Line Transversals of Balls and Smallest Enclosing Cylinders in Three Dimensions[*]

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Abstract

We establish a near-cubic upper bound on the complexity of the space of line transversals of a collection of n balls in three dimensions, and show that the bound is almost tight, in the worst case. We apply this bound to obtain a near-cubic algorithm for computing a smallest infinite cylinder enclosing a given set of points or balls in 3-space. We also present an approximation algorithm for computing a smallest enclosing cylinder.

1 Introduction

Line transversals in three dimensions. Let $\mathcal{S}$ be a collection of n compact convex sets with non-empty interiors in $\mathbb{R}^3$. A line ℓ is called a *(line) transversal* of $\mathcal{S}$ if it intersects every member of $\mathcal{S}$. Let $\mathcal{T}(\mathcal{S})$ denote the set of all line transversals of $\mathcal{S}$. Since lines in 3-space can be parametrized by 4 real parameters, $\mathcal{T}(\mathcal{S})$ is a 4-dimensional set. For example, one may use the parameterization $(\xi_1, \xi_2, \xi_3, \xi_4)$, where the equations of the line are given by $y = \xi_1 x + \xi_2$, $z = \xi_3 x + \xi_4$; this excludes lines parallel to the yz-plane, which can be treated separately, in a simpler manner, since they can be parametrized by only three real

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parameters. We assume that each $S \in \mathcal{S}$ has "constant description complexity," meaning that it is a semialgebraic set defined by a constant number of polynomial equalities and inequalities of constant maximum degree. In this case, $\mathcal{T}(\mathcal{S})$ is also a semialgebraic set, whose boundary $\partial \mathcal{T}(\mathcal{S})$ consists of transversals that are tangent to at least one set $S \in \mathcal{S}$. We can measure the *combinatorial complexity* of $\mathcal{T}(\mathcal{S})$ by the number of its faces of all dimensions, where a *face* is a connected component of a portion of $\partial \mathcal{T}(\mathcal{S})$ consisting of lines tangent to a fixed subset of $\mathcal{S}$. Assuming general position, a j-dimensional face of $\mathcal{T}(\mathcal{S})$, for $j = 0, 1, 2, 3$, consists of line transversals tangent to a fixed set of $4 - j$ members of $\mathcal{S}$; see [1, 22].

Wenger [28] obtained an $O(n^4)$ upper bound on the number of connected components of $\mathcal{T}(\mathcal{S})$ for the case where the sets in $\mathcal{S}$ are pairwise disjoint. Pellegrini [19, 20, 21] and Pellegrini and Shor [22] have studied several combinatorial and algorithmic problems involving line transversals of a set $\mathcal{S}$ of triangles and convex polytopes in 3-space. They obtained a lower bound of $\Omega(n^3)$ and an upper bound of $n^3 \cdot 2^{O(\sqrt{\log n})}$ on the complexity of $\mathcal{T}(\mathcal{S})$, where $\mathcal{S}$ is a set of convex polytopes in $\mathbb{R}^3$ with a total of n vertices. The upper bound was later improved to $O(n^3 \log n)$ by Agarwal [1]. Pellegrini and Shor also gave an $O(n^{3+\varepsilon})$-time algorithm, for any $\varepsilon > 0$, for determining whether $\mathcal{S}$ admits any line transversal, i.e., whether $\mathcal{T}(\mathcal{S}) \neq \emptyset$. No sub-quartic upper bound was known for collections $\mathcal{S}$ of more general convex sets, even for collections of balls. An easier, related problem is to bound the complexity of the space of *plane transversals* of $\mathcal{S}$. It is shown in [3] that if each object in $\mathcal{S}$ has constant description complexity, then the combinatorial complexity of the space of plane transversals of $\mathcal{S}$ is $O(n^{2+\varepsilon})$. See Goodman et al. [12] and Wenger [29] for recent surveyes on geometric transversal theory.

In this paper we study the case where $\mathcal{S}$ is a collection of n balls in $\mathbb{R}^3$. We show that the complexity of $\mathcal{T}(\mathcal{S})$ is $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$.[1] The same bound also holds if $\mathcal{S}$ if a set of n homothets of a convex body of simple shape. We also show that this bound is nearly tight by exhibiting a collection $\mathcal{S}$ of n balls for which the complexity of $\mathcal{T}(\mathcal{S})$ is $\Omega(n^3)$.

We conjecture that such a near-cubic bound on the complexity of $\mathcal{T}(\mathcal{S})$ holds for arbitrary collections $\mathcal{S}$ of n simply-shaped convex sets in $\mathbb{R}^3$. As already mentioned, this holds for the case of polyhedra.

Smallest enclosing cylinders. Next, we study the following problem: Let P be a set of n points in $\mathbb{R}^3$. We aim to find an infinite circular cylinder C of the smallest possible radius that contains P; we refer to C as a *smallest enclosing cylinder* of P. More generally, we may consider the case where P is a collection of balls, and we wish to find a cylinder of smallest radius that either (a) contains all the balls in P, or (b) intersects every ball in P.

These problems arise in statistical analysis and computational metrology. In statistical analysis, given a set P of points in $\mathbb{R}^3$, we wish to fit a line ℓ through P so that the maximum

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[1]Throughout this paper, ε denotes an arbitrarily small positive constant. The constant of proportionality depends on ε, and tends to ∞ as $\varepsilon \downarrow 0$.

distance between ℓ and the points of P is minimized, which is equivalent to finding a smallest enclosing cylinder of P. In computational metrology, this problem arises in measuring the quality of a product [8, 10, 26, 27]. P is a set of points sampled from a cylindrical object being manufactured. A smallest cylinder enclosing P serves as a calibration of how close is the manufactured object to the desired shape. In order to capture measurement errors (in both statistical analysis and computational metrology), we can replace each point $p \in P$ by a ball of radius δ_p centered at p. In that case, the problem (a) seeks the smallest cylinder that is guaranteed to contain all the points, regardless of their exact position; and a solution to problem (b) gives an underestimate on the radius of a smallest cylinder containing S.

This problem is closely related to the space of line transversals of balls in $\mathbb{R}^3$. Indeed, suppose, for specificity, that P is the set $\{B(x_1, r_1), \ldots, B(x_n, r_n)\}$, where $B(x, r)$ denotes the closed ball of radius r centered at x. Suppose that we conduct a binary search on the radius r of the cylinder, and that we need a procedure that can determine, for a given r, whether there exists a cylinder of radius r that contains (resp. intersects) all the balls of S. This decision problem is equivalent to the problem of determining whether the sets

$$P_r^- = \{B(x_1, r - r_1), \ldots, B(x_n, r - r_n)\}$$

and

$$P_r^+ = \{B(x_1, r + r_1), \ldots, B(x_n, r + r_n)\},$$

respectively, have a line transversal.

This indeed is the approach that we use to solve this problem. We first present an algorithm for determining whether a given set of n balls in 3-space has a line transversal. The running time of the algorithm is $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$. We then combine this procedure with parametric search to obtain a near-cubic solution to computing the smallest enclosing cylinder. This improves a recent $n^4 \log^{O(1)} n$ solution of Schömer et al. [23].

We also present an $O((1 + 1/\delta^2)n)$-time algorithm to compute a cylinder enclosing a set of balls in $\mathbb{R}^3$, whose radius is at most $(1+\delta)r^*$, where r^* is the radius of a smallest enclosing cylinder of the set. Schömer et al. [23] gave an $O((1 + 1/\delta^2)n \log 1/\delta)$-time algorithm to compute a cylinder of radius $r^* + \delta$ enclosing a set of n points in $\mathbb{R}^3$, assuming that the input points lie inside a unit sphere. They also presented two other algorithms for this problem that run in time $O(n^4 \log(1/\delta))$ and $O((1 + 1/\delta)n^3 \log(1/\delta))$.

2 Geometric Preliminaries

In this section we introduce a few technical concepts and results that we will be using in subsequent sections. Let $\mathcal{F}$ be a set of n d-variate, possibly partially defined functions. The *lower envelope* of $\mathcal{F}$ is defined as $\mathcal{F}^L(x) = \min_{f \in \mathcal{F}} f(x)$, and the *upper envelope* of $\mathcal{F}$ is defined as $\mathcal{F}^U(x) = \max_{f \in \mathcal{F}} f(x)$; in case of partially defined functions, the minimum and maximum are taken only over those functions that are defined at x. The decomposition of

$\mathbb{R}^d$ into maximal connected regions over each of which a fixed subset of functions attains the lower (resp. upper) envelope of $\mathcal{F}$ is called the *minimization* (resp. *maximization*) *diagram* of $\mathcal{F}$. A recent result of Sharir [24] shows that if the graphs of the functions in $\mathcal{F}$ have constant description complexity, i.e., if the graph of each function is a semialgebraic set defined by a constant number of polynomial equalities and inequalities of constant maximum degree, then the complexity of the lower and upper envelopes of $\mathcal{F}$ is $O(n^{d+\varepsilon})$, for any $\varepsilon > 0$. Moreover, Agarwal et al. [2] have shown that, for $d = 3$, $\mathcal{F}$ can be preprocessed in $O(n^{3+\varepsilon})$ time into a data structure of size $O(n^{3+\varepsilon})$ so that for a query point $x \in \mathbb{R}^3$, $\mathcal{F}^L(x)$ and $\mathcal{F}^U(x)$ and one of the functions attaining each envelope can be determined in $O(\log^2 n)$ time.

Let $\mathcal{B}$ be a given set of n balls in $\mathbb{R}^3$. We use the parameterization $(\xi_1, \xi_2, \xi_3, \xi_4)$ of lines in $\mathbb{R}^3$ defined in the introduction. For each ball $B \in \mathcal{B}$, let σ_B denote the surface in $\mathbb{R}^4$ consisting of all lines tangent to B. We can represent σ_B as the union of the graphs of two partially-defined functions $\xi_4 = F_B(\xi_1, \xi_2, \xi_3)$, $\xi_4 = G_B(\xi_1, \xi_2, \xi_3)$, where $F_B(\xi_1, \xi_2, \xi_3)$ (resp. $G_B(\xi_1, \xi_2, \xi_3)$) is the ξ_4-coordinate of the line $(\xi_1, \xi_2, \xi_3, \xi_4)$ that is tangent to B from below (resp. from above). The functions F_B and G_B are defined whenever the line $y = \xi_1 x + \xi_2$ intersects the xy-projection of B (a condition independent of ξ_3); otherwise we put $F_B = +\infty$ and $G_B = -\infty$. The graphs of F_B and G_B are semialgebraic sets of constant description complexity. Let $\mathcal{F} = \{F_B \mid B \in \mathcal{B}\}$ and $\mathcal{G} = \{G_B \mid B \in \mathcal{B}\}$. The following observation is immediate:

Observation 2.1 *A line* $(\xi_1, \xi_2, \xi_3, \xi_4)$ *intersects* B *if and only if*

$$F_B(\xi_1, \xi_2, \xi_3) \le \xi_4 \le G_B(\xi_1, \xi_2, \xi_3).$$

Thus $(\xi_1, \xi_2, \xi_3, \xi_4)$ *is a transversal of* $\mathcal{B}$ *if and only if*

$$\max_{B \in \mathcal{B}} F_B(\xi_1, \xi_2, \xi_3) \le \xi_4 \le \min_{B \in \mathcal{B}} G_B(\xi_1, \xi_2, \xi_3).$$

In other words, $\mathcal{T}(\mathcal{B})$ is the region 'sandwiched' between the upper envelope $\mathcal{F}^U$ of $\mathcal{F}$ and the lower envelope $\mathcal{G}^L$ of $\mathcal{G}$.

Since the functions in $\mathcal{F}, \mathcal{G}$ are partial trivariate algebraic functions, of constant description complexity, the result of Agarwal et al. [2] mentioned at the beginning of this section implies the following fact.

Observation 2.2 *A set* $\mathcal{B}$ *of* n *balls in* $\mathbb{R}^3$ *can be preprocessed in time* $O(n^{3+\varepsilon})$, *for any* $\varepsilon > 0$, *into a data structure of size* $O(n^{3+\varepsilon})$, *so that we can determine in* $O(\log^2 n)$ *time whether a query line* ℓ *is a transversal of* $\mathcal{B}$.

This observation, however, falls short of giving a bound on the complexity of $\mathcal{T}(\mathcal{B})$. In order to bound this complexity, we need a bound on the complexity of the 'sandwich'

region enclosed between the lower envelope $\mathcal{G}^L$ and the upper envelope $\mathcal{F}^U$. It is still an open problem whether a near-cubic bound holds for the case of trivariate functions. However, such a result was recently established in [3] for bivariate functions, and will be used in the next section:

Theorem 2.3 (Agarwal et al. [3]) *The complexity of the region enclosed between the upper envelope of n bivariate functions of constant description complexity and the lower envelope of n other bivariate functions of constant description complexity is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. This region can be computed, in an appropriate model of computation, in time $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$.*

3   Complexity of the Set of Line Transversals

Let $\mathcal{B}$ be a set of n balls of arbitrary radii in $\mathbb{R}^3$. In this section we bound the combinatorial complexity of $\mathcal{T}(\mathcal{B})$. The following theorem states the main result.

Theorem 3.1 *Let $\mathcal{B}$ be a collection of n balls in $\mathbb{R}^3$. Then the complexity of $\mathcal{T}(\mathcal{B})$ is $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$.*

Proof: We assume that the balls in $\mathcal{B}$ are in general position, i.e., that no line is tangent to more than four balls, and that the point $(\xi_1, \xi_2, \xi_3, \xi_4)$ corresponding to a line tangent to exactly k balls, $k = 1, \dots, 4$, lies in a face of co-dimension k in the arrangement of the set of surfaces $\{\sigma_B \mid B \in \mathcal{B}\}$. Following the same argument as in [24], the upper bound can be extended to the situation when the balls are not in general position.

For a set $\mathcal{B}$ of balls in general position, a simple counting argument shows that the number of faces of $\mathcal{T}(\mathcal{B})$ that are not incident to any vertex of $\mathcal{T}(\mathcal{B})$ is bounded by $O(n^3)$, and that the number of all other faces of $\mathcal{T}(\mathcal{B})$ is proportional to the number of vertices in $\mathcal{T}(\mathcal{B})$. It thus suffices to bound the number of vertices in $\mathcal{T}(B)$. We will refer to a transversal as *extreme* if it is tangent to four balls in $\mathcal{B}$, so that it corresponds to a vertex of $\mathcal{T}(\mathcal{B})$ (where four of the graphs of the functions F_B, G_B intersect).

Let the set $\mathcal{B} = \{B_1, \dots, B_n\}$ of balls be ordered so that the radii r_i of the balls B_i form a nonincreasing sequence. Fix a direction $\mathbf{n}$ in $\mathbb{R}^3$ not parallel to the x-axis, and consider the family of all lines in direction $\mathbf{n}$. Let $H_{\mathbf{n}}$ be some plane orthogonal to $\mathbf{n}$, and let $B^{(\mathbf{n})}$ denote the orthogonal projection of a ball B to $H_{\mathbf{n}}$. Let $\mathcal{B}^{(\mathbf{n})}$ denote the collection of n disks in $H_{\mathbf{n}}$ resulting from the projection of the balls in $\mathcal{B}$.

Suppose that ℓ is an extreme transversal of $\mathcal{B}$ in direction $\mathbf{n}$, and let $B_{i_1}, B_{i_2}, B_{i_3}, B_{i_4}$ be the four balls of $\mathcal{B}$ to which ℓ is tangent. Without loss of generality, assume that $i_1 < i_2 < i_3 < i_4$, i.e., B_{i_1} has the largest radius among these four balls. Clearly, the point $\ell \cap H_{\mathbf{n}}$ must lie in each of the disks of $\mathcal{B}^{(\mathbf{n})}$ and on the boundary of the four disks $B_{i_k}^{(\mathbf{n})}$,

for $k = 1, \ldots, 4$. Our strategy is to fix the ball B_{i_1}, and to analyze the number of extreme transversals ℓ with these properties, that is,

($\star$) *extreme transversals that are tangent to B_{i_1} and to three other balls of $\mathcal{B}$ whose indices are larger than i_1.*

We will show that the number of such extreme transversals is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. This will clearly imply the bound asserted in the theorem. In fact, we will count the number of extreme transversals of $\mathcal{B}_{i_1} = \{B_{i_1}, B_{i_1+1}, \ldots, B_n\}$ that are tangent to B_{i_1}. Since any transversal of $\mathcal{B}$ satisfying property ($\star$) is also an extreme transversal of $\mathcal{B}_{i_1}$, what we are counting is a superset of transversals satisfying property ($\star$) (clearly, not every extreme transversal of $\mathcal{B}_{i_1}$ is necessarily a transversal of $\mathcal{B}$ satisfying ($\star$)).
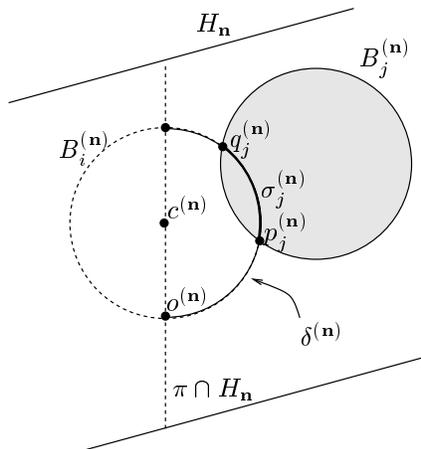

Figure 1: Projection of B_i and B_j onto $H_{\mathbf{n}}$.

Fix a ball B_i, for some $i = 1, \ldots, n-3$, and let $\mathbf{n}$ be an arbitrary direction not parallel to the x-axis. Intersect $B_i^{(\mathbf{n})}$ with the plane Π passing through its center and parallel to the yz-plane (since we have excluded the case in which $\mathbf{n}$ is parallel to the x-axis, $B_i^{(\mathbf{n})} \not\subset \Pi$). Let $\delta^{(\mathbf{n})}$ denote the semicircle bounding the portion of $B_i^{(\mathbf{n})}$ that lies in the positive halfspace bounded by Π. Let $c^{(\mathbf{n})}$, $o^{(\mathbf{n})}$ denote the center of $B_i^{(\mathbf{n})}$ and the clockwise endpoint of $\delta^{(\mathbf{n})}$ (when viewed in direction $\mathbf{n}$), respectively. For a point $p \in \delta^{(\mathbf{n})}$, we define $\varphi^{(\mathbf{n})}(p) = \tan((\angle o^{(\mathbf{n})} c^{(\mathbf{n})} p)/2)$. Since $0 \leq \angle o^{(\mathbf{n})} c^{(\mathbf{n})} p \leq \pi$, $\varphi^{(\mathbf{n})}$ increases monotonically with the length of the arc of $\delta^{(\mathbf{n})}$ from $o^{(\mathbf{n})}$ to p. See Figure 1.

For each $j > i$, we define two functions $f_j(\mathbf{n})$, $g_j(\mathbf{n})$, as follows. Let $\sigma_j^{(\mathbf{n})} = \delta^{(\mathbf{n})} \cap B_j^{(\mathbf{n})}$. Since $r_j \leq r_i$, $\sigma_j^{(\mathbf{n})}$ must be a (possibly empty) connected arc. Let $p_j^{(\mathbf{n})}, q_j^{(\mathbf{n})}$ denote the clockwise and counterclockwise endpoints of $\sigma_j^{(\mathbf{n})}$, respectively. We define $f_j(\mathbf{n}) = \varphi^{(\mathbf{n})}(p_j^{(\mathbf{n})})$ and $g_j(\mathbf{n}) = \varphi^{(\mathbf{n})}(q_j^{(\mathbf{n})})$. If $\sigma_j^{(\mathbf{n})}$ is empty, we put $f_j(\mathbf{n}) = \infty$ and $g_j(\mathbf{n}) = 0$. It is a routine

task to show that the graphs of f_j and g_j are semialgebraic sets of constant description complexity, provided that $\mathbf{n}$ is specified appropriately.

Let ℓ be an extreme transversal in direction $\mathbf{n}$ that is tangent to B_i and to three other balls with larger indices, such that $\ell \cap H_\mathbf{n}$ lies in $\delta^{(\mathbf{n})}$. Then the parameter $t = \varphi^{(\mathbf{n})}(\ell \cap H_\mathbf{n})$ must satisfy

$$\max_{j>i} f_j(\mathbf{n}) \le t \le \min_{j>i} g_j(\mathbf{n}) \ .$$

Moreover, the point $(\mathbf{n}, t)$ must be a vertex of the region enclosed between the upper envelope of the f_j's and the lower envelope of the g_j's, since it lies on (at least) three of the surfaces and the balls in $\mathcal{B}$ are assumed to be in general position. By Theorem 2.3, the number of such vertices is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$.

Repeating the same analysis, but replacing $\delta^{(\mathbf{n})}$ by the other semicircle bounding $B_i^{(\mathbf{n})}$, we conclude that the number of extreme transversals that are tangent to B_i and to three other balls with larger indices is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. As observed above, this completes the proof of the theorem. $\square$

Remark 3.2 The crucial observation in the above proof is that, for every direction $\mathbf{n}$, $\sigma_j^{(\mathbf{n})} = \delta^{(\mathbf{n})} \cap B_j^{(\mathbf{n})}$ is a connected arc. A similar property also holds for a set of *homothets* of a compact convex set. More precisely, let $\mathcal{S}$ be a set of n homothets of a semialgebraic compact convex set of constant description complexity in $\mathbb{R}^3$. Assume that the sets in $\mathcal{S}$ are sorted in a nonincreasing order of their scaling factors. For a direction $\mathbf{n}$, let $S_i^{(\mathbf{n})}$ denote the projection of S_i onto a plane $H_\mathbf{n}$ orthogonal to $\mathbf{n}$, and let $\Pi_\mathbf{n}$ be the plane parallel to the yz-plane for which the length of $\Pi_\mathbf{n} \cap S_i^{(\mathbf{n})}$ is maximized over all planes parallel to the yz-plane. Let $\delta^{(\mathbf{n})}$ be the portion of $\partial S_i^{(\mathbf{n})}$ lying in the closed positive halfspace bounded by $\Pi_\mathbf{n}$. Then for any $j > i$, $\delta^{(\mathbf{n})} \cap S_j^{(\mathbf{n})}$ is a connected arc. This follows from the observations that (a) $\partial S_i^{(\mathbf{n})}$ and $\partial S_j^{(\mathbf{n})}$, being homothetic to each other, intersect at two points [13], and (b) if these two points lie on $\delta^{(\mathbf{n})}$ then $S_j^{(\mathbf{n})} \cap \partial S_i^{(\mathbf{n})} \subseteq \delta^{(\mathbf{n})}$, for otherwise $S_j^{(\mathbf{n})} \cap \Pi_\mathbf{n}$ would be longer than $S_i^{(\mathbf{n})} \cap \Pi_\mathbf{n}$, which is impossible since $S_j^{(\mathbf{n})}$ has a smaller scaling factor than that of $S_i^{(\mathbf{n})}$. Now an easy modification of the above proof shows that the complexity of $\mathcal{T}(\mathcal{S})$ is also $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$.

4 Testing for the Existence of a Line Transversal

Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a collection of n balls in 3-space in general position, as described in the beginning of the proof of Theorem 3.1. We present a near-cubic time algorithm for determining whether $\mathcal{T}(\mathcal{B}) \ne \emptyset$. If $n < 4$, we can explicitly compute $\mathcal{T}(\mathcal{B})$ in $O(1)$ time, so assume $n \ge 4$.

We begin by extending the notion of extreme transversals so that there always exists an

extreme transversal whenever $\mathcal{T}(\mathcal{B}) \neq \emptyset$.

Recall the parameterization $(\xi_1, \xi_2, \xi_3, \xi_4)$ of the set of lines, as defined in the introduction. Then $\mathcal{T}(\mathcal{B})$ is the region "sandwiched" between two envelopes in $\mathbb{R}^4$. Consider a connected component C of the sandwich region and let w be the highest point of C, if such a point exists (the remaining case will be discussed below). Then w is a vertex, a ξ_4-extreme point on a curve, a ξ_4-extreme point on a 2-dimensional face, or a ξ_4-extreme point on a 3-dimensional face. We extend the definition of extreme transversals, so that they also include lines that correspond to these points. For example, the second type corresponds to a line transversal of $\mathcal{B}$ touching three fixed balls and, among nearby such transversals, locally maximizing ξ_4. Clearly the number of extreme transversals that are not vertices of $\mathcal{T}(\mathcal{B})$ (i.e., the extreme transversals that are tangent to fewer than four balls) is $O(n^3)$ and a superset of them can be trivially computed in $O(n^3)$ time, by considering all individual surfaces, intersections of pairs and intersections of triples of surfaces, and by computing locally ξ_4-extreme points on each such set (assuming general position, there are only $O(1)$ such points on each of these sets).

Suppose C is unbounded in the ξ_4-direction. Then C contains a transversal parallel to the yz-plane: The unboundedness implies that there are lines in C whose projection on the xz-plane has arbitrarily large z-intercept. The projection of such a transversal must be nearly vertical (in the xz-plane), because of the compactness of the balls. As $\mathcal{T}(\mathcal{B})$ is a closed set, in the limit we get a line transversal orthogonal to the x-axis.

In the three-dimensional space of lines perpendicular to the x-axis, we choose another parameterization and repeat the process, identifying local extrema in the last parameter. It is easily verified that no more than $O(n^3)$ potential extreme transversals can be generated by carrying out this process to its logical conclusion; a superset of these transversals, of size $O(n^3)$, can be computed in $O(n^3)$ time.

The following lemma follows from the definition of extreme transversals given above.

Lemma 4.1 *If $\mathcal{T}(\mathcal{B})$ is nonempty, then each connected component of $\mathcal{T}(\mathcal{B})$ contains at least one extreme transversal of $\mathcal{B}$, in the extended sense just defined. Moreover, if $\mathcal{T}(\mathcal{B})$ has nonempty interior, then the closure of each connected component of its interior contains at least one point corresponding to an extreme transversal.*

In view of the lemma, the emptiness of $\mathcal{T}(\mathcal{B})$ can be tested by determining whether there exists an extreme transversal of $\mathcal{B}$. The algorithm works in two phases. In the first phase we compute a superset of extreme transversals (i.e., we compute a set of lines that are guaranteed to contain all extreme transversals of $\mathcal{B}$), and in the second phase we check whether any of them is indeed a transversal of $\mathcal{B}$. As discussed above, there are $O(n^3)$ lines that touch fewer than four balls and could potentially be extreme transversals of $\mathcal{B}$; they can be computed in $O(n^3)$ time. Let $\mathcal{L}_1$ denote the set of these lines. (This set also includes extreme lines perpendicular to the x-axis.)

We next compute a set $\mathcal{L}_2$ of $O(n^{3+\varepsilon})$ lines that touch four balls of $\mathcal{B}$ and include all extreme transversals that correspond to vertices of $\mathcal{T}(\mathcal{B})$, by proceeding along the same lines as the proof of Theorem 3.1. For each B_i (and for each of the two semicircles, $\delta^{(\mathbf{n})}$ and its complement, in the corresponding planes of projection), we define the two collections of bivariate functions, $\{f_j(\mathbf{n})\}_{j>i}$, and $\{g_j(\mathbf{n})\}_{j>i}$, as in the preceding section. Let ℓ be an an extreme transversal that is tangent to B_i at a point that projects to the semicircle under consideration and that is also tangent to three other balls with larger indices. Then ℓ corresponds to a vertex of the region sandwiched between the lower envelope of the g_j's and the upper envelope of the f_j's. Using the algorithm described in [3] (see Theorem 2.3), we can compute all these vertices in time $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. We repeat this step for each ball B_i and for each of the two corresponding semicircles, and obtain, in overall time $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$, a set of lines that contains all extreme line transversals corresponding to vertices of $\mathcal{T}(\mathcal{B})$.

Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ be the set of all lines computed by the above procedures. As already observed, some of the lines may not be transversals to the entire set of balls, so we need to test whether any of these lines is indeed a transversal of $\mathcal{B}$. By Observation 2.2, we can preprocess $\mathcal{B}$, in $O(n^{3+\varepsilon})$ time, into a data structure of size $O(n^{3+\varepsilon})$, so that we can determine in $O(\log^2 n)$ time whether a line $\ell \in \mathcal{L}$ is a transversal of $\mathcal{B}$. Hence, we can identify all extreme transversals of $\mathcal{B}$ in $O(n^{3+\varepsilon})$ additional time, for any $\varepsilon > 0$. We therefore conclude:

Theorem 4.2 *Given a set $\mathcal{B}$ of n balls in 3-space in general position, we can determine in $O(n^{3+\varepsilon})$ time, for any $\varepsilon > 0$, whether $\mathcal{B}$ admits a line transversal.*

Remark 4.3 (i) If all balls in $\mathcal{B}$ have the same radius, then for each i, we can compute the region lying between the lower envelope of $\{g_j(\mathbf{n}) \mid 1 \le j \ne i \le n\}$ and the upper envelope of $\{f_j(\mathbf{n}) \mid 1 \le j \ne i \le n\}$. In this case the collection of these sandwich regions represents the boundary of $\mathcal{T}(\mathcal{B})$. It is therefore sufficient to determine whether any of these sandwich regions is nonempty (and thus the second "filtering" phase is not required).

(ii) The above algorithm can handle some limited degeneracies in the input. For example, the algorithm works even if more than 4 balls are tangent to a line, as long as the number of such balls is bounded by a constant. We will need this extension in Section 6.

5   Lower Bound

In this section we describe a construction of a family $\mathcal{B}$ of n balls in $\mathbb{R}^3$, for which $\mathcal{T}(\mathcal{B})$ has complexity $\Omega(n^3)$. We also present an easier construction of a set $\mathcal{B}$ of n balls of the same radius, with $\mathcal{T}(\mathcal{B})$ consisting of $\Omega(n^2)$ connected components. The constructions are inspired by those of Pellegrini [21].

We begin by describing the second, simpler construction. Consider a plane π through the origin O in $\mathbb{R}^3$. Let $R \gg 1$ be a quantity to be fixed later. Place two balls of radius R tangent to π at O from the opposite sides of π, and then move them apart, perpendicular to π, so that they lie at distance 1 from π. Denote the resulting two balls by B_π^1 and B_π^2. Note that no line transversal of $\{B_\pi^1, B_\pi^2\}$ is parallel to π. Moreover, any line that passes through O and makes an angle larger than $\varepsilon = \varepsilon(R)$ with π is a transversal of this set. Note that $\varepsilon(R) \to 0$ as $R \to \infty$.

Consider a set of $n/2$ planes $\{\pi_1, \ldots, \pi_{n/2}\}$ passing through the origin, with no three planes sharing a common line. Put

$$\mathcal{B} = \{B_{\pi_i}^1, B_{\pi_i}^2 \mid i = 1, \ldots, n/2\}.$$

We claim that $\mathcal{T}(\mathcal{B})$ has $\Omega(n^2)$ connected components, for some sufficiently large R. Indeed, no line parallel to any of the planes is a transversal of $\mathcal{B}$. Let $\mathbb{S}$ denote the sphere of directions in $\mathbb{R}^3$ — the unit-radius sphere centered at the origin; each direction in 3-space can be represented as a point on $\mathbb{S}$. The locus of orientations of lines parallel to π_i is a great circle $C_i = C(\pi_i)$ on $\mathbb{S}$, and the $n/2$ great circles $\{C_1, \ldots, C_{n/2}\}$ in general position corresponding to the $n/2$ planes $\{\pi_1, \ldots, \pi_{n/2}\}$ induce an arrangement $\mathcal{A}$ on $\mathbb{S}$ with $\Theta(n^2)$ faces. To finish the argument it is sufficient to show that, for a sufficiently large R, each face of $\mathcal{A}$ contains at least one line orientation corresponding to a transversal of $\mathcal{B}$. Indeed, any line passing through O and making an angle more than ε with the planes is a transversal of $\mathcal{B}$. Therefore every point on $\mathbb{S}$ outside the union of $n/2$ bands of half-width ε, each centered around one of the great circles C_i, is the orientation of a line transversal to $\mathcal{B}$ through O. Picking R large enough we can assure that the bands are sufficiently narrow so that every face of $\mathcal{A}$ contains a point outside of their union. This completes the construction of a family of balls for which $\mathcal{T}(\mathcal{B})$ has $\Omega(n^2)$ connected components.

To obtain $\mathcal{T}(\mathcal{B})$ with a large number of extreme transversals, we proceed as follows: For any $\delta > 0$, let $K_\delta \subseteq \mathbb{S}$ be the *closed spherical cap* centered at the north pole $(0, 0, 1)$ with angular opening 2δ, i.e., K_δ is the set of unit vectors whose angles with the positive z-axis are at most δ. Let $\{\pi_1, \ldots, \pi_{n/4}\}$ be a set of $n/4$ planes passing through the origin and let $C = \{C(\pi_i) \mid i = 1, \ldots, n/4\}$ be the set of great circles as defined in the previous paragraph. We choose the planes so that most (i.e., $\Theta(n^2)$) of the faces of the arrangement of great circles $\mathcal{A} = \mathcal{A}(C)$ on $\mathbb{S}$ lie in a spherical cap K_δ centered at the north pole, for some $\delta \ll 1$. Clearly this requires a choice of $\varepsilon < \delta/n$ and a correspondingly large R. (Notice also that $\mathcal{A}$ is centrally symmetric, with center at the origin. Two opposite points of $\mathbb{S}$ correspond to lines with opposite orientation. So, in fact, we are considering an arrangement in which two opposite spherical caps contain most of the features.) Let $\mathcal{B}' = \{B_{\pi_i}^1, B_{\pi_i}^2 \mid i = 1, \ldots, n/4\}$ be the resulting set of $n/2$ spheres.

Consider now the set of lines passing through a point O' near O. The set of orientations of lines missing both balls $B_{\pi_i}^1, B_{\pi_i}^2$ and passing through O' is a slightly distorted version of the band of half-width ε around the great circle C_i. Indeed, the orientations of the lines tangent to $B_{\pi_i}^1$ and passing through O' sweep out a cone corresponding to a circle near the

great circle C_i, so the claim follows. In particular, the orientations of all transversals of $\mathcal{B}$ passing through O' is a complement of a set of narrow (in general, asymmetric) bands around the same family of great circles $\{C_i\}$. It is easy to verify that there are a sufficiently small neighborhood N of O and a choice of R, so that the following holds: There is a fixed set of $\Theta(n^2)$ orientations θ, one in each face of $\mathcal{A}$ contained in the spherical cap K_δ, so that each θ corresponds to a different component of $\mathcal{T}(\mathcal{B}')$ and such that for every point $O' \in N$ the line passing through O' in direction θ is a transversal of $\mathcal{B}'$.

Now we construct a collection $\mathcal{B}'' = \{B_1'', \ldots, B_{n/2}''\}$ of $n/2$ additional small congruent balls around O, so that the following conditions hold.

1. For each $i = 1, \ldots, n/2$, B_i'' is contained in N and the distance from O to the center of B_i'' is half the common radius of these balls.

2. The centers of the balls lie at the vertices of a regular $(n/2)$-gon in the xy-plane centered at O.

Put $I = \bigcap_i B_i''$ and $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$. Note that I is a scaled copy of a set I_0 that depends only on n and not on δ. Fix a face $f \subset K_\delta$ of $\mathcal{A}$. The preceding discussion implies that there is a connected component Q of $\mathcal{T}(\mathcal{B})$ that contains only lines whose orientation lies in f. We claim that each such component Q contains at least $n/2$ distinct 2-dimensional faces of $\mathcal{T}(\mathcal{B})$ on its boundary. This implies the claimed lower bound, since the number of faces f in K_δ is $\Theta(n^2)$.

As observed above, there exists an orientation $\theta_f \in f$ so that for all points $O' \in I$ the line $\ell(O', \theta_f)$ through O' in direction θ_f is a transversal of $\mathcal{B}'$ and thus of $\mathcal{B}$. Let I^* denote the projection of I onto a plane perpendicular to θ. I^* is a scaled copy of the corresponding projection of I_0. Since θ is near-vertical, I^* has the shape of a convex "$n/2$-gon" bounded by circular arcs, provided δ is sufficiently small. For any vertex v_i, for $i = 1, \ldots, n/2$, of this 'circular polygon', $\ell(v_i, \theta_f)$ is a transversal of $\mathcal{B}$ lying on the boundary of $\mathcal{T}(\mathcal{B})$. For every i, $\ell(v_i, \theta_f)$ touches a different pair of balls of $\mathcal{B}''$, and does not touch any other ball of $\mathcal{B}''$. Therefore, the lines $\{\ell(v_i, \theta_f) \mid i = 1, \ldots, n/2\}$ are witnesses of the presence of $n/2$ different 2-dimensional faces on the boundary of Q, which completes the argument. Hence we have shown:

Theorem 5.1 (i) *There exists a set of n balls in $\mathbb{R}^3$ whose transversal space has $\Omega(n^3)$ complexity.*

(ii) *There exists a set of n unit-radius balls in $\mathbb{R}^3$, for which the space of line transversals consists of $\Omega(n^2)$ connected components.*

6 Computing a Smallest Enclosing Cylinder

Let $\mathcal{B} = \{B(x_1, r_1), \ldots, B(x_n, r_n)\}$ be a set of n balls in 3-space, where $B(x_i, r_i)$ is the ball of radius r_i centered at x_i and $r_i \geq r_j$ whenever $i \leq j$. We assume that the balls in $\mathcal{B}$ are in general position in the sense that any cylinder is tangent to at most five balls of $\mathcal{B}$. We wish to compute a smallest cylinder containing all balls of $\mathcal{B}$. (The case of a smallest cylinder that intersects all the balls in $\mathcal{B}$ can be handled in essentially the same manner.) For a real parameter $r \geq \max_i r_i$, let $\mathcal{B}_r^- = \{B(x_i, r - r_i) \mid 1 \leq i \leq n\}$, and recall that there exists a cylinder of radius r containing all balls of $\mathcal{B}$ if and only if $\mathcal{B}_r^-$ has a line transversal. By our assumption, a transversal of $\mathcal{B}_r^-$ is tangent to at most five balls. We thus need to find the smallest value r^* of r such that $\mathcal{B}_r^-$ admits a line transversal. We accomplish this by applying the parametric search technique of Megiddo [15]. First, we need an 'oracle' for determining whether a given r is equal to, greater than, or smaller than r^*. This can be done using Theorem 4.2. In fact, Theorem 4.2 can be extended so that it can also determine whether a given r is equal to r^*, as follows. It can be shown that the interior of $\mathcal{T}(\mathcal{B}_r^-)$ is empty if and only if $r = r^*$. In view of Lemma 4.1, it suffices to test for the emptiness condition locally near each extreme transversal ℓ. This local test has to consider only the $O(1)$ balls to which ℓ is tangent (by our general-position assumption, at most five balls are tangent to ℓ), and to determine whether the interior of the space of line transversals of these balls is empty; this can be done in constant time. The oracle thus runs in $O(n^{3+\varepsilon})$ time, for any $\varepsilon > 0$.

Next, we simulate the first phase of the oracle (i.e., the algorithm that computes a superset of extreme transversals) generically on $\mathcal{B}_{r^*}^-$, without knowing the value of r^*, and maintain an interval I that is guaranteed to contain r^*; initially, I is set to $[\max_i r_i, \infty]$. The algorithm for computing extreme transversals, as described above, is comparison-based, in the sense that its control flow is governed by comparisons, each of which amounts to testing the sign of some low-degree polynomial in r^*. Moreover, the algorithm uses the value of r^* only to resolve such comparisons. Since r^* is unknown, resolving a comparison involves determining the sign of a corresponding polynomial $p(r)$ at $r = r^*$. Whenever the simulation of the algorithm encounters such a comparison, we compute the roots $\rho_1, \ldots, \rho_k$ of the corresponding polynomial $p(r)$. Applying Theorem 4.2 to $\mathcal{B}_{r_i}^-$, for each $1 \leq i \leq k$, we determine, in $O(n^{3+\varepsilon})$ time, whether $\rho_i = r^*$, $\rho_i < r^*$, or $\rho_i > r^*$. If any ρ_i is equal to r^*, we stop the entire procedure. Otherwise, we have an interval (ρ_i, ρ_{i+1}) that contains r^*. We compute the sign of $p(r^*)$, by evaluating $p(r)$ at any point in (ρ_i, ρ_{i+1}), and thus resolve the comparison. Next, we shrink the interval I to $I \cap (\rho_i, \rho_{i+1})$, and continue the simulation of the generic algorithm.

The traditional parametric search should simulate both phases of the algorithm, but the following lemma shows that simulating the first phase is sufficient, in the sense that the algorithm always terminates before finishing the simulation of the first phase. (Recall that if all balls in $\mathcal{B}$ have the same radius, then the second phase is not required anyway.)

Lemma 6.1 *The generic algorithm that simulates the first phase of the oracle always performs a comparison whose corresponding polynomial vanishes at r^*.*

Proof: Suppose that the generic algorithm does not make such a comparison. Then, at the end of the simulation we have an open interval I_f containing r^*. Since r is an indeterminate in the simulation, each potential extreme transversal computed by the generic algorithm can be parametrized as $\ell(r) = (\xi_1(r), \xi_2(r), \xi_3(r), \xi_4(r))$ for $r \in I_f$, where the ξ_i's are univariate algebraic functions of r. Each such line $\ell(r)$ is tangent to the same subset of balls $B(x_{i_1}, r - r_{i_1}), \ldots, B(x_{i_k}, r - r_{i_k})$, for every $r \in I_f$. Let $\mathcal{L}$ be the set of parametrized lines computed by the generic algorithm. Then, for any $r_0 \in I_f$, the potential extreme transversals of $\mathcal{B}_{r_0}^-$ computed by (the non-generic version of) the first phase is the set $\mathcal{L}(r_0) = \{\ell(r_0) \mid \ell \in \mathcal{L}\}$.

Let ℓ_0 be an extreme transversal line of $\mathcal{B}_{r^*}^-$, and let $\ell_0(r)$ be the corresponding parametrized line in $\mathcal{L}$ (so that $\ell_0 = \ell_0(r^*)$). For the sake of simplicity, assume that ℓ_0 is a vertex of $\mathcal{T}(\mathcal{B}_{r^*}^-)$, which is an intersection point of four surfaces. The generic algorithm computes a superset of extreme transversals of $\mathcal{B}_{r^*}^-$, so $\ell_0 \in \mathcal{L}$. Since r^* is the radius of a smallest enclosing cylinder of $\mathcal{B}$, $\ell_0(r)$ is not a transversal of $\mathcal{B}_r^-$, for any $r < r^*$. Let $B(x_i, r^* - r_i)$ be the largest (i.e., the first in the list) ball that ℓ_0 is tangent to, so ℓ_0 intersects the interiors of $B(x_1, r^* - r_1), \ldots, B(x_{i-1}, r^* - r_{i-1})$. Hence, for any $r_0 \in I_f$ smaller than r^*, $\ell_0(r_0)$ still intersects the interiors of $B(x_1, r_0 - r_1), \ldots, B(x_{i-1}, r_0 - r_{i-1})$. However, by definition of r^*, $\ell_0(r_0)$ is not a transversal of $\mathcal{B}_{r_0}^-$. Therefore $\ell_0(r_0)$ is not a transversal, and thus not an extreme transversal, of the set $\{B(x_i, r_0 - r_i), \ldots B(x_n, r_0 - r_n)\}$ and would not have been computed by the non-generic oracle at r_0. This contradicts the fact that $\ell_0(r_0) \in \mathcal{L}(r_0)$. Hence, the generic algorithm always performs a comparison whose corresponding polynomial vanishes at r^*. $\square$

To make this parametric searching efficient, we perform the generic simulation of the first phase of the oracle using a parallel implementation [15]. For each ball B_i, the first phase computes the region between the upper envelope and the lower envelope of two families of bivariate functions, using the algorithm of Agarwal et al. [3]. The only step in this algorithm that is nontrivial to parallelize is the construction of the overlay of two planar maps, each of which is a minimization diagram of a family of bivariate functions. The sequential algorithm in [3] uses a sweep-line algorithm to compute the overlay, which is difficult to parallelize. Instead, we compute the overlay using segment trees, which is easier to parallelize, as described in [4]. The parallel version of the overlay procedure runs in $O(\log n)$ time using $O(n^{2+\varepsilon})$ processors, under Valiant's comparisons model, which is the model that is relevant for the generic simulation. Plugging this procedure into the algorithm of Agarwal et al. [3], we can compute the region lying between the upper and lower envelopes in $O(\log^2 n)$ time using $O(n^{2+\varepsilon})$ processors, for any $\varepsilon > 0$; see [3] for details. Executing this procedure for all balls of $\mathcal{B}$ in parallel, we can compute the set $\mathcal{L}$ of potential extreme transversals in $O(\log^2 n)$ time, using $O(n^{3+\varepsilon})$ processors, for any $\varepsilon > 0$.

Finally, simulating the parallel algorithm generically, as described in [15], we can com-

pute a smallest enclosing cylinder of $\mathcal{B}$ in time $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$. Hence, we obtain the following result:

Theorem 6.2 *Given a set $\mathcal{B}$ of n balls in $\mathbb{R}^3$, the minimum-radius cylinder containing (resp. intersecting) every ball of $\mathcal{B}$ can be computed in time $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$.*

7 An Approximation Algorithm

Since computing a smallest enclosing cylinder exactly is expensive, it is desirable to design more efficient algorithms that solve the following approximation problem: Given a set S of n balls (or points) in $\mathbb{R}^3$ and a real parameter $\delta > 0$, compute a cylinder containing (or intersecting) every ball (or point) in S, whose radius is at most $(1 + \delta)r^*$, where r^* is the radius of a smallest enclosing (or intersecting) cylinder of S. Schömer et al. [23] consider a somewhat different problem. They give an $O((1 + 1/\delta^2)n \log(1/\delta))$-time algorithm for computing a cylinder that contains S and has radius at most $r^* + \delta \cdot \operatorname{diam}(S)$. If the shape of S is sufficiently "round," then r^* and $\operatorname{diam}(S)$ are comparable in size, so the output of the algorithm of [23] is comparable with what we want. However, if S is "long and skinny" then r^* can be arbitrarily small compared to $\operatorname{diam}(S)$, in which case the value returned by the algorithm of [23] will be much larger than $(1+\delta)r^*$. We present an $O((1+1/\delta^2)n)$-time algorithm that computes a cylinder of radius at most $(1 + \delta)r^*$ containing (or intersecting) S.

For the sake of simplicity, we describe the algorithm for the case when S is a set of n points in $\mathbb{R}^3$. The modifications for the more general case of balls are straightforward. Let $Q(s, r)$ be the axis-parallel cube of side $2r$ centered at s, i.e.,

$$Q(s, r) = \{p : |p_x - s_x|, |p_y - s_y|, |p_z - s_z| \le r\}.$$

For a direction $\mathbf{n}$, let $S^{(\mathbf{n})}$ denote the orthogonal projection of S onto the plane $H_\mathbf{n}$ normal to $\mathbf{n}$ and passing through the origin. If a cylinder of radius r with axis in direction $\mathbf{n}$ contains S, then there is a disk of radius r within $H_\mathbf{n}$ that contains $S^{(\mathbf{n})}$. We now describe the algorithm.

ALGORITHM: SMALL ENCLOSING CYLINDER

1. Compute the diameter of S.

2. Perform a linear transformation on S so that the two points $p = (0, 0, 0)$ and $q = (0, 0, 1)$ form a diametral pair, i.e., $p, q \in S$ and $\operatorname{diam}(S) = d(p, q)$.

3. Compute the smallest value r_0 such that the set $\{Q(s, r_0) \mid s \in S\}$ of cubes admits a line transversal. Since a cube of side length $2r$ has an inscribed ball of radius r and a circumscribing ball of radius $\sqrt{3}r$, we have

$$r^*/\sqrt{3} \le r_0 \le r^*.$$

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4. Set $\nu = \delta r_0$ and $b = \min\{4\sqrt{3}r_0, \pi/2\}$. Choose a set Δ of $O(1/\delta^2)$ points on K_b, the spherical cap with angular opening b and centered at the north pole q, so that for any point $\mathbf{n} \in K_b$, there is a point $\mathbf{n}' \in \Delta$ with $\angle\mathbf{n}p\mathbf{n}' \leq \nu$; Δ can be computed by drawing a grid of longitudes and latitudes on K_b of size $c/\delta \times c/\delta$, where $c > 0$ is an appropriate absolute constant.

5. For each $\mathbf{n} \in \Delta$, compute the smallest disk, $D_{\mathbf{n}}$, enclosing $S^{(\mathbf{n})}$. Let $r_{\mathbf{n}}$ denote the radius of $D_{\mathbf{n}}$, and let $\ell_{\mathbf{n}}$ be the line passing through the center of $D_{\mathbf{n}}$ in direction $\mathbf{n}$.

6. Let $\mathbf{n}_0 \in \Delta$ be a direction with the minimum value of $r_{\mathbf{n}_0}$. Then the cylinder of radius $r_{\mathbf{n}_0}$ with $\ell_{\mathbf{n}_0}$ as its axis is the desired cylinder.

Lemma 7.1 *The above algorithm computes a cylinder of radius $(1 + \delta)r^*$ enclosing S.*

Proof. Let $\mathbf{n}^*$ be the direction of the axis of a smallest cylinder enclosing S. We claim that $\mathbf{n}^* \in K_b$. Indeed, if $\mathbf{n}^* \notin K_b$, then the angle between the positive z-axis and $\mathbf{n}^*$ is more than b. Let p^*, q^* be the orthogonal projections of p and q onto $H_{\mathbf{n}^*}$. Since S is contained in a cylinder of radius r^* whose axis is in direction $\mathbf{n}^*$, $d(p^*, q^*) \leq 2r^*$. On the other hand,

$$d(p^*, q^*) > \sin b \geq b/2 = 2\sqrt{3}r_0 \geq 2r^*,$$

a contradiction. Hence, $\mathbf{n}^* \in K_b$.

By construction, there is a direction $\mathbf{n}_0 \in \Delta$ so that the angle between $\mathbf{n}^*$ and $\mathbf{n}_0$ is at most ν. Let w be a point of S, and let w_0, w^* denote the corresponding projections of w in $S^{(\mathbf{n}_0)}, S^{(\mathbf{n}^*)}$, respectively. By construction, we have $|ww_0|, |ww^*| \leq |wp| \leq 1$ (recall that the diameter of S is assumed to be 1). Since the angle between the directed segments ww_0 and ww^* is at most ν, it follows that $|w_0w^*| \leq \nu$. Since $S^{(\mathbf{n}^*)}$ is contained in a disk of radius r^*, it follows that all the points of $S^{(\mathbf{n}_0)}$ are contained in a ball of radius $r^* + \nu \leq (1 + \delta)r^*$; all these points being coplanar, they are also contained in a disk of radius $(1 + \delta)r^*$. In other words, $r_{\mathbf{n}_0} \leq (1 + \delta)r^*$. Let c be the center of this disk. Then the cylinder of radius $r_{\mathbf{n}_0} \leq (1 + \delta)r^*$ whose axis is the line passing through c in direction $\mathbf{n}_0$ contains S. $\qquad\square$

The diameter of S can be computed in $O(n \log n)$ randomized expected time [9], and r_0 can be computed in $O(n)$ time using the algorithm by Amenta [5]. For each $\mathbf{n} \in \Delta$, $D_{\mathbf{n}}$ can be computed either in linear worst-case time using Megiddo's deterministic algorithm [16, 17] or in linear expected time using the randomized algorithm by Matoušek et al. [14]. Hence, the overall expected running time of this randomized implementation of the algorithm is $O(n \log n + n/\delta^2)$. A closer look at the proof of Lemma 7.1 shows that in Step 1 it suffices to compute a pair of points $p, q \in S$ that are separated by a distance of at least $\mathrm{diam}(S)/2$. Such a pair of points can easily be computed deterministically in worst-case time $O(n)$: choose an arbitrary point p of S and compute its farthest neighbor q in S (if there is more than one such point, choose any of them); by the triangle inequality, $d(p, q) \geq \mathrm{diam}(S)/2$. This makes the algorithm fully deterministic, and improves its total

running time to $O((1+1/\delta^2)n)$. Moreover, the algorithm can easily be extended to the case in which S is a set of balls instead of a set of points. One notable difference is that Step 5 has to be replaced by the algorithm of Megiddo [18] for finding, in linear time, the smallest disk containing a given set of disks. Hence, we obtain the following result:

Theorem 7.2 *Given a set S of n balls in $\mathbb{R}^3$ and a parameter $\delta > 0$, a cylinder containing (or intersecting) every member of S, whose radius is at most $1 + \delta$ times that of a smallest enclosing (or intersecting) cylinder of S, can be computed in (deterministic) time $O((1 + 1/\delta^2)n)$.*

8 Conclusion

In this paper we obtained near-cubic bounds for the complexity of the space of line transversals of a collection $\mathcal{B}$ of n balls in $\mathbb{R}^3$ or of n homothets of any convex body of constant description complexity. We also developed near-cubic algorithms for determining whether $\mathcal{T}(\mathcal{B}) = \emptyset$ and for computing a smallest cylinder enclosing (or intersecting) every member of $\mathcal{B}$. We finally presented a linear-time approximation algorithm for computing a small cylinder enclosing $\mathcal{B}$. The main open problem is to obtain near-cubic bounds for the case of collections of general convex sets of simple shape in $\mathbb{R}^3$. We conjecture that such bounds hold in fairly general situations, but the proof techniques used in this paper do not seem to extend to more general sets. In the light of the discussion in Section 2, this conjecture would automatically be established if one manages to extend the results of [3] concerning overlay of envelopes to four dimensions.

Another open problem is to extend the results of this paper to higher dimensions. Since lines in d dimensions have $2d - 2$ degrees of freedom, we conjecture that the complexity of the space of line transversals of a collection of n balls, or of convex polyhedra with a total of n faces (of all dimensions), in $\mathbb{R}^d$ is close to $O(n^{2d-3})$.

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