends on both $\mathcal{U}_L$ and $\mathcal{L}_L$. We therefore store both the envelopes of L separately, using the above data structure. For a given value t, one can then compute, in $O(\log n)$ time, $g(t)$ and its derivative at t (if $g(t)$ is a vertex of the chain, then we compute the slope of the two edges incident upon the vertex). Using this operation as the primitive, a type (ii) query can be answered in $O(\log^2 n)$ time. Our overall algorithm remains the same. We now define an event to be the time at which a point is inserted or deleted or at which the combinatorial structure of the extent changes. Hence, we obtain the following.

THEOREM 3.3. *Given a set P of n points in $\mathbb{R}$ under linear motion, one can maintain an outer or a strong inner ε-extent of P, so that it can be updated in $O(\log^2 n)$ at each event.*

Unfortunately, the data structure of Overmars and van Leeuwen is too complicated to be of practical use. We can use a simpler hierarchical data structure to maintain $B^\varepsilon(t)$. This data structure is especially suitable for applications (such as R-trees) in which we store a set of points in a tree T, each of whose node v is associated with the subset S_v of points stored in the subtree rooted at that node, and in which we wish to maintain a bounding rectangle $B_v^\varepsilon \subseteq (1+\varepsilon)B(S_v)$ at each node v. For simplicity, we describe the algorithm for one dimensional case. For higher dimensions, we repeat the same algorithm for each axis.

We set $\delta = \varepsilon/(c \log_r n)$, where c is a sufficiently large constant. For each node v of T, let L_v be the set of lines corresponding to the trajectories of the points in S_v. If the height of v is i, we maintain an outer $(2i\delta)$-approximation $\mathcal{J}_v$ of L_v and also the upper and lower envelopes of L_v. (Note that we do not maintain an outer $2i\delta$-extent using the on-line algorithm described in Section 3.1.) The insertion/deletion procedure visits a path Π from the root to a leaf of the tree. After having inserted or deleted a point at a leaf, we recompute the extent information at each node v on Π in a bottom-up manner, as follows. Let $J = \bigcup_w \mathcal{J}_w$, where w is a child of v; $|J| = O(r/\sqrt{\delta})$. We compute in $O(r/\sqrt{\delta}\log(r/\sqrt{\delta}))$ time an outer δ-approximation $\mathcal{J}_v$ of $\mathcal{J}$ and the upper and lower envelopes of $\mathcal{J}_v$. It is easily seen that $\mathcal{J}_v$ is an outer $(2i\delta)$-approximation of L_v. Since Π has $O(\log_r n)$ nodes, an update operation takes $O((r/\log(r))((\log n)^{3/2}/\sqrt{\varepsilon})\log((r \log n)/\varepsilon))$. We thus obtain the following.

THEOREM 3.4. *For a tree T storing n points $P(t)$, with a maximal out-degree r and depth $O(\log_r(n))$, one can perform insertion/deletion operations so that the time spent on updating the inner/outer ε-extent of the nodes is $O((r/\log(r))((\log n)^{3/2}/\sqrt{\varepsilon})\log((r \log n)/\varepsilon))$.*

A similar idea can be used to maintain the outer and inner ε-approximations of a set of hyperplanes under insertions and deletions. Omitting all the details we conclude the following.

THEOREM 3.5. *For a tree T storing n hyperplanes in $\mathbb{R}^d$, with a maximal out-degree r, and depth $O(\log_r(n))$, one can perform insertion/deletion operations so that the time spent on updating the inner/outer ε-approximate extent of the nodes is*

$$O\left(\frac{r}{\varepsilon^{d-1}} \log^{d-1} n \log_r n\right)$$

time. Each node stores a set of $O(1/\varepsilon^{d-1})$ hyperplanes that ε-approximates the extent of the hyperplanes in this subtree.

4 Maintaining Other Extent Measures

In this section we show how the algorithms described in the previous section can be used to maintain the diameter, width, and the smallest enclosing disk of a set of points moving linearly in $\mathbb{R}^2$.

Diameter. Let P be a set of n linearly moving points in the plane, and let $\varepsilon > 0$ be a parameter. We choose a sequence $\mathbf{n}_1, \ldots, \mathbf{n}_k \in \mathbb{S}^1$ of $k = O(1/\sqrt{\varepsilon})$ directions so that the angle between two consecutive directions is at most $\sqrt{\varepsilon}/2$. Let $P^i(t)$ denote the projection of P on a line ℓ_i in direction $\mathbf{n}_i$. Choose $\delta = \varepsilon/2$. We maintain a strong inner δ-extent $\beta_i^\delta(t)$ of $P^i(t)$.

LEMMA 4.1. $\max_{1\le i\le k} |\beta_i^\delta(t)| \ge (1-\varepsilon)\,\mathrm{diam}(P(t))$.

Proof. Suppose $\mathrm{diam}(P(t)) = d(p(t), q(t))$. Let $\mathbf{n}_i$ be the direction closest to the line $p(t)q(t)$, and let p^i, q^i be the projections of p and q on ℓ_i. The angle between ℓ_i and $p(t)q(t)$ is at most $\sqrt{\varepsilon}/2$. Therefore

$$\begin{aligned} |\beta_i^\delta(t)| &\ge |p^i(t)q^i(t)| \ge d(p(t),q(t))\cos(\sqrt{\varepsilon}/2) \\ &= \mathrm{diam}(P(t))(1 - 2\sin^2(\sqrt{\varepsilon}/4)) \\ &\ge (1-\varepsilon)\,\mathrm{diam}(P(t)). \end{aligned}$$

□

Hence, it suffices to maintain the maximum of the set $\mathcal{B}(t) = \{|\beta_i^\delta(t)| \mid 1 \le i \le k\}$. Recall that $|\beta_i^\delta|$ is a piecewise-linear function with a total of $O(\sqrt{1/\varepsilon})$ vertices. By Theorem 3.2, the total time spent in computing the set $\mathcal{B}$ is $O((n/\sqrt{\varepsilon})\log n)$.

We can use a kinetic data structure by Basch *et al.* [7] to maintain the maximum of $\mathcal{B}$. This structure processes at most

$$O\left(\frac{1}{\sqrt{\varepsilon}}\frac{1}{\sqrt{\varepsilon}}\log\left(\frac{1}{\varepsilon}\right)\alpha\left(\frac{1}{\varepsilon}\right)\right) = O\left(\frac{\log(1/\varepsilon)\alpha(1/\varepsilon)}{\varepsilon}\right)$$

events, and it spends $O(\log(1/\varepsilon))$ time at each of these events. Hence, we obtain the following.

THEOREM 4.1. *Given a set P of n linearly moving points in the plane and parameter $\varepsilon > 0$, there is a kinetic data structure for maintaining an ε-approximation of the diameter of S that processes $O(\log(1/\varepsilon)\alpha(1/\varepsilon)/\varepsilon)$ events and spends a total of $O((n/\sqrt{\varepsilon})\log(n))$ time on these events.*

REMARK 4.1. We can also insert or delete a point in $O(\log^2(n)/\sqrt{\varepsilon})$ time. Moreover the above algorithm can be extended to $\mathbb{R}^d$. We now have to choose a set of $O(1/\varepsilon^{(d-1)/2})$ directions, so the number of events processed by the data structure is $O(\log(1/\varepsilon)\alpha(1/\varepsilon)/\varepsilon^{d/2})$.

Smallest enclosing disk. Let $\mathcal{B}$ be the set as defined above. For each i, let $p_i(t)$ and $q_i(t)$ be the endpoints of the δ-inner extent $\beta_i^\delta(t)$. Set

$$W(t) = \{p_i(t), q_i(t) \mid 1 \le i \le k\}.$$

For a set X in the plane, let $r(X)$ be the radius of the smallest disk enclosing X. We can prove the following.

LEMMA 4.2. $r(W(t)) \ge (1-\varepsilon)r(P(t))$.

We can therefore maintain the smallest enclosing disk of $W(t)$. Unfortunately, the best known kinetic data structure to maintain the smallest enclosing disk of a point set, which basically maintains the farthest-point Voronoi diagram of the point set, processes cubic number of events. Whenever the set W changes, we restart from the beginning, spending $O(\log(1/\varepsilon)/\sqrt{\varepsilon})$ time reconstructing the data structure. Hence, the total number of events processed by the structure is $O((1/\sqrt{\varepsilon})^3 \cdot (1/\sqrt{\varepsilon}) \cdot (1/\sqrt{\varepsilon})) = O(1/\varepsilon^{5/2})$. We thus obtain the following.

THEOREM 4.2. *Given a set P of n linearly moving points in the plane and parameter $\varepsilon > 0$, there is a kinetic data structure for maintaining an ε-approximation of the smallest enclosing disk of S that processes $O(1/\varepsilon^{5/2})$ events and spends a total of $O((n/\sqrt{\varepsilon})\log(n))$ time at these events.*

5 Handling Higher Degree Motion

In this section we consider the case in which the degree of motion of P is k for some constant $k \ge 1$. As in Section 2, we focus on points in $\mathbb{R}$, that is, $p_i(t) = a_i^0 + \sum_{j=1}^k a_i^j t^j$. The goal is to compute an outer ε-extent of $B(t)$. Instead of working with a set of curves (graphs of trajectories of points) in the parametric xt-plane, we use the so-called *linearization* technique (see e.g. [15, 3]). We map each point $p_i \in P$ to a hyperplane h_i in $\mathbb{R}^{k+1}$ $h_i : x_{k+1} = a_i^0 + \sum_{j=1}^k a_i^j x_j$. Set $\mathcal{H} = \{h_1, \ldots, h_n\}$. Let $\mu : \mathbb{R} \to \mathbb{R}^k$ be the curve $\mu(t) = (t, t^2, \ldots, t^k)$, then $p_i(t) = h_i(\mu(t))$. For brevity, let $\mathcal{U}(\mathbf{x}) = \mathcal{U}_\mathcal{H}(\mathbf{x})$, $\mathcal{L}(\mathbf{x}) = \mathcal{L}_\mathcal{H}(\mathbf{x})$, and $I(\mathbf{x}) = I_\mathcal{H}(\mathbf{x})$. The lower (resp. upper) endpoint of $B(t)$ is $L(\mu(t))$ (resp. $U(\mu(t))$), and $|B(t)| = I(\mu(t))$. Using Theorem 2.1, we can compute an outer ε-approximation $\mathcal{J}$ of $\mathcal{H}$ of size $O(1/\varepsilon^{k/2})$. The restriction of $I_\mathcal{J}(x)$ to the curve μ gives an outer ε-extent of P, whose combinatorial structure changes roughly $O(1/\varepsilon^{k/2})$ times.

A drawback of this approach is that the bound depends on k, the degree of motion. In this section we describe a different approach that can maintain an ε-extent of P whose combinatorial structure changes $O((1/\varepsilon)\log 1/\varepsilon)$ times. In particular, we compute two piecewise-linear functions $\mathcal{L}^\varepsilon, \mathcal{U}^\varepsilon : \mathbb{R}^k \to \mathbb{R}$ so that

(C1) $\mathcal{L}(\mathbf{x}) - (\varepsilon/2)I(\mathbf{x}) \le \mathcal{L}^\varepsilon(\mathbf{x}) \le \mathcal{L}(\mathbf{x})$, and

(C2) $\mathcal{U}(\mathbf{x}) \le \mathcal{U}^\varepsilon(\mathbf{x}) \le \mathcal{U}(\mathbf{x}) + (\varepsilon/2)I(\mathbf{x})$.

We will show that the restriction of $I^\varepsilon(\mathbf{x}) = \mathcal{U}^\varepsilon(\mathbf{x}) - \mathcal{L}^\varepsilon(\mathbf{x})$ to the curve μ has $O((1/\varepsilon)\log 1/\varepsilon)$ breakpoints. We first prove the following result.

THEOREM 5.1. *Given a set of n hyperplanes in $\mathbb{R}^{k+1}$ and a parameter $\varepsilon > 0$, one can compute in time $O(n/\varepsilon^k \log 1/\varepsilon)$ two piecewise-linear functions $\mathcal{L}^\varepsilon, \mathcal{U}^\varepsilon : \mathbb{R}^k \to \mathbb{R}$ that satisfy conditions (C1) and (C2) and that consist of $O(1/\varepsilon^k \log(1/\varepsilon))$ k-simplices.*

Note that, unlike Section 2, we do not compute a small set of hyperplanes whose lower and upper envelopes approximate the extent of $\mathcal{H}$.

This theorem is proved in three stages. In the first stage, we apply a transformation on $\mathcal{H}$ so that the resulting set of hyperplanes are "well behaved." We then draw a d-dimensional box $\mathcal{C}$ on the hyperplane $x_{k+1} = 0$ and construct $\mathcal{U}^\varepsilon(\mathbf{x})$ and $\mathcal{L}^\varepsilon(\mathbf{x})$ for $\mathbf{x} \in \mathcal{C}$. We then compute the approximate envelopes on the boundary $\partial\mathcal{C}$, by invoking the algorithm recursively in one lower dimension, and then extend them to the exterior of $\mathcal{C}$. Because of lack of space, we simply describe the properties of the transformed hyperplanes and briefly describe the construction inside $\mathcal{C}$.

DEFINITION 5.1. For positive constants α, β, a set $\mathcal{H}$ of hyperplanes in $\mathbb{R}^{k+1}$ is (α,β)-*normalized* if the following conditions hold:

(i) $I(\mathbf{x})$ is minimum at the origin o;

(ii) the hyperplane $x_{k+1} = 0$ is in $\mathcal{H}$;

(iii) for any point $v \in \mathbb{R}^k$, there exists a hyperplane $h \in \mathcal{H}$, such that $|h(v) - h(o)| \ge \alpha||v||$, where o denotes the origin; and

(iv) for any vector $v \in \mathbb{R}^k$ and for any $h \in \mathcal{H}$, $|h(v) - h(o)| \le \beta||v||$.

The proof of the following lemma is omitted.

LEMMA 5.1. *One can compute in linear time a transformation T so that $\mathcal{H}' = T(\mathcal{H})$ is $(c_k, \sqrt{k+1})$-normalized, where c_k is a positive constant depending only on k.*

LEMMA 5.2. *Let $\mathcal{H}$ be a set of (α,β)-normalized hyperplanes in $\mathbb{R}^{k+1}$, and let $\mathcal{C}$ be the hypercube of side length $12M/(\varepsilon\alpha)$ centered at the origin, where $M = I(0)$ is the minimal extent of $\mathcal{H}$. We compute in time $O(n/\varepsilon^k \log 1/\varepsilon)$ two piecewise-linear functions $\mathcal{U}^\varepsilon, \mathcal{L}^\varepsilon$ (as defined above) of complexity $O(1/\varepsilon^k \log 1/\varepsilon)$ that satisfy (C1) and (C2) for points inside $\mathcal{C}$.*

Proof. Let $\mathcal{C}_0$ be the cube of side $M_0 = 12M/\alpha$ centered at the origin, and let $\mathcal{G}_0$ be a uniform grid in $\mathcal{C}_0$ with distance $\delta_0 = \varepsilon M/(12(k+1)\beta)$ between grid points. For each point z of $\mathcal{G}_0$, we compute $z^+ = \mathcal{U}(z)$ and $z^- = \mathcal{L}(z)$. Next, each cell of G_0 (which is a subcube of side length δ_0) is partitioned into simplices in some canonical way. This results into a simplicial decomposition Ξ of $\mathcal{C}_0$ into $O(1/\varepsilon^k)$ simplices. We lift each such simplex $\Delta = \mathrm{conv}(z_1, \ldots, z_{k+1})$ to two k-simplices $\Delta^+ = \mathrm{conv}(z_1^+, \ldots, z_{k+1}^+)$ and $\Delta^- = \mathrm{conv}(z_1^-, \ldots, z_{k+1}^-)$ in $\mathbb{R}^{k+1}$. We claim that Δ^+, Δ^- approximate $\mathcal{U}$ and $\mathcal{L}$ inside Δ.

By the convexity of the upper (reps. lower) envelope of $\mathcal{H}$, $\Delta^+(\mathbf{x}) \ge \mathcal{U}(\mathbf{x})$ and $\Delta^-(\mathbf{x}) \le \mathcal{L}(\mathbf{x})$ for all $\mathbf{x} \in \Delta$. Thus, it remains to bound the error incurred by this approximation. The error for each of Δ^+ andΔ^- is clearly bounded by $\beta\,\mathrm{diam}(\Delta)$ because $\mathcal{H}$ is (α,β)-normalized. Thus, for any $\mathbf{x} \in \Delta$,

$$\begin{aligned}
\Delta^+(\mathbf{x}) &\le \mathcal{U}(\mathbf{x}) + \beta\,\mathrm{diam}(\Delta) \\
&\le \mathcal{U}(\mathbf{x}) + \beta\frac{\sqrt{k+1}\varepsilon M}{12(k+1)\beta} \\
&\le \mathcal{U}(\mathbf{x}) + \frac{\varepsilon M}{12\sqrt{k+1}} \\
&\le \mathcal{U}(\mathbf{x}) + \frac{\varepsilon}{2}I(\mathbf{x})
\end{aligned}$$

since $I(\mathbf{x}) \ge M$. Similarly, we can prove that $\Delta^-(\mathbf{x}) \ge \mathcal{L}(\mathbf{x}) - \varepsilon I(\mathbf{x})/2$.

We continue in the same way as follows: Let $\mathcal{C}_i$ be the cube centered at the origin of side $M_i = 12 \cdot 2^{i-1}M/\alpha$, for $i = 1, \ldots, \lceil \log 1/\varepsilon \rceil$. Note, that for $\mathbf{x} \in \mathcal{C}_i \setminus \mathcal{C}_{i-1}$, we have $I(\mathbf{x}) \ge M_i\alpha/2$, since $\mathcal{H}$ is (α,β)-normalized. Let $\mathcal{G}_i$ be a uniform grid in $\mathcal{C}_i$ with distance $\delta_i = \varepsilon 2^{i-1}M/(12(k+1)\beta)$ between the grid points. For each point $z \in G_i \cap (\mathcal{C}_i \setminus \mathcal{C}_{i-1})$, we compute the $\mathcal{U}(z)$ and $\mathcal{L}(z)$ and compute approximations of $\mathcal{U}$ and $\mathcal{L}$ inside $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ as earlier. Arguing as above, we conclude that the resulting functions satisfy (C1) and (C2) inside $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$.

Thus, the resulting piecewise-linear functions (defined by the lifted simplices of the resulting simplicial decomposition of $\mathcal{C}$), ε-approximate the extent of $\mathcal{H}$, and they can be computed in $O(n/\varepsilon^k \log 1/\varepsilon)$ time, since $\mathcal{U}(z), \mathcal{L}(z)$, for any point $z \in \mathbb{R}^k$ can be computed in $O(n)$ time. □

We omit the details of computing $\mathcal{U}^\varepsilon$ and $\mathcal{L}^\varepsilon$ in the exterior of $\mathcal{C}$. This completes the proof of Theorem 5.1.

Finally, we observe the the curve μ intersects only $O(1/\varepsilon)$ cells in $\mathcal{C}_0$ and in $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ for $i \geq 1$. Therefore, the restriction of $\mathcal{U}^\varepsilon$ and $\mathcal{L}^\varepsilon$ to μ has complexity only $O(\log(1/\varepsilon)/\varepsilon)$. Putting everything together and omitting all further details, we obtain the following.

THEOREM 5.2. *Given a set of P of n points in $\mathbb{R}^d$ with motion of degree k, one can maintain an ε-approximate smallest bounding rectangle of P, in a total of $O((n/\varepsilon)\log(1/\varepsilon))$ time, whose combinatorial structure changes $O(\log(1/\varepsilon)/\varepsilon)$ times; the constant of proportionality depends on k.*

Using the above theorem in conjunction with Theorem 3.5, we can expedite the time spent in updating the rectangle.

THEOREM 5.3. *Given a set of P of n points in $\mathbb{R}^d$ with motion of degree k, one can maintain an ε-approximation $B^\varepsilon(t)$ of the smallest enclosing rectangle of P, whose combinatorial structure changes $O(\log(1/\varepsilon)/\varepsilon)$ times; the constant of proportionality depends on k. We can compute the new rectangle after insertion/deletion in time*

$$O\left(\frac{\log^{k+1} n}{\varepsilon^k} + \frac{1}{\varepsilon^{k+1}} \log \frac{1}{\varepsilon}\right).$$

Proof. We use the algorithm of Theorem 3.5 to maintain an $\varepsilon/2$-approximate extent to the given set of hyperplanes (with $r = 2$). For the extent stored in the root of our tree, we use the above grid approximation, to compute the extent of the points along the curve μ, this results in $O(1/\varepsilon \log(1/\varepsilon))$ events, and since the number of hyperplanes stored in the root of the tree is $O(1/\varepsilon^k)$, the bound follows. □

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