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Counting Facets and Incidences*

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Abstract. We show that m distinct cells in an arrangement of n planes in $\mathbb{R}^3$ are bounded by $O(m^{2/3}n + n^2)$ faces, which in turn yields a tight bound on the maximum number of facets bounding m cells in an arrangement of n hyperplanes in $\mathbb{R}^d$, for every $d \geq 3$. In addition, the method is extended to obtain tight bounds on the maximum number of faces on the boundary of all nonconvex cells in an arrangement of triangles in $\mathbb{R}^3$. We also present a simpler proof of the $O(m^{2/3}n^{d/3} + n^{d-1})$ bound on the number of incidences between n hyperplanes in $\mathbb{R}^d$ and m vertices of their arrangement.

1. Introduction

Given a collection $\mathscr{H}$ of hyperplanes in $\mathbb{R}^d$, the *arrangement* of $\mathscr{H}$, denoted $\mathscr{A}(\mathscr{H})$, is the decomposition of $\mathbb{R}^d$ induced by the hyperplanes of $\mathscr{H}$. The *cells* of $\mathscr{A}(\mathscr{H})$ are the connected components of $\mathbb{R}^d - \bigcup_{\pi \in \mathscr{H}} \pi$. Each cell of $\mathscr{A}(\mathscr{H})$ is the interior of a convex polytope. The *k-faces* of $\mathscr{A}(\mathscr{H})$ are the k-faces of cells in $\mathscr{A}(\mathscr{H})$. The 0-faces, 1-faces, and $(d-1)$-faces of $\mathscr{A}(\mathscr{H})$ are called *vertices*, *edges*, and *facets*, respectively. It is well known that $\mathscr{A}(\mathscr{H})$ has a total of $O(n^d)$ faces. See [E] for more information on arrangements.

In this paper we consider two problems related to arrangements of hyperplanes. The first problem concerns counting the maximum number of facets in m distinct cells of $\mathscr{A}(\mathscr{H})$, and the second question involves counting the maximum number of incidences between the hyperplanes and m distinct vertices of $\mathscr{A}(\mathscr{H})$. In the last few years, these problems have received considerable attention. It is well known

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that the second problem can be reduced to the first one, and therefore an upper bound on the number of facets yields the same asymptotic upper bound on the number of incidences [E, Lemma 6.14].

For $d = 2$, both of the above questions are well understood and asymptotically tight bounds are known; see [CEG$^+$] and [ST]. As for higher dimensions, very recently Edelsbrunner *et al.* [EGS] obtained almost tight bounds (within a logarithmic factor) on the maximum number of facets in m distinct cells of arrangements of n hyperplanes in $\mathbb{R}^d$, which in turn yielded the same upper bound on the number of incidences between n hyperplanes and m distinct vertices of their arrangement.

Below we obtain tight bounds on both of these quantities. We extend our method to obtain tight bounds on the maximum number of faces of all dimensions in all nonconvex cells of arrangements of triangles in $\mathbb{R}^3$. We also give a rather simple direct argument for counting the maximum number of incidences between n hyperplanes and m distinct vertices of their arrangement.

The paper is organized as follows. In Section 2 we prove an upper bound on the maximum number of 2-faces bounding m distinct cells in arrangements of n planes in $\mathbb{R}^3$. In Section 3 we extend our method to bound the maximum number of faces all dimensions bounding all nonconvex cells in arrangements of n triangles in $\mathbb{R}^3$, and in Section 4 we generalize the result of Section 2 to higher dimensions. In Section 5 we present the direct proof for bounding the maximum number of hyperplanes incident to m distinct vertices of $\mathcal{A}(\mathcal{H})$. Finally, we conclude in Section 6 by mentioning some open problems.

2. Complexity of Many Cells

Let Π be a collection of planes in $\mathbb{R}^3$ and let P be a collection of points, none on a plane of Π. A cell in $\mathcal{A}(\Pi)$ is *nonempty* if it contains at least one point of P. Let $\mathbf{K}(P, \Pi)$ be the number of 2-faces bounding nonempty cells of $\mathcal{A}(\Pi)$, and let

$$\mathbf{K}(m, n) = \max_{\substack{|P|=m \\ |\Pi|=n}} \mathbf{K}(P, \Pi).$$

In this section we obtain an asymptotically tight bound on $\mathbf{K}(m, n)$ in terms of m and n. By Euler's formula, the number of edges and vertices of a cell in $\mathcal{A}(\Pi)$ is proportional to the number of its 2-faces, therefore $\mathbf{K}(m, n)$ also bounds the number of faces of all dimensions of these cells.

It is easy to show that, for $m \le n$, $\mathbf{K}(m, n) = \Theta(mn)$ and, for $m \ge n^3$, $\mathbf{K}(m, n) = \Theta(n^3)$ (see, for example, Section 6.4 of [E]). For values of m between n and n^3, on the other hand, Edelsbrunner and Haussler [EH] proved that $\mathbf{K}(m, n) = \Omega(m^{2/3}n + n^2)$. An upper bound of $O(m^{2/3}n \log n + n^2)$ on $\mathbf{K}(m, n)$, for this range of values of m, was recently established by Edelsbrunner *et al.* [EGS]. In this paper we show $\mathbf{K}(m, n) = \Theta(m^{2/3}n + n^2)$.

Our proof has the same flavor as that of Edelsbrunner *et al.*, so we first briefly describe the basic idea of their argument. Π is partitioned arbitrarily into two subsets Π_1 and Π_2, each containing at most $\lceil n/2 \rceil$ planes, $\mathbf{K}(P, \Pi_1)$ and $\mathbf{K}(P, \Pi_2)$ are bounded recursively, and then $\mathbf{K}(P, \Pi)$ is estimated in terms of $\mathbf{K}(P, \Pi_1)$ and $\mathbf{K}(P, \Pi_2)$, using the following lemma:

Lemma 2.1 [EGS]. *Given two arrangements $\mathscr{A}(\Pi_1)$, $\mathscr{A}(\Pi_2)$ of planes in $\mathbb{R}^3$ and a set P of m points, none of which lie on a plane of $\Pi = \Pi_1 \cup \Pi_2$, the total number of two-dimensional faces bounding the nonempty cells of $\mathscr{A}(\Pi)$ is*

$$\mathbf{K}(P, \Pi) = \mathbf{K}(P, \Pi_1) + \mathbf{K}(P, \Pi_2) + O(m^{2/3}n + n^2), \tag{2.1}$$

where $n = |\Pi|$.

We are able to obtain a better upper bound on $\mathbf{K}(m, n)$ by combining the arbitrary partition of Π used by Edelsbrunner *et al.* with a more careful partitioning scheme that increases the number of subproblems created, but allows better control over the size of an individual subproblem. Namely, we show

Lemma 2.2.

$$\mathbf{K}(m, n) = \sum_{i=1}^{16} \mathbf{K}(m_i, n_i) + O(m^{2/3}n + n^2), \tag{2.2}$$

for some nonnegative integers n_i, m_i, $i = 1, \ldots, 16$, with $m_i \leq 7m/8 + 10$, $n_i \leq n/2 + 1$, and $\sum_{i=1}^{16} n_i = n$.

Proof. Let Π be a collection of n planes and let P be a set of m points in $\mathbb{R}^3$. Since none of the given points lie on a plane of Π, P can be slightly perturbed without affecting the statement of the theorem. In particular, we assume the points of P to be in general position.

We begin, as in [EGS], by arbitrarily subdividing Π into two subsets Π', Π'', each of size at most $n/2 + 1$. We proceed to analyze $\mathbf{K}(P, \Pi')$ and $K(P, \Pi'')$ separately and then use Lemma 2.1 to estimate $\mathbf{K}(P, \Pi)$.

By the "Eight-Partition Theorem" of Yao *et al.* [YDEP], there exist three planes H_1, H_2, and H_3 that split the space into eight open octants $Q_1, \ldots, Q_8$, such that each octant contains at most $m/8$ points of P. By the general position assumption, none of the three planes contains more than three points of P. In particular, after a slight perturbation, none of the three planes is parallel to a plane of Π or contains a point of P, and each octant contains between $m/8 - 9$ and $m/8 + 9$ points.

We divide Π' into eight subsets, $\Pi_1, \ldots, \Pi_8$, some of which may be empty, each associated with an octant. A plane $\pi \in \Pi'$ is associated with Q_i if the tetrahedron bounded by H_1, H_2, H_3, and π is contained in Q_i. Let Π_i be the set of planes associated with Q_i, and let P_i be the set of points lying in Q_i.

Let $Q_{i'}$ be the octant diagonally opposite to Q_i, i.e., for each plane H_j, Q_i and $Q_{i'}$ lie on different sides of H_j. Observe that a plane π associated with Q_i does not intersect $Q_{i'}$, and therefore all the points of $P_{i'}$ lie in a single cell of $\mathscr{A}(\Pi_i)$. As a result, the points of P lie in at most $|P - P_{i'}| + 1$ distinct cells of $\mathscr{A}(\Pi_i)$, thus

$$\mathbf{K}(P, \Pi_i) \le \mathbf{K}(m_i, n_i),$$

where $n_i = |\Pi_i|$ and $m_i \le |P - P_{i'}| + 1 \le 7m/8 + 10$.

We use this estimate on the number of two-dimensional faces bounding nonempty cells of $\mathscr{A}(\Pi_i)$, for $i = 1, \ldots, 8$, and then obtain the corresponding bound for $\mathscr{A}(\Pi')$ by repeatedly merging the resulting eight arrangements in pairs and invoking Lemma 2.1 at each merge step. It is easily seen that by applying (2.1) seven times, we obtain

$$\mathbf{K}(P, \Pi') \le \sum_{i=1}^{8} \mathbf{K}(m_i, n_i) + O(m^{2/3}n + n^2),$$

where $m_i \le 7m/8 + 10$, $n_i \le n/2 + 1$, and $\sum_i n_i = |\Pi'|$. The argument is completed by repeating the analysis for $\mathbf{K}(P, \Pi'')$ and invoking Lemma 2.1 one final time. □

Theorem 2.3. *There exist constants A, $B > 0$, such that*

$$\mathbf{K}(m, n) \le Am^{2/3}n + Bn^2. \tag{2.3}$$

Proof. Observe that $\mathbf{K}(m, n)$ never exceeds $\min\{mn, n^3\}$. Let $B \ge 10$. Then the lemma is true for $n \le 10$, because $Bn^2 \ge n^3 \ge \mathbf{K}(m, n)$. Next, for $n > 10, m \le 100$,

$$Bn^2 \ge 10n^2 > 100n \ge mn \ge \mathbf{K}(m, n).$$

Finally, for $n > 10$, $m > 100$, we prove the claim by induction on m. Assume that (2.3) holds for all $m' < m$; now let $m' = m > 100$. By Lemma 2.2, we have

$$\mathbf{K}(m, n) \le \sum_{i=1}^{16} \mathbf{K}(m_i, n_i) + C(m^{2/3}n + n^2),$$

where $m_i \le 7m/8 + 10$, $n_i \le n/2 + 1$, $\sum n_i = n$, and $C > 0$ is an absolute constant. Since $m > 100$, $7m/8 + 10 < m$ and therefore by the inductive hypothesis we obtain

$$\begin{aligned}\mathbf{K}(m, n) &\le \sum_{i=1}^{16} (Am_i^{2/3}n_i + Bn_i^2) + C(m^{2/3}n + n^2) \\ &\le A\left(\frac{7m}{8} + 10\right)^{2/3} n + B\left(\frac{n}{2} + 1\right)n + C(m^{2/3}n + n^2) \\ &= Am^{2/3}n\left[\left(\frac{7}{8} + \frac{10}{m}\right)^{2/3} + \frac{C}{A}\right] + Bn^2\left(\frac{1}{2} + \frac{1}{n} + \frac{C}{B}\right).\end{aligned}$$

Put $A = 60C$ and $B = \max\{10, 4C\}$. Then

$$\left(\frac{7}{8}+\frac{10}{m}\right)^{2/3}+\frac{C}{A}<(0.975)^{2/3}+\frac{1}{60}<1 \quad \text{and} \quad \frac{1}{2}+\frac{1}{n}+\frac{C}{B}\le\frac{3}{4}+\frac{1}{n}<1,$$

as $m > 100$ and $n > 10$. This concludes the inductive argument. □

Remarks. (1) Alternatively, the upper bound can be proven without the initial partitioning of Π into Π' and Π'' in the proof of Lemma 2.2, but it requires a variant of Lemma 2.1 in which the n^2 term is replaced by a term proportional to $|\Pi_1|\cdot|\Pi_2|$. This somewhat stronger version is given in an earlier draft of [EGS] and requires a more ingenious proof.

(2) The main difference between our approach and the approach of Edelsbrunner *et al.* is the way the problem is partitioned into subproblems. In their case the problem is divided arbitrarily into two subproblems, each receiving half of the planes. As a result, the entire point set must be passed to each subproblem. We, on the other hand, use a more clever partitioning scheme, so that we can control the number of planes as well as the number of points being passed to each subproblem.

The above theorem in conjunction with the lower bound of Edelsbrunner and Haussler [EH] implies that

Corollary 2.4. *For* $n < m < n^3$, $\mathbf{K}(m, n) = \Theta(m^{2/3}n + n^2)$.

3. Triangles in Space

The question of counting the number of faces bounding a set of cells in an arrangement of planar triangles suspended in $\mathbb{R}^3$ has been considered by Aronov and Sharir [AS1]. More formally, let $\mathscr{T}$ be a collection of two-dimensional triangles in $\mathbb{R}^3$. Let $\mathscr{A}(\mathscr{T})$ be their arrangement, i.e., the subdivision of $\mathbb{R}^3$ induced by them. In particular, a *cell* in $\mathscr{A}(\mathscr{T})$ is a connected component in the complement of the union of the triangles, a *2-face* is a maximal connected portion of the relative interior of a triangle that is free of points of other triangles, and an *exposed segment* of $\mathscr{A}(\mathscr{T})$ is a maximal section of a triangle edge free of points of other triangles. Let $e(\mathscr{T})$ be the number of exposed segments in $\mathscr{A}(\mathscr{T})$. Given a set of points P, none in the relative interior of a triangle of $\mathscr{T}$, a cell of $\mathscr{A}(\mathscr{T})$ is *nonempty* if it either contains a point of P or if its boundary contains an exposed segment containing a point of P. Let $\hat{\mathbf{K}}(P, \mathscr{T})$ be the number of faces bounding the nonempty cells of $\mathscr{A}(\mathscr{T})$ and put

$$\hat{\mathbf{K}}(m, \mathscr{T}) = \max_{|P|=m} \hat{\mathbf{K}}(P, \mathscr{T}) \quad \text{and} \quad \hat{\mathbf{K}}(m, n) = \max_{|\mathscr{T}|=n} \hat{\mathbf{K}}(m, \mathscr{T}).$$

Once again, $\hat{\mathbf{K}}(m, n)$ is trivially $\Theta(n^3)$ whenever $m \ge n^3$. For large m, one would expect $\hat{\mathbf{K}}(m, n)$ and $\mathbf{K}(m, n)$ to behave somewhat similarly, while for smaller values of m the two quantities may differ substantially. Indeed, Aronov and Sharir showed a lower bound of $\Omega(m^{2/3}n + n^2\alpha(n))$ on $\hat{\mathbf{K}}(m, n)$, where $\alpha(n)$ is the inverse Ackermann's function, and also demonstrated, for $m \geq n^2$, an upper bound of $O(m^{2/3}n \log n)$.[1] We improve the latter bound to $\hat{\mathbf{K}}(m, n) = O(m^{2/3}n)$ by applying the technique used in the previous section, thereby providing a tight bound on $\hat{\mathbf{K}}(m, n)$ for $n^2 \leq m \leq n^3$. Our analysis hinges on the following lemma:

Lemma 3.1 [AS1, Lemma 1]. *Given arrangements $\mathscr{A}(\mathscr{T}_1)$, $\mathscr{A}(\mathscr{T}_2)$ of triangles in $\mathbb{R}^3$, $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$, $|\mathscr{T}| = n$, and a set P of $m \geq e(\mathscr{T})$ points. Assume furthermore that no point of P lies in the relative interior of a triangle of $\mathscr{T}$ and there is at least one point of P on each exposed segment of $\mathscr{A}(\mathscr{T})$. Then*

$$\hat{\mathbf{K}}(P, \mathscr{T}) = \hat{\mathbf{K}}(P, \mathscr{T}_1) + \hat{\mathbf{K}}(P, \mathscr{T}_2) + O(m^{2/3}n + n^2 \log n).$$

Notice that the restriction on P in the above lemma will, in the worst case, limit the range of values of m to between n^2 and n^3, since the maximum value of $e(\mathscr{T})$ is $\Theta(n^2)$. As in the proof of Lemma 2.2, we deduce

Lemma 3.2. *For any collection $\mathscr{T}$ of n triangles in $\mathbb{R}^3$ and for any $m \geq e(\mathscr{T})$, there is a partition of T into 16 subsets T_i and a set of nonnegative integers $m_i \geq e(\mathscr{T}_i)$, such that*

$$\hat{\mathbf{K}}(m, \mathscr{T}) = \sum_{i=1}^{16} \hat{\mathbf{K}}(m_i, \mathscr{T}_i) + O(m^{2/3}n + n^2 \log n),$$

with $m_i \leq 7m/8 + o(m)$, $|\mathscr{T}_i| \leq n/2 + 1$.

Proof. The argument used in the proof of Lemma 2.2 hinges on the fact that an "eight-partition" of the points with three planes is possible in which no point lies on the partitioning planes and every octant contains at least $m/8 - 9$ points. However, for the proof to go through essentially unchanged, it is enough to guarantee that each octant contains at least $m/8 - o(m)$ points. Such a partition was obtained by perturbing the planes given by the "perfect" Eight-Partition Theorem of Yao *et al.* [YDEP]. Since in this lemma the points are not in general position, such a perturbation may reduce the number of points in an octant by more than a constant amount. However, we may assume that the triangles are in general position. In this case, the only degeneracy in our set of points comes from the fact that the points lying on exposed segments belonging to the same triangle are coplanar (and, in fact, collinear if they lie on the same triangle edge). Hence, the partitioning planes of the perfect eight-partition can contain $O(n)$ points, as there are no more than a linear number of exposed segments per triangle.

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[1] The results of Aronov and Sharir are somewhat more general, but our methods do not immediately yield an improvement of these more general bounds.

Perturbing them will produce octants with at least $m/8 - O(n)$ points, which is good enough as long as $n = o(m)$. Thus the argument goes through essentially unchanged if, say, $e(\mathscr{T}) > n \log n$, as $m \geq e(\mathscr{T})$. Now observe that the complexity of the *entire arrangement* is always $O(n\, e(\mathscr{T}))$. This can be seen as follows: $e(\mathscr{T})$ is essentially the number of *pairs* of intersecting triangles in the arrangement and the complexity of the arrangement is dominated by the number of intersections of *triples* of triangles, i.e., intersections of a pair of intersecting triangles with a third triangle; the number of such events is clearly $O(n\, e(\mathscr{T}))$; see [AS1] for details. Thus the lemma holds for $e(\mathscr{T}) \leq n \log n$ as well. □

A slight modification of the proof of Theorem 2.3 yields

Theorem 3.3. *There exist constants A, $B > 0$, such that*

$$\hat{\mathbf{K}}(m, \mathscr{T}) \leq Am^{2/3}n + Bn^2 \log n,$$

for any arrangement of a set $\mathscr{T}$ of n triangles and any m between $e(\mathscr{T})$ and n^3.

Together with the results of Aronov and Shairir, Theorem 3.3 implies

Corollary 3.4.

1. *For $n^2 \leq m \leq n^3$, $\hat{\mathbf{K}}(m, n) = \Theta(m^{2/3}n)$.*
2. *The maximum number of faces of all dimensions bounding the nonconvex cells in an arrangement of n triangles in $\mathbb{R}^3$ is $\Theta(n^{7/3})$.*

Remarks. (1) The question of behavior of $\hat{\mathbf{K}}(m, n)$ for smaller values of m remains unresolved. Aronov and Sharir [AS1] conjecture that the $\Omega(m^{2/3}n + n^2\alpha(n))$ lower bound is close to the correct answer. It was recently shown that $\hat{\mathbf{K}}(1, n) = O(n^2 \log n)$ [AS2], which appears to support this conjecture.

(2) The bounds in this section can be easily extended to collections of arbitrary two-dimensional convex objects in $\mathbb{R}^3$. For the purpose of counting faces of all dimensions, we must introduce an additive term proportional to the "boundary complexity" of the objects. For example, the total number of 2-faces on the boundary of all nonconvex cells in an arrangement of n convex polygons with a total of N edges is $\Theta(n^{7/3})$ in the worst case, while the total number of faces of all dimensions is $\Theta(n^{7/3} + N)$. Aronov and Sharir remark that such an extension is possible for their results, while our refined analysis is essentially independent of the shape of the two-dimensional objects under consideration, as long as the analogue of Lemma 3.1 can be shown to hold.

4. Facet Counts in Higher Dimensions

In this section we apply the result of Section 2 to obtain tight bounds on the number of facets bounding many cells in arrangements of hyperplanes in $\mathbb{R}^d$.

Given a set $\mathcal{H}$ of n hyperplanes and a set P of m points in $\mathbb{R}^d$ ($d \geq 3$), none on a hyperplane of $\mathcal{H}$, let $\mathbf{K}^d(P, \mathcal{H})$ be the number of facets bounding nonempty cells of $\mathcal{A}(\mathcal{H})$, and define

$$\mathbf{K}^d(m, n) = \max_{\substack{|P|=m \\ |\mathcal{H}|=n}} \mathbf{K}^d(P, \mathcal{H}).$$

Unlike the three-dimensional case, the bound on $\mathbf{K}^d(m, n)$ does not imply the same asymptotic bound on the number of faces of all dimensions bounding m distinct cells.

It is easy to show that, for $m \geq n^d$, $\mathbf{K}^d(m, n) = \Theta(n^d)$, and, for $m \leq n^{d-2}$, $\mathbf{K}^d(m, n) = \Theta(mn)$; see, for example, Chapter 6 of [E]. Thus we restrict ourselves to the case when $n^{d-2} < m < n^d$. For this range of values of m, a lower bound of $\Omega(m^{2/3}n^{d/3} + n^{d-1})$ can be obtained in a manner similar to the three-dimensional case (see Theorem 6.19 of [E]). Edelsbrunner *et al.* [EGS] have shown an upper bound of $O(m^{2/3}n^{d/3} \log n + n^{d-1})$ and observed that removal of the logarithmic factor from the upper bound in $\mathbb{R}^3$ would produce a similar improvement in the upper bound on $\mathbf{K}^d(m, n)$, for all $d \geq 3$. Thus Corollary 2.4 implies

Theorem 4.1. *For any fixed value of* $d \geq 3$ *and* $n^{d-2} \leq m \leq n^d$,

$$\mathbf{K}^d(m, n) = \Theta(m^{2/3}n^{d/3} + n^{d-1}).$$

5. Incidences between Points and Hyperplanes

In this section we consider a problem involving the number of incidences between points and hyperplanes in $\mathbb{R}^d$. If $d \geq 3$ and m points and n hyperplanes are chosen arbitrarily, it is easy to see that the maximum number of incidences between them is mn—choose m points on a line and then choose n hyperplanes passing through that line. However, imposing some condition on points and hyperplanes, we may be able to obtain an $o(mn)$ upper bound on the number of incidences. We consider the following variant of the problem, analyzed in [EGS]: given a set $\mathcal{H}$ of n hyperplanes and a set P of m vertices of $\mathcal{A}(\mathcal{H})$, let $\mathbf{I}^d(P, \mathcal{H})$ denote the number of incidences between $\mathcal{H}$ and P in $\mathbb{R}^d$, and let

$$\mathbf{I}^d(m, n) = \max_{\substack{|P|=m \\ |\mathcal{H}|=n}} \mathbf{I}^d(P, \mathcal{H}).$$

By definition $m \leq \binom{n}{d}$. It is easily seen that, for $m < n^{d-2}$, $\mathbf{I}^d(m, n) = \Theta(mn)$, so we concentrate on the case $n^{d-2} < m < \binom{n}{d}$. It is known that

$$\mathbf{I}^d(m, n) = \Omega(m^{2/3}n^{d/3} + n^{d-1}),$$

for this range of values of m, and that $\mathbf{I}^d(m, n) < \frac{1}{2}\mathbf{K}^d(m, n)$ [E, Lemma 6.14]. Therefore, from Theorem 2.3 we obtain

Theorem 5.1. *For $d \geq 2$, $n^{d-2} \leq m \leq n^d$,*

$$\mathbf{I}^d(m, n) = \Theta(m^{2/3}n^{d/3} + n^{d-1}). \tag{5.1}$$

In this section we give a rather simple direct proof of the upper bound by induction on the dimension d, using the following result of Szemerédi and Trotter [ST] (see also [B]), known as the weak Dirac conjecture.

Lemma 5.2 [ST]. *In any set P of n points in the plane, not all collinear, there is a point that lies on at least cn distinct lines determined by the set P, for some constant $c > 0$.*

We need the dual version of the above lemma, which states that if L is a set of n lines in the plane, not all parallel or passing through a common point, then there is a line in L that meets other lines of L in at least cn distinct points.

Proof of Theorem 5.1. Let $\mathscr{H}$ be a set of n hyperplanes in $\mathbb{R}^d$ and let P be a set of m vertices of $\mathscr{A}(\mathscr{H})$. Szemerédi and Trotter [ST] (see also Clarkson *et al.* [CEG$^+$]) have proved that the maximum number of incidences between m points and n lines in the plane is $O(m^{2/3}n^{2/3} + m + n)$, which implies the upper bound for $d = 2$. Assume that the upper bound holds for all $d' < d$, and consider $d' = d$.

We partition P into n subsets $P_1, \ldots, P_n$, each associated with a hyperplane of $\mathscr{H}$, and for each subset P_i we estimate $\mathbf{I}^d(P_i, \mathscr{H})$ by reducing it to counting the number of incidences between points and $(d-2)$-dimensional hyperplanes in $\mathbb{R}^{d-1}$. The proof proceeds as follows.

Let p be a point of P, and let $\mathscr{H}_p = \{\pi_1, \ldots, \pi_t\}$ be the set of hyperplanes incident to p. Choose a two-dimensional plane H that is not parallel to any hyperplane of $\mathscr{H}_p$ or any line determined by the intersection of a set of $d-1$ hyperplanes of $\mathscr{H}_p$, and does not contain any point of P. Then $L_p = \{\pi \cap H \mid \pi \in \mathscr{H}_p\}$ is a set of t lines. Since p is a vertex of $\mathscr{A}(\mathscr{H})$, $\bigcap_{\pi \in \mathscr{H}_p} \pi$ is a single point (namely, p), and not a higher-dimensional flat, and therefore not all of the lines of L_p are concurrent. Moreover, the choice of H ensures that no two lines of L_p are parallel. Hence, by Lemma 5.2, there is a line $l_p \in L_p$ that intersects other lines of L_p in $k \geq ct$ distinct points. Let $\pi_p \in \mathscr{H}_p$ be the hyperplane corresponding to l_p. Assign point p to π_p.

Since $l_p = \pi_p \cap H$ contains k distinct intersection points with lines in L_p, the hyperplanes of $\mathscr{H}_p - \{\pi_p\}$ meet π_p in exactly k distinct $(d-2)$-dimensional hyperplanes, and all these hyperplanes intersect at p. Thus

$$\mathbf{I}^{d-1}(\{p\}, \{\pi \cap \pi_p \mid \pi \in \mathscr{H}_p - \{\pi_p\}\}) = k \geq ct = c \cdot \mathbf{I}^d(\{p\}, \mathscr{H}).$$

Repeat the same procedure for all points of P. Let P_i be the set of m_i points assigned to the hyperplane π_i. For $i = 1, \ldots, n$ put

$$\Pi_i = \{\pi_i \cap \pi_j | \pi_j \in \mathscr{H} - \{\pi_i\}\}.$$

It is a set of $n_i < n$ hyperplanes in $\mathbb{R}^{d-1}$. It follows from the above discussion that $\mathbf{I}^{d-1}(P_i, \Pi_i) \geq c \cdot \mathbf{I}^d(P_i, \mathscr{H})$, and therefore

$$\mathbf{I}^d(P, \mathscr{H}) = \sum_{i=1}^{n} \mathbf{I}^d(P_i, \mathscr{H}) \leq \frac{1}{c} \sum_{i=1}^{n} \mathbf{I}^{d-1}(P_i, \Pi_i) \leq \frac{1}{c} \sum_{i=1}^{n} \mathbf{I}^{d-1}(m_i, n_i).$$

By inductive hypothesis, we thus obtain

$$\begin{aligned} \mathbf{I}^d(P, \mathscr{H}) &= \sum_{i=1}^{n} O(m_i^{2/3} n_i^{(d-1)/3} + n_i^{d-2}) \\ &= O\left(\sum_{i=1}^{n} m_i^{2/3} n^{(d-1)/3} + n^{d-1} \right) \\ &= O\left(n^{d/3} \left(\sum_{i=1}^{n} m_i \right)^{2/3} + n^{d-1} \right). \end{aligned}$$

Hence

$$\mathbf{I}^d(m, n) = O(m^{2/3} n^{d/3} + n^{d-1}),$$

completing our proof of the upper bound. For the lower bound construction, see the proofs of Theorems 6.18 and 6.19 in [E]. □

6. Conclusion

In this paper we proved a tight bound of $\Theta(m^{2/3} n^{d/3} + n^{d-1})$ on the maximum number of facets bounding $n^{d-2} \leq m \leq n^d$ distinct cells in arrangements of n hyperplanes in $\mathbb{R}^d$. This, in turn, yielded a tight bound on the number of incidences between n hyperplanes in $\mathbb{R}^d$ and m vertices in their arrangement. We employed the same technique to obtain a tight bound on the number of faces bounding all nonconvex cells in arrangements of triangles in $\mathbb{R}^3$. Recently there have been some interesting developments in this area:

(i) Aronov *et al.* [AMS] proved that the maximum number of faces of all dimensions bounding m distinct cells in arrangements of n hyperplanes in $\mathbb{R}^d$ is $O(m^{1/2} n^{d/2} \log^{(\lfloor d/2 \rfloor - 1)/2} n)$ and $\Omega(m^{1/2} n^{d/2 - 1/4})$, except when d is odd and $m \leq n$.

(ii) Aronov and Sharir [AS2] showed that the complexity of a single cell in an arrangement of n simplices in $\mathbb{R}^d$ is $O(n^{d-1} \log n)$, which is within a polylogarithmic factor of the known lower bound $\Omega(n^{d-1} \alpha(n))$ [PS].

We conclude with some open questions.

1. How does $\hat{\mathbf{K}}(m, n)$ behave for small values of m?
2. What is the maximum number of facets bounding all nonconvex cells or, more generally, any m cells in arrangements of n simplices in $\mathbb{R}^d$? The lower bound of $\Omega(m^{2/3}n^{d/3} + n^{d-1})$ on $\mathbf{K}^d(m, n)$ is easily seen to apply here as well.

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