Exact and Approximation Algorithms for Minimum-Width Cylindrical Shells*

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Abstract

Let S be a set of n points in $\mathbb{R}^3$. Let $\omega^* = \omega^*(S)$ be the width (i.e., thickness) of a minimum-width infinite cylindrical shell (the region between two co-axial cylinders) containing S. We first present an $O(n^5)$-time algorithm for computing ω^*, which as far as we know is the first nontrivial algorithm for this problem. We then present an $O(n^{2+\delta})$-time algorithm, for any $\delta > 0$, that computes a cylindrical shell of width at most $26(1 + 1/n^{4/9})\omega^*$ containing S.

1 Introduction

Given a line ℓ in $\mathbb{R}^3$ and two real numbers $0 \le r \le R$, the *cylindrical shell* $\Sigma(\ell, r, R)$ is the closed region lying between the two co-axial cylinders of radii r and R with ℓ as their axis, i.e.,

$$\Sigma(\ell, r, R) = \{p \in \mathbb{R}^3 \mid r \le d(p, \ell) \le R\},$$

where $d(p, \ell)$ is the Euclidean distance between point p and line ℓ. The *width* of $\Sigma(\ell, r, R)$ is $R-r$. Let S be a set of n points in $\mathbb{R}^3$. How well S fits a cylindrical surface can be measured by computing a cylindrical surface $\mathcal{C} = \mathcal{C}(S)$ so that the maximum distance between any point of S and $\mathcal{C}$ is minimized. If ℓ and ρ are the axis and the radius of $\mathcal{C}$ and δ is the maximum distance between $\mathcal{C}$ and S, then $S \subset \Sigma(\ell, \rho - \delta, \rho + \delta)$. Hence, the problem of approximating S by a cylindrical surface is equivalent to computing a cylindrical shell, $\Sigma^*(S)$, of the minimum width that contains S.

The main motivation for computing a minimum-width cylindrical shell comes from computational metrology. In order to measure the quality of a manufactured cylinder Γ, we sample a set S of points on the surface of Γ using coordinate measuring machines and then fit a cylindrical surface through S so that the maximum distance between the points of S and the cylinder is minimized. For example, this is one of the criteria suggested in the recent ASME Y14.5M standard to determine how closely Γ resembles a cylinder [14, 15].

In the last few years much work has been done on measuring the circularity of a planar point set, which is defined as the width of the thinnest annulus that contains the point set [2, 5, 9, 10, 11]. The best known algorithm runs in $O(n^{3/2+\delta})$, for any $\delta > 0$ [5], and near-linear approximation algorithms are proposed in [2, 9]. In three dimensions, the minimum-width spherical shell (a region enclosed between two concentric spheres) containing an n-element point set S can be computed in time $O(n^{3-\frac{1}{19}+\delta})$, for any $\delta > 0$ [2]. The same paper also presents near-linear algorithms that compute an approximation to the minimum-width enclosing spherical shell in any dimension. There has also been some work on computing the smallest cylinder enclosing a point set in $\mathbb{R}^3$ [1, 13]. Agarwal *et al.* [1] developed an $O(n^{3+\delta})$-time algorithm, for any $\delta > 0$, for computing the smallest enclosing cylinder. They also proposed a $(1+\varepsilon)$-approximation algorithm (i.e., an algorithm that produces an enclosing cylinder whose radius is at most $(1+\varepsilon)$ times the minimum radius) that runs in $O(n/\varepsilon^2)$ time.

Finding the minimum-width cylindrical shell that contains a given set of points is harder than computing a minimum-width enclosing spherical shell, computing a smallest enclosing cylinder, or computing a thinnest annulus containing a planar point set. Actually, the second and third problems are special cases of computing a thinnest cylindrical shell — finding a smallest enclosing cylinder is the same as finding a minimum-width cylindrical shell whose inner radius is 0; and finding a thinnest cylindrical shell with axis parallel to a given direction $\mathbf{n}$ is the same as finding a thinnest annulus containing the projection of S in direction $\mathbf{n}$ onto a plane orthogonal to $\mathbf{n}$. Since a cylindrical shell is specified by six parameters — four parameters define the axis of

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the shell, and the remaining two define the inner and outer radii of the shell, $\Sigma^*(S)$ is "defined" by a subset $A \subset S$ of six points, in the sense that $\Sigma^*(S)$ is one of the $O(1)$ cylindrical shells that contain A on their inner and outer boundaries. This suggests the following naive procedure for computing $\Sigma^*(S)$: For each subset $A \subseteq S$ of size six, compute the $O(1)$ cylindrical shells containing A on their inner and outer boundary. For each such shell Σ, check in $O(n)$ time whether $S \subset \Sigma$. Return the thinnest among them that contains S. This naive approach leads to an $O(n^7)$ algorithm for computing $\Sigma^*(S)$. As the first result of this paper, we describe, in Section 2, an improved $O(n^5)$-time algorithm for computing $\Sigma^*(S)$. We are not aware of any faster algorithm for the exact problem.

Since computing $\Sigma^*(S)$ is so expensive, we develop an efficient approximation algorithm for computing a cylindrical shell that contains S and has width at most $c\omega^*$, where $\omega^* = \omega^*(S)$ is the width of $\Sigma^*(S)$ and c is a constant. We first prove in Section 3 a *Helly-type* theorem for $\Sigma^*(S)$, which we believe to be of independent interest, and which, roughly speaking, says the following: Let $A \subseteq S$ be a subset of four points so that the volume of the tetrahedron spanned by A is close to the largest volume of a simplex spanned by any four points of S. Then $\omega^*(S) \le c \cdot \max_{p\in S} \omega^*(A \cup \{p\})$, for a constant $c > 1$. The constant that our analysis yields is about 26, but we believe that the theorem also holds with a much smaller constant. Using this observation, we develop in Section 4 an $O(n^{2+\delta})$-time algorithm, for any $\delta > 0$, to compute a cylindrical shell of width at most about $26\omega^*$ that contains S. We believe that our approach can be strengthened to compute in near-linear time a cylindrical shell of width $O(\omega^*)$ that contains S, but at present there are some technical difficulties that we have not overcome yet (see Remark 4.4 for more details). We also believe that our technique can be enhanced to yield a near-quadratic algorithm that approximates the minimum width of an enclosing cylindrical shell by a factor of at most $1 + \varepsilon$, for any $\varepsilon > 0$. This approach also faces some technical difficulties that we are currently studying.

2 Computing $\Sigma^*(S)$

In this section we describe an $O(n^5)$-time algorithm for computing $\Sigma^*(S)$. Without loss of generality assume that the axis of $\Sigma^*(S)$ is not parallel to the xy-plane; the case of a horizontal axis can be handled by a simpler algorithm, whose details are omitted. A cylinder C with a nonhorizontal axis a can be parameterized by a five-tuple (a_1, a_2, a_3, a_4, r), where r is the radius of C and where the axis of C is the line $a = \{p + tq \mid t \in \mathbb{R}\}$, $p = (a_1, a_2, 0)$ is the intersection point of a with the xy-plane, and $q = (a_3, a_4, 1)$ is the direction vector of a. Let x be a point in $\mathbb{R}^3$. Changing the coordinate system so that p maps to the origin, we observe that the projection of x on the axis a is $((x-p)\cdot q/\|q\|)q/\|q\| = ((x-p)\cdot q/\|q\|^2)q$. Hence, distance between x and the line a is

$$d(x, a) = \left\| (p - x) - \frac{(p-x)\cdot q}{\|q\|^2} q \right\|.$$

Since x lies in the cylinder C if and only if $d(x, a) \le r$, after some algebraic manipulation, we obtain that $x = (x_1, x_2, x_3)$ lies inside C if and only if

$$f(x_1, x_2, x_3, a_1, a_2, a_3, a_4) \le (a_3^2 + a_4^2 + 1)r^2,$$

where

$$\begin{aligned} f(x_1, x_2, x_3, a_1, a_2, a_3, a_4) \ = \\ &[(a_4^2+1)a_1^2 + (a_3^2+1)a_2^2 - 2a_1a_2a_3a_4] + \\ &2[a_2a_3a_4 - a_1(a_4^2+1)]x_1 + 2[a_1a_3a_4 - \\ &a_2(a_3^2+1)]x_2 + 2[a_1a_3 + a_2a_4]x_3 - \\ &2[a_3a_4]x_1x_2 - 2[a_3]x_1x_3 - 2[a_4]x_2x_3 + [1](x_1^2+x_2^2) + \\ &[a_3^2](x_2^2+x_3^2) + [a_4^2](x_1^2+x_3^2)\,. \qquad (1) \end{aligned}$$

Hence, a point x lies in a cylindrical shell $\sigma = (a_1, a_2, a_3, a_4, r, R)$ with axis $a = (a_1, a_2, a_3, a_4)$, parametrized as above, inner radius r, and outer radius R if and only if

$$\begin{aligned} r^2(a_3^2 + a_4^2 + 1) &\le f(x_1, x_2, x_3, a_1, a_2, a_3, a_4) \\ &\le R^2(a_3^2 + a_4^2 + 1). \qquad (2) \end{aligned}$$

Let us set

$$\begin{aligned} \varphi_1(\sigma) &= a_2a_3a_4 - a_1(a_4^2+1), \\ \varphi_2(\sigma) &= a_1a_3a_4 - a_2(a_3^2+1), \\ \varphi_3(\sigma) &= a_1a_3 + a_2a_4, \\ \varphi_4(\sigma) &= a_3a_4, \quad \varphi_5(\sigma) = a_3 \\ \varphi_6(\sigma) &= a_4, \quad \varphi_7(\sigma) = a_3^2, \quad \varphi_8(\sigma) = a_4^2 \\ \varphi_9(\sigma) &= r^2(a_3^2+a_4^2+1) - (a_4^2+1)a_1^2 - (a_3^2+1)a_2^2 + \\ &\quad 2a_1a_2a_3a_4, \\ \varphi_{10}(\sigma) &= R^2(a_3^2+a_4^2+1) - (a_4^2+1)a_1^2 - (a_3^2+1)a_2^2 + \\ &\quad 2a_1a_2a_3a_4, \end{aligned}$$

$$\begin{aligned} \psi_0(x) &= (x_1^2+x_2^2), \quad \psi_1(x) = 2x_1, \\ \psi_2(x) &= 2x_2, \quad \psi_3(x) = 2x_3, \\ \psi_4(x) &= -2x_1x_2, \quad \psi_5(x) = -2x_1x_3 \\ \psi_6(x) &= -2x_2x_3, \quad \psi_7(x) = x_2^2 + x_3^2 \\ \psi_8(x) &= x_1^2 + x_3^2. \end{aligned}$$

Then the constraint (2) can be rewritten as a linear constraint

$$H_x(\sigma) : \varphi_9(\sigma) \le \psi_0(x) + \sum_{i=1}^{8} \varphi_i(\sigma)\psi_i(x) \le \varphi_{10}(\sigma).$$

For any point $p \in \mathbb{R}^3$, define the wedge $H_p \subset \mathbb{R}^{10}$, formed by the intersection of two halfspaces, as

$$H_p = \left\{ (y_1, \ldots, y_{10}) \mid y_9 \le \psi_0(p) + \sum_{i=1}^{8} y_i \psi_i(p) \le y_{10} \right\}.$$

Set $\varphi(\sigma) = \langle \varphi_1(\sigma), \ldots, \varphi_{10}(\sigma) \rangle \in \mathbb{R}^{10}$. Let $P = \bigcap_{p \in S} H_p$ be the convex polyhedron defined by the intersection of the $2n$ corresponding halfspaces. P has $O(n^5)$ faces and can be computed in $O(n^5)$ time [8]. A cylindrical shell (with nonhorizontal axis) σ contains S if and only $\varphi(\sigma) \in P$.

Let $\Psi \subseteq \mathbb{R}^4 \times (\mathbb{R}^+)^2$ denote the 6-dimensional set of all cylindrical shells (with nonhorizontal axis) that contain S. Then $\varphi(\Psi)$ is the intersection of P with the 6-dimensional surface $\Phi = \{\varphi(\sigma) \mid \sigma \in \mathbb{R}^4 \times (\mathbb{R}^+)^2\}$. After having computed P, Ψ can be computed in $O(n^5)$ time, e.g., by triangulating P into $O(n^5)$ simplices and then, for every simplex τ in the triangulation, computing $\tau \cap \Phi$. Finally, for each simplex τ, we compute in $O(1)$ time the minimum-width cylindrical shell σ such that $\varphi(\sigma) \in \tau \cap \varphi(\Psi)$. Hence, we can conclude the following.

Theorem 2.1 *Given a set S of n points in $\mathbb{R}^3$, a minimum-width cylindrical shell containing S can be computed in $O(n^5)$ time.*

3 A Helly-like Property of Cylindrical Shells

Let S be a set of n points in $\mathbb{R}^3$, and let $t > 1$ be a constant. For any finite point set $X \subset \mathbb{R}^3$ of at least four points, let $\mu(X)$ denote the volume of the largest volume simplex spanned by four points of X. Let Δ be a tetrahedron spanned by points of S so that its volume is $\mu(S)/t$. Let $A = \{a_1, \ldots, a_4\} \subseteq S$ denote the set of vertices of Δ. The simplex Δ has the following useful property.

Lemma 3.1 *Let f be any k-flat, for $k = 0, 1, 2$. Then for any $p \in S$ we have*

$$d(p, f) \le (4t-1) \cdot \max_{1 \le i \le 4} d(a_i, f). \quad (3)$$

Proof: Let $K \subset \mathbb{R}^3$ be the locus of all points q so that each of the simplices $a_1a_2a_3q$, $a_1a_2a_4q$, $a_1a_3a_4q$, and $a_2a_3a_4q$ has volume at most $t \cdot \mathrm{Vol}(\Delta)$; see Figure 1. By assumption, we have $S \subset K$. Let h_i be the plane containing $A \setminus \{a_i\}$, and let Λ_i be the slab bounded by two planes parallel to h_i and at distance $t \cdot d(a_i, h_i)$ from it. $K = \bigcap_{i=1}^4 \Lambda_i$; see Figure 1. Using barycentric coordinates, we can represent any point $q \in K$ as $q = \sum_{i=1}^4 \lambda_i a_i$, where $\sum_{i=1}^4 \lambda_i = 1$ and $|\lambda_i| \le t$, for $i = 1, \ldots, 4$. Let b_i, for $i = 1, \ldots, 4$, be the point in f nearest to a_i, and put $q^* := \sum_{i=1}^4 \lambda_i b_i \in f$. We then have

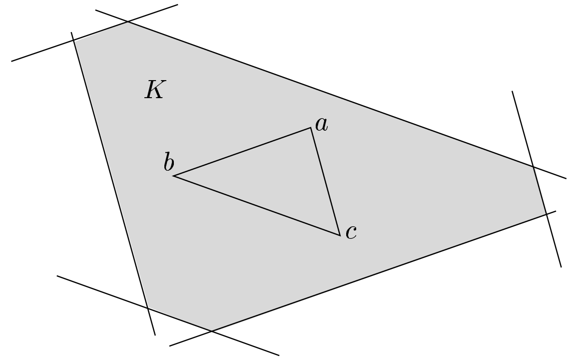

Figure 1: A two dimensional version of the region K, for t slightly larger than 1

$$\begin{aligned} d(q,f) &\le d(q,q^*) \\ &= d\left(\sum_{i=1}^4 \lambda_i a_i, \sum_{i=1}^4 \lambda_i b_i\right) \\ &= \left\| \sum_{i=1}^4 \lambda_i (a_i - b_i) \right\| \\ &\le \sum_{i=1}^4 |\lambda_i| d(a_i, f) \\ &\le (4t-1) \cdot \max_{1 \le i \le 4} d(a_i, f), \end{aligned}$$

for each $q \in K$, where the last inequality follows by observing that $\max \sum_{i=1}^4 |\lambda_i|$, subject to $\sum_{i=1}^4 \lambda_i = 1$ and $|\lambda_i| \le t$ for $i = 1, \ldots, 4$, is $4t - 1$. This implies the assertion of the lemma. □

Fix a direction $\mathbf{n} \in \mathbb{S}^2$, and let $\pi = \pi^{(\mathbf{n})}$ be the plane normal to $\mathbf{n}$ and passing through the origin. For a point $x \in \mathbb{R}^3$, let x^* denote its orthogonal projection onto π. Set $S^* = \{p^* \mid p \in S\}$. Similarly, define A^* to be the projection of A onto π.

Corollary 3.2 *(i) Let o and ρ be the center and radius of the smallest disk enclosing A^*. Then S^* is contained in the disk of radius $(4t-1)\rho$ centered at o.*

(ii) For any line ℓ lying in π,

$$\max_{p \in S} d(p^*, \ell) \le (4t-1) \max_{a \in A} d(a^*, \ell).$$

Proof: Part (i) follows by applying Lemma 3.1 to the line in direction $\mathbf{n}$ and passing through o. The second part is proved by applying Lemma 3.1 to the plane orthogonal to π and passing through ℓ. □

The next theorem is the main result of this section.

Theorem 3.3 *Suppose there exists $\omega > 0$ such that for each $p \in S^*$ there exists an annulus of width ω that encloses $A^* \cup \{p\}$. Then there exists an annulus of width at most $26t\omega$ that encloses S^*.*

We need the following geometric lemma to prove the above theorem. Let $D(x, \delta)$ denote the disk of diameter δ centered at a point x.

Lemma 3.4 *Let abc be a triangle in the plane, and let $\tau \geq 1$ and $0 < \omega <$ Width $(\triangle abc)/3.25$ be two parameters. Define $\Delta = \Delta(\tau)$ to be the locus of all points x such that the area of each of the triangles $\triangle abx$, $\triangle acx$, $\triangle bcx$ is at most τ times the area of $\triangle abc$. Let C and C' be two circles, each of which meets all three disks $D(a, \omega)$, $D(b, \omega)$, $D(c, \omega)$. Then, for any $z \in C \cap \Delta$ we have*

$$d(z, C') \leq (6.5\tau + 3.6)\omega$$

(see Figure 2(i)).

Remark 3.5 Informally, the lemma asserts that if two circles are close to each other near three points a, b, c then they remain close to each other within Δ. Without such a confinement, the assertion may fail, as is easily checked.

Proof: We parametrize points on C using *inversion*, as follows. Pick points $u \in C \cap \mathcal{D}(a, \omega)$, $v \in C \cap \mathcal{D}(b, \omega)$, $w \in C \cap \mathcal{D}(c, \omega)$. Without loss of generality, we may assume that the order of u, v, w and z along C in the clockwise direction is u, v, z, w. Write $v = u + p$, $w = u + q$, and $z = u + \zeta$. Apply an inversion to the plane that takes u to infinity. For example, using complex numbers, we may use the transformation $\xi \mapsto 1/(\xi - u)$. This transformation maps C to a straight line containing the images $1/p$, $1/q$, and $1/\zeta$ of v, w, and z, respectively, so that $1/\zeta$ lies *between* $1/p$ and $1/q$. Hence there is a real parameter $\lambda \in [0, 1]$, such that

$$\frac{1}{\zeta} = \frac{\lambda}{p} + \frac{1-\lambda}{q}, \tag{4}$$

or

$$\zeta = \frac{pq}{\lambda q + (1-\lambda)p}.$$

The following geometric interpretation will be useful in the subsequent analysis. Put $s = \lambda q + (1-\lambda)p$ and $x = u + s$. The point x lies on the edge vw of the triangle uvw and splits it in the ratio $\lambda : (1-\lambda)$; that is $|x - v| = \lambda|w - v|$ and $|x - w| = (1-\lambda)|w - v|$. Since $pq = \zeta s$ (or $p/s = \zeta/q$), the triangles $\triangle vux$ and $\triangle zuw$ are similar. Analogously, we can prove that the triangles $\triangle wux$ and $\triangle zuv$ are similar. See Figure 2(ii).

This implies that

$$\frac{\lambda|w-v|}{|s|} = \frac{|w-z|}{|q|} \text{ and } \frac{(1-\lambda)|w-v|}{|s|} = \frac{|v-z|}{|p|}. \tag{5}$$

Since u, v, z, w are cocircular, $\measuredangle vuw = \pi - \measuredangle vzw$, therefore $\sin(\measuredangle vuw) = \sin(\measuredangle vzw)$. Multiplying the two equalities in (5), we obtain

$$\begin{aligned} \lambda(1-\lambda)|w-v|^2 &= |s|^2 \cdot \frac{|v-z||w-z|}{|p||q|} \\ &= |s|^2 \cdot \frac{|v-z| \cdot |w-z| \sin(\measuredangle vzw)}{|p| \cdot |q| \sin(\measuredangle vuw)} \\ &= |s|^2 \cdot \frac{\text{Area}\,(\triangle vwz)}{\text{Area}\,(\triangle uvw)}, \end{aligned}$$

We will prove below in Corollary 3.7 that

$$\text{Area}\,(\triangle vwz) \leq \frac{289}{81}\tau \cdot \text{Area}\,(\triangle uvw). \tag{6}$$

Intuitively, this is true because the area of the triangle uvw (resp. vwz) is a good approximation of the area of abc (resp. bcz); a rigorous proof is given in Lemma 3.6 below.

We thus have

$$\lambda(1-\lambda)|w-v|^2 \leq \frac{289}{81}\tau|s|^2. \tag{7}$$

Let $\theta = \measuredangle uvw$. Using the law of cosines, we have

$$|s|^2 = |p|^2 + \lambda^2|w-v|^2 - 2\lambda|p||w-v|\cos\theta$$

and

$$|q|^2 = |p|^2 + |w-v|^2 - 2|p||w-v|\cos\theta.$$

Subtracting λ times the second equality from the first, we obtain

$$|s|^2 - \lambda|q|^2 = |p|^2 + \lambda^2|w-v|^2 - \lambda|p|^2 - \lambda|w-v|^2,$$

or

$$|s|^2 = \lambda|q|^2 + (1-\lambda)|p|^2 - \lambda(1-\lambda)|w-v|^2. \tag{8}$$

Combining (7) and (8), we obtain

$$\lambda|q|^2 + (1-\lambda)|p|^2 \leq \left(\frac{289}{81}\tau + 1\right)|s|^2. \tag{9}$$

Apply a symmetric transformation to parametrize C': Pick points $u' \in C' \cap \mathcal{D}(a, \omega)$, $v' \in C' \cap \mathcal{D}(b, \omega)$, $w' \in C' \cap \mathcal{D}(c, \omega)$. Write $v' = u' + p'$, $w' = u' + q'$, and put

$$z' = u' + \frac{p'q'}{\lambda q' + (1-\lambda)p'} \in C'.$$

Set

$$\delta = \frac{pq}{\lambda q + (1-\lambda)p} - \frac{p'q'}{\lambda q' + (1-\lambda)p'}.$$

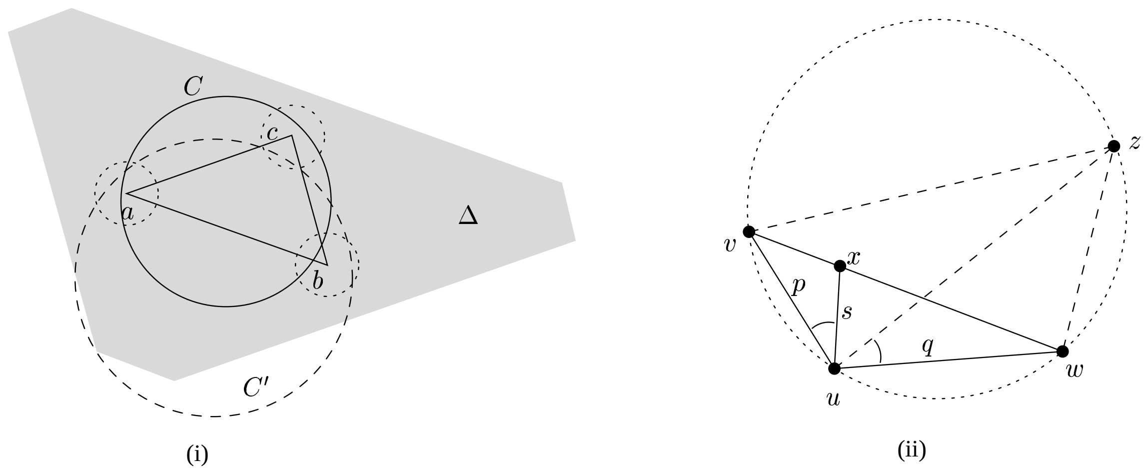

Figure 2: (i) Setup of the lemma; (ii) geometric interpretation of the inversion

Put $\xi = p' - p$ and $\eta = q' - q$. Observe that $|\xi|, |\eta| \le \omega$. We have

$$|\delta| = \left| \frac{pq}{\lambda q + (1-\lambda)p} - \frac{(p+\xi)(q+\eta)}{\lambda(q+\eta) + (1-\lambda)(p+\xi)} \right| \le \frac{|\lambda q + (1-\lambda)p| \cdot |\xi| \cdot |\eta| + \lambda |q|^2 |\xi| + (1-\lambda)|p|^2|\eta|}{|\lambda q + (1-\lambda)p| \cdot |\lambda(q+\eta) + (1-\lambda)(p+\xi)|}.$$

Hence the denominator in the expression for δ is at least $|s|(|s| - \omega)$. Moreover, $|s|$ is larger than the height to vw in the triangle uvw. As we will show below in Lemma 3.6, this height is at least Width $(\triangle abc) - \omega \ge 2.25\omega$ (again, this holds because $\triangle uvw$ is a good approximation of $\triangle abc$). Therefore

$$|\delta| \le \frac{|s|\omega^2 + \omega(\lambda|q|^2 + (1-\lambda)|p|^2)}{|s|(|s|-\omega)}.$$

Using (9) and the fact that $|s| \ge 2.25\omega$, we obtain

$$\begin{aligned} |\delta| &\le \left(\frac{1}{(|s|/\omega) - 1} + \frac{(289/81)\tau + 1}{1 - (\omega/|s|)} \right) \omega \\ &\le \left(\frac{4}{5} + \frac{9\,((289/81)\tau + 1)}{5} \right) \omega \\ &\le \left(\frac{289}{45}\tau + 2.6 \right) \omega. \end{aligned}$$

Therefore,

$$\begin{aligned} d(z, C') &\le d(z, z') \le d(u, u') + |\delta| \\ &\le \left(\frac{289}{45}\tau + 3.6 \right) \omega \\ &\le (6.5\tau + 3.6)\omega. \end{aligned}$$

This completes the proof of the lemma. □

We still need to establish the following lemma.

Lemma 3.6 (a) Area $(\triangle uvw) \ge \frac{81}{169}$Area $(\triangle abc)$.

(b) Area $(\triangle vwz) \le \frac{289\tau}{169}$Area $(\triangle abc)$.

(c) $|$Width $(\triangle uvw) -$ Width $(\triangle abc)| \le \omega$.

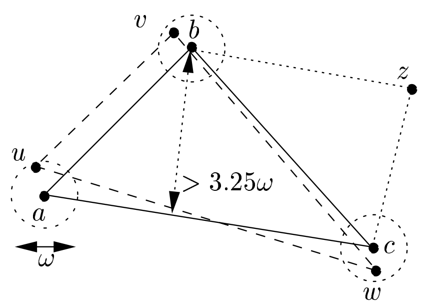

Figure 3: Illustration to Lemma 3.6.

Proof: For a segment e, let ℓ_e be the line supporting e. To prove (a), without loss of generality, let uv be the longest edge in $\triangle uvw$. Then the orthogonal projection w^* of w on the line ℓ_{uv} lies on the segment uv itself, so that $w^* = \lambda u + (1-\lambda)v$ for some λ, $0 \le \lambda \le 1$. Let $w' = \lambda a + (1-\lambda)b$. Then

$$\begin{aligned} d(w, \ell_{ab}) &\le |ww'| \le |ww^*| + |w^*w'| \\ &= |ww^*| + |\lambda(a-u) + (1-\lambda)(b-v)|. \end{aligned}$$

Since $|ua|, |vb| \le \omega/2$,

$$d(w, \ell_{ab}) \le d(w, uv) + \omega/2. \qquad (10)$$

Let h be the distance between c and the line supporting ab. Then

$$h = d(c, \ell_{ab}) \le d(w, \ell_{ab}) + |cw| \le d(w, uv) + \omega.$$

Therefore

$$\begin{aligned}\text{Area}\,(\triangle uvw) &= \frac{1}{2}|uv|\cdot d(w,uv)\\ &\geq \frac{1}{2}(|ab|-\omega)(h-\omega)\\ &= \frac{h\cdot|ab|}{2}\left(1-\frac{\omega}{h}\right)\left(1-\frac{\omega}{|ab|}\right)\\ &\geq \frac{81}{169}\text{Area}\,(\triangle abc).\end{aligned}$$

The last inequality follows from the fact that

$$h, |ab| \geq \text{Width}\,(\triangle abc) \geq 3.25\omega.$$

Next, we prove (b). Obviously, Area($\triangle vwz$) is maximum when z lies on a vertex of the region K. Since Width$(\triangle abc) \geq 3.25\omega$ and v, w lie inside the disks of radius $\omega/2$ centerted at b and c, respectively (i.e., the slope of vw is roughly the same as that of bc), Area$(\triangle vwz)$ is maximum when z lies at a vertex of K that is incident upon the edge parallel to bc and lying on the opposite side of a; see Figure 4. In this case $d(z, \ell_{bc}) = \tau\, d(a, \ell_{bc})$. If the projection of z on ℓ_{vw} lies on the segment vw itself, then, as in (10), $d(z, \ell_{vw}) \leq d(z, \ell_{bc}) + \omega/2$. Therefore,

$$\begin{aligned}\text{Area}\,(\triangle vwz) &= \frac{1}{2}|vw| d(z,\ell_{vw})\\ &\leq \frac{1}{2}(|bc|+\omega)(d(z,\ell_{bc})+\omega/2)\\ &\leq \text{Area}\,(\triangle bcz)\left(1+\frac{\omega}{|bc|}\right)\left(1+\frac{\omega}{d(z,\ell_{bc})}\right).\end{aligned}$$

Since $|bc| \geq 3.25\omega$ and $d(z, \ell_{bc}) = \tau\, d(a, \ell_{bc}) \geq 3.25\omega$,

$$\text{Area}\,(\triangle uvw) \leq \frac{289}{169}\text{Area}\,(\triangle bcz) = \frac{289}{169}\tau\text{Area}\,(\triangle abc).$$

If the projection of z on ℓ_{vw} does not lie on the segment vw, then either $\angle vwz$ or $\angle wvz$ is obtuse. Assume that $\measuredangle vwz > \pi/2$. Since v, u, w, and z lie on the circle C in that order in counterclockwise direction, all of them lie on a semicircle of C. Therefore, $\angle vuw > \pi/2$, which implies that vw is the longest edge of $\triangle uvw$. Therefore

$$|ac| \leq |uw| + \omega \leq |vw| + \omega \leq |bc| + 2\omega.$$

Similarly, we can show that $|ab| \leq |bc| + 2\omega$. In other words, the length of each edge (and thus also its height to bc) in $\triangle abc$ is at most $bc + 2\omega$. Using a simple algebraic calculation, it can be proved that

$$|bz| \leq \tau(|bc| + 2\omega).$$

Since Area $(\triangle bcz) = \tau$Area $(\triangle abc)$, we obtain

$$\tau\cdot|bc| d(a,\ell_{bc}) = |bz|\cdot d(c,\ell_{bz})$$

or

$$d(c,\ell_{bz}) = \frac{\tau|bc|d(a,\ell_{bc})}{|bz|}.$$

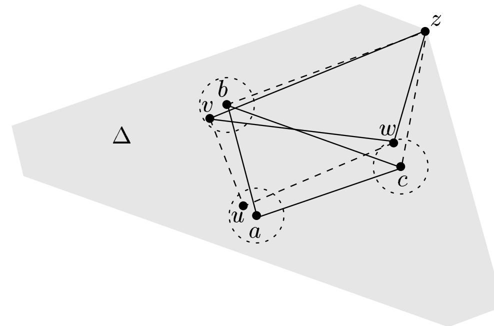

Figure 4: Illustration to Lemma 3.6(b).

On the other hand,

$$\begin{aligned}\text{Area}\,(\triangle vwz) &= \frac{1}{2}|vz| d(w,vz)\\ &\leq \frac{1}{2}\left(|bz|+\frac{\omega}{2}\right)d(w,vz)\\ &\leq \text{Area}\,(\triangle bcz)\frac{d(w,vz)}{d(c,\ell_{bz})}\left(1+\frac{\omega}{2|bz|}\right).\end{aligned}$$

Using the fact that $|bz| \geq d(z, \ell_{bc}) \geq 3.25\omega$ and $d(w, vz) \leq d(c, \ell_{bz}) + \omega$, and substituting the value of $d(c, \ell_{bz})$, we obtain

$$\begin{aligned}&\frac{d(w,vz)}{d(c,\ell_{bz})}\left(1+\frac{\omega}{2|bz|}\right)\\ &\leq \left(1+\frac{\omega}{d(c,\ell_{bz})}\right)\left(1+\frac{\omega}{2|bz|}\right)\\ &\leq \left(1+\frac{\omega|bz|}{\tau|bc|d(a,\ell_{bc})}\right)\left(1+\frac{\omega}{2|bz|}\right)\\ &\leq 1+\frac{\omega}{2|bz|}+\frac{\omega}{d(a,\ell_{bc})}\frac{|bc|+2\omega}{|bc|}+\frac{\omega}{2|bc|}\frac{\omega}{d(a,\ell_{bc})}\\ &\leq 1+\frac{\omega}{2\cdot 3.25\omega}+\frac{4}{13}\left(1+\frac{2\omega}{|bc|}\right)+\frac{8}{169}\\ &\leq \frac{203}{169}+\frac{4}{13}\left(1+\frac{8}{13}\right) = \frac{287}{169}.\end{aligned}$$

Hence,

$$\text{Area}\,(\triangle vwz) \leq \frac{287}{169}\text{Area}\,(\triangle bcz) < \frac{289\tau}{169}\text{Area}\,(\triangle abc).$$

Finally, to prove (c), suppose that the width of $\triangle uvw$ is the height to edge uv (which is then the longest edge). Arguing as above, we have

$$\begin{aligned}\text{Width}\,(\triangle abc) &\leq d(c,ab) \leq d(w,uv)+\omega\\ &= \text{Width}\,(\triangle uvw)+\omega.\end{aligned}$$

The reverse inequality is proved in exactly the same manner. $\square$

The first two parts of the above lemma imply the following.

Corollary 3.7 Area $(\triangle vwz) < \dfrac{289}{81}\tau \cdot$ Area $(\triangle uvw)$.

Proof of Theorem 3.3. If Width $(A^*) \le 6.5\omega$, then Lemma 3.1 implies that the width of S^* is at most $6.5(4t-1)\omega$. Since a slab can be regarded as a degenerate annulus, S^* can be enclosed by an annulus of width at most $26t\omega$. So assume that Width $(A^*) \ge 6.5\omega$.

Suppose, without loss of generality, that $\triangle a_1^* a_2^* a_3^*$ is the largest-area triangle spanned by three points of A^*. We have

$$\text{Width}\,(\triangle a_1^* a_2^* a_3^*) \ge \text{Width}\,(A^*)/2 \ge 3.25\omega.$$

Fix a point $q \in S^*$. By Corollary 3.2(ii), the area of each of the triangles $\triangle a_1^* a_2^* q$, $\triangle a_1^* a_3^* q$, $\triangle a_2^* a_3^* q$ is at most $(4t-1)\cdot$ Area $(\triangle a_1^* a_2^* a_3^*)$. Let $\mathcal{A}$ be an annulus of width ω that contains $A^* \cup \{q\}$, and let C be the mid-circle of $\mathcal{A}$. Let $\mathcal{A}^*$ be the annulus of width $26t\omega$ that has C as its mid-circle. We claim that $\mathcal{A}^*$ contains S^*. Indeed, let q' be any point of S^*, and let $\mathcal{A}'$ be an annulus of width ω that contains $A^* \cup \{q'\}$. Let C' be the mid-circle of $\mathcal{A}'$. Clearly, C, C', and $\triangle a_1^* a_2^* a_3^*$ satisfy the conditions in Lemma 3.4 (with $\tau = 4t-1$), which implies

$$d(q, C) \le (6.5(4t-1) + 3.6)\omega \le 26t\omega,$$

implying that $q \in \mathcal{A}^*$, as claimed. □

4 Approximating the Minimum-Width Cylindrical Shell

In this section we apply the results of the preceding section to obtain an algorithm for computing a cylindrical shell of width at most $O(\omega^*(S))$ that encloses an n-element point set $S \subset \mathbb{R}^3$. We first describe an algorithm for computing a subset $A \subseteq S$ of four points so that $\mu(A) \ge (1-\varepsilon)\mu(S)$, for some constant $\varepsilon > 0$; recall that $\mu(X)$ is the maximum volume of a simplex spanned by the points of X.

Lemma 4.1 *Given a set of n points in $\mathbb{R}^3$ and a parameter $\varepsilon > 0$, we can compute in $O(n\log(1/\varepsilon) + (1/\varepsilon)^{4.5}\log(1/\varepsilon))$ time a subset A of four points so that $\mu(A) \ge (1-\varepsilon)\mu(S)$.*

Proof (Sketch): We first compute a box B enclosing S whose volume is at most $1+\varepsilon$ times the minimum volume of any box containing S. This can be done in $O(n + 1/\varepsilon^{4.5})$ time using the algorithm by Barequet and Har-Peled [7]. Suppose, with no loss of generality, that B is axis-aligned and the coordinates of the endpoints of its main diagonal are $(0,0,0)$ and (l_x, l_y, l_z). Choose a sufficiently large constant $c > 1$ and set $\delta = \varepsilon/c$. Draw a three-dimensional grid

$$\{[i\delta l_x, (i+1)\delta l_x] \times [j\delta l_y, (j+1)\delta l_y] \times [k\delta l_z, (k+1)\delta l_z] \mid 0 \le i,j,k \le \lceil 1/\delta \rceil\}$$

of size $O(1/\delta^3)$. Let Q be the set of grid vertices adjacent to the grid cells that contain at least one point of S. Q can be computed in $O(n\log(1/\varepsilon) + 1/\varepsilon^3)$ time. We then compute, in $O((1/\delta^2)\log(1/\delta))$ time, the set $V \subseteq Q$ of vertices of the convex hull of Q. By a result of Andrews [6], $|V| = O(1/\delta^{3/2})$. Next, we compute in $O(|V|^3 \log |V|)$ time the largest volume tetrahedron $q_1 q_2 q_3 q_4$ spanned by V (we omit details of the rather straightforward algorithm for doing so). Let $a_i \in S$ be a nearest neighbor of q_i, for $i = 1, \ldots, 4$. We return $A = \{a_1, a_2, a_3, a_4\}$. Using a somewhat tedious analysis, similar to the one in [7], it can be shown that $\mu(A) \ge (1-\varepsilon)\mu(S)$. □

Set $\varepsilon = (1/n)^{4/9}$ and compute in $O(n^2 \log n)$ time a set $A \subseteq S$ of four points such that $\mu(A) \ge (1-\varepsilon/2)\mu(S)$, using the above lemma. Let $\mathbb{S}^2$ denote the unit sphere of directions in $\mathbb{R}^3$. For each $q \in S$ we define a real-valued function F_q on $\mathbb{S}^2$, so that, for $\mathbf{n} \in \mathbb{S}^2$, $F_q(\mathbf{n})$ is the width of a thinnest annulus within the plane $\pi^{(\mathbf{n})}$ that contains the orthogonal projections of $A \cup \{q\}$ on the plane $\pi^{(\mathbf{n})}$. Clearly, F_q is a piecewise-algebraic function of "constant description complexity" (in the terminology of [12]). Let E denote the pointwise maximum of $\{F_q\}_{q\in S}$, let $\mathbf{n} \in \mathbb{S}^2$ be a direction that minimizes E, and let $\omega = E(\mathbf{n})$.

Lemma 4.2 $\omega \le \omega^*(S) \le 26(1 + 1/n^{4/9})\omega$.

Proof: The fact that $\omega = \min_{\mathbf{v}\in\mathbb{S}^2} \max_{q\in S} F_q(\mathbf{v})$ implies that for each $\mathbf{v} \in \mathbb{S}^2$ there exists $q \in S$ such that any cylindrical shell that contains $A \cup \{q\}$ and has axis-direction $\mathbf{v}$ must have width at least ω. Hence the minimum width of a cylindrical shell that encloses S is at least ω.

On the other hand, since $\mu(A) \ge (1-\varepsilon/2)\mu(S)$, which corresponds to setting $t = 1/(1-\varepsilon/2) \le (1 + 1/n^{4/9})$ in Lemma 3.4, Theorem 3.3 implies that there exists a cylindrical shell with axis-direction $\mathbf{n}$ and width at most $26(1+1/n^{4/9})\omega$ that contains S. □

The algorithm is now straightforward. We compute E in $O(n^{2+\delta})$ time, for any $\delta > 0$, using, e.g., the algorithm of [4], and then examine each vertex, edge, and face of (the graph of) E to find the global minimum of E. Suppose the minimum is attained at some direction $\mathbf{n}$. We project S orthogonally onto $\pi^{(\mathbf{n})}$, and compute the minimum-width annulus $\mathcal{A}$ within $\pi^{(\mathbf{n})}$ that contains the projected set S^*. This can be done in time $O(n^2)$ [10]. (Alternatively, we can compute the radius ρ and the mid circle C^* of the minimum width annulus containing $A^{(\mathbf{n})}$ and set $\mathcal{A}$ to be the annulus of width $26(1+1/n^{4/9})\rho$ and with mid circle C^*.) We then "lift" $\mathcal{A}$ in the direction $\mathbf{n}$ to obtain a cylindrical shell, of the same width, that encloses S. By the preceding analysis, we obtain the following.

Theorem 4.3 *Given a set S of n points in $\mathbb{R}^3$, one can compute, in $O(n^{2+\delta})$ time, for any $\delta > 0$, a cylindrical shell that contains S, whose width is at most $26(1 + 1/n^{4/9})\omega^*(S)$.*

Remark 4.4 We believe that our approach can be strengthened to give a near-linear-time algorithm. Intuitively, we need to show that one does not have to search over all directions $\mathbf{n} \in \mathbb{S}^2$. Instead, we conjecture that it suffices to search over the 1-dimensional locus of axis directions of cylinders that pass through four points of S that span the largest-, or nearly largest-volume simplex spanned by S. However, at present we have some technical difficulties in proving this claim.

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