Segmenting Object Space by Geometric Reference Structures

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A model for segmentation of an object space by an array of binary, radiation-field sensors and geometric reference structures is described. Given a family of binary, radiation-field sensors and a geometric reference structure, we refer to the set of sensor states induced by a source at point p as the signature of p. We study the segmentation of an object space into signature cells and prove near optimal bounds on the number of distinct signatures induced by a point source, as a function of sensor and reference structure complexity. We also show that almost any family of signatures can be implemented under this model.

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1. INTRODUCTION

A sensor system implements a mapping between an object state and a sensor state. In the case of radiation-field sensors, objects and sensors are embedded in a physical space and the mapping is mediated by field propagation. Conventionally, radiation-field sensors have been designed to implement simple

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isomorphisms of the object state (as in photography) or continuous transformations of the object distribution (as in tomography). The range of sensor mappings has dramatically expanded over the past decade, however, with the development of multiscale and distributed sensor models. This article considers a particularly simple radiation sensor system, with strictly geometric field propagation. In this case the object sensor mapping is determined by the visibility of object points from a particular detector. The visibility is modulated by optical structures (also known as *reference structures*), placed between detectors and objects, which can absorb, reflect, or refract radiation. Object estimation under this approach, referred to as *reference structure tomography* (RST) [Brady et al. 2004; Potuluri et al. 2003], is based on sensor data and prior knowledge of the reference structure. *Geometric* RST describes reference structure tomography under a geometrical optics model for field propagation. It may be regarded as a generalization of coded aperture imaging [Fenimore 1978; Gottesman and Fenimore 1989], in which a 2D mask modulates projections from source points onto a detector array and the impulse response is equal to the aperture.

While most analyses of sensor networks focus on the communications and processing topology of the network, analysis of the mapping that the sensor array implements on the embedding space is also critical in describing the system's functionality. It is important to formalize the concept of sensor spatial response as a fundamental design characteristic of distributed sensor systems, and to understand the complexity of the mapping between the object and sensor states for a particular class of sensor response functions. For example, in the context of RST, the mappings between object and sensor states that can be realized depends on the complexity of the reference structure. One of the main challenges in the design of reference structures is to narrow the gap between the physically realizable mappings and the logically desirable mappings. Surprisingly little is known about this problem.

In this article we study a simple model consisting of geometric radiation field propagation and opacity-based field modulation. As illustrated in Figure 1, under this model the "object space" is segmented into cells, each with a constant *signature.* The number, structure, and distribution of these cells is of profound interest to the design and data analysis of a sensor system. We illustrate the physical constraints on realizable mappings by bounding the number of distinct *signatures* that can be realized for a particular reference structure geometry.

2. OUR MODEL AND RESULTS

The context for the work described here includes the rapidly developing field of computational sensing and imaging [Mait et al. 2003]. As researchers in this field span a wide range from physical optoelectronics to computational geometry, we first formally define our sensor system model.

Our model. Let $\mathbb{P} \subseteq \mathbb{R}^d$ denote the *radiation* space: objects, occluders (reference structures), and field sensors are embedded in the radiation space. The subspace where the source objects may lie is referred to as the *object space* and is denoted by Σ. We assume that a source object is a point and that there is only one source object present at any given time. Finally, we use $\mathbb{X} \subseteq \mathbb{P}$ to

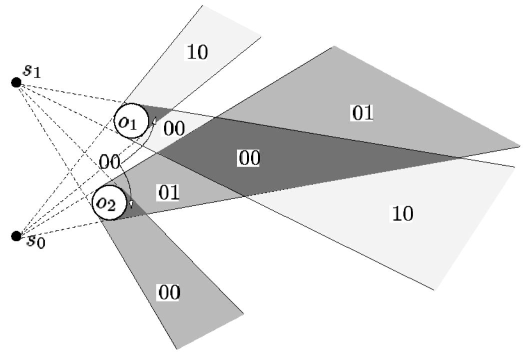

Fig. 1. A segmentation of the object space by radiation sensors and occluders (reference structures). $\Pi(\mathcal{S}, \mathcal{O}) = \{00, 01, 10, 11\}$. The nonshaded area has signature 11; the regions having the same signature may be disconnected.

denote the subspace where sensors are located; let $\mathcal{S} = \{s_0, \ldots, s_{m-1}\}$ be a set of m point sensors located in $\mathbb{X}$. Source objects generate radiation signals that propagate through the radiation space along the rays. The radiation field is modulated by optical elements that refract, diffract, or absorb. Here we consider particularly simple optical elements consisting of absorptive *obscurants* (or *occluders*) that block field propagation. Let $\mathcal{O} = \{O_0, \ldots, O_{n-1}\}$ denote a family of n occluders in $\mathbb{P}$. We assume that each o_i is a convex, semialgebraic set of constant description complexity.[1] Each sensor point locally samples the field state, for example, in Figure 1, the field state is sampled at s_0 and s_1. We assume that the field state is binary, with a value of 1 if any active object point is visible and 0 if no active object point is visible. Geometrically, for a point $p \in \Sigma$, a sensor $s_i \in \mathcal{S}$ returns 1 (resp. 0) if p is visible (resp. not visible) from s_i, that is, the segment $s_i p$ does not intersect the interior of any occluder. We use $\chi_i(p)$ to denote the value returned by sensor s_i. We define a function $\chi : \Sigma \to \{0, 1\}^m$ where $\chi(p) = \chi_{m-1}(p) \cdots \chi_0(p)$; $\chi(p)$ is the *signature* of p. Let $\Pi(\mathcal{S}, \mathcal{O}, \Sigma) = \{\chi(p) \mid p \in \Sigma\}$ denote the set of signatures realized by $\mathcal{S}$ and $\mathcal{O}$ in Σ. Set $\pi(\mathcal{S}, \mathcal{O}, \Sigma) = |\Pi(\mathcal{S}, \mathcal{O}, \Sigma)|$. If $\Sigma = \mathbb{R}^d$, we define

$$\pi_d(m, n) = \max_{\substack{|\mathcal{S}|=m \\ |\mathcal{O}|=n}} \pi(\mathcal{S}, \mathcal{O}, \mathbb{R}^d).$$

That is, $\pi_d(m, n)$ is the maximum number of distinct signatures realized by a system with m sensors and n occluders. If we regard each sensor as a light source emitting the light of a distinct primary color, then $\pi_d(m, n)$ is the maximum number of distinct colors that can be generated by m light sources using n occluders in $\mathbb{R}^d$.

Our results. Our main result is a lower bound on the number of distinct signatures that can be realized:

[1]A set is *semialgebraic* if it can be expressed as a formula that is a Boolean combination of polynomial inequalities. A semialgebraic set has *constant description complexity* if it can be defined by polynomials whose number and maximum degree are bounded by some constant.

THEOREM 2.1. *For $m, d \geq 1$ and for any n in range $1 \leq n \leq 2^{m/16}$,*

$$\pi_d(m, n) = \Omega\left(\left(\frac{mn}{\log n}\right)^d\right).$$

Next, using the standard techniques from computational geometry [Sharir and Agarwal 1995], we prove that the above bound is almost tight.

THEOREM 2.2. *For any $d \geq 1$ and for any $m, n \geq 1$, $\pi_d(m, n) = O((mn)^d)$.*

Finally, we show that one can realize almost any family of signatures even if Σ is a line.

THEOREM 2.3. *Let $\mathbb{X}$ be the x-axis, and let Σ be the line $y = 2$. Given any set $\Pi \subseteq \{0, 1\}^m$, there exists a set $\mathcal{S}$ of m sensors and a set $\mathcal{O}$ of occluders so that $\Pi(\mathcal{S}, \mathcal{O}, \Sigma) = \Pi \cup \{0^m\}$.*

Related work. Although the early work on the analysis of sensor networks focused on communication and processing topology of a network, the problem of placing sensor nodes to optimize the communication cost, the coverage, and data collection has been studied in the last few years. For example, see Chakrabarty et al. [2002], Gonzalez-Banos and Latombe [2001], Guestrin et al. [2005], Henderson et al. [2005] and the references therein for a sample of such results. A simple example of how sensor placement can affect the system's ability to locate and track an object is presented in the work of Chakrabarty et al. [2002]. They consider sensors with uniform spatial response over a finite circular neighborhood and derive limits on the sensor density necessary to localize targets.

The problem of sensor placement can be viewed as a generalization of the well known *art gallery problem* in computational geometry, which asks for placing the minimum number of guards so that each point in a polygonal region is visible from at least one of the guards [O'Rourke 1987; Urrutia 2000]. The algorithms for the art gallery problem have been used for sensor placement as well; see for example, Gonzalez-Banos and Latombe [2001].

Theorem 2.1 is proved using the so called *probabilistic method*, a powerful technique that has been applied to a wide spectrum of problems [Alon and Spencer 1992]. Agarwal et al. [1994] used this technique for proving the lower bound on a visibility problem. Although there are some similarities in their construction and the one presented here, the two proofs are very different.

Our upper bound is similar to a result by Suri and O'Rourke [1986], who proved an upper bound on the complexity of the region in $\mathbb{R}^2$ that is visible from a line segment in the presence of occluders. See also O'Rourke [1987].

3. LOWER BOUNDS

We first prove the bound for the case in which Σ is a line, and then we cascade multiple copies of this construction to prove the lower bound for $d > 1$.

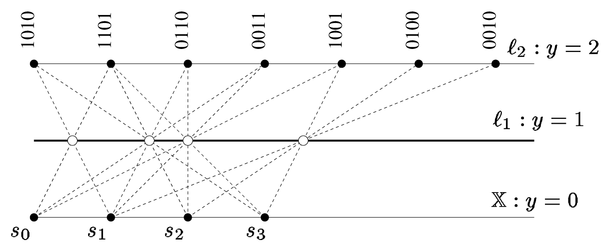

Fig. 2. 1D lower-bound construction; signature of a point is $\chi(p) = \chi_3(p)\chi_2(p)\chi_1(p)\chi_0(p)$.

3.1 One Dimensional Object Space

In our construction, $\mathbb{X}$ is the x-axis, Σ is the line $\ell_2 : y = 2$, and all the occluders, each of which is an interval, are placed on the line $\ell_1 : y = 1$. First we consider the case $n \geq m$, that is, there are more occluders than sensors. Without loss of generality, we can assume that $m \geq m_0$ for sufficiently large constant $m_0 \geq 1$.

We set $\mathcal{S} = \{s_0, \ldots, s_{m-1}\}$, where $s_i = (2i, 0)$. $\mathcal{O}$ is a set of intervals. For simplicity, we describe the construction with degenerate occluders in the sense that $\ell_1 \setminus \bigcup \mathcal{O}$ is a set of points called *pin holes* and denoted by $\mathcal{H}$. We later modify the construction by replacing each pin hole by an interval. The points in $\mathcal{H}$ are chosen as follows. Set

$$N = \frac{mn}{16 \ln n}. \tag{1}$$

We select each $i \in [1..N]$ in $\mathcal{H}$ with probability

$$p = \frac{n}{2N} = \frac{8 \ln n}{m}. \tag{2}$$

Since $n \leq 2^{m/16}$, $p \leq 1/2$. Let $x_1 < \cdots < x_u$ be the set of selected numbers. Using Chernoff's bound [Alon and Spencer 1992], one can argue that the probability of $u > n$ is at most $1/4$. We set $\mathcal{H} = \{(x_1, 1), \ldots, (x_u, 1)\}$, and we set o_i to be the interval $((x_i, 1), (x_{i+1}, 1))$. We include two additional occluders, $o_0 = ((-\infty, 1), (x_1, 1))$ and $o_u = ((x_u, 1), (\infty, 1))$.

Finally, for $j \geq 0$, let $\sigma_j = (2j, 2) \in \Sigma$ be the point sources whose signatures we consider. We prove that for $0 \leq a < b \leq N - m$, the probability of $\chi(\sigma_a) = \chi(\sigma_b)$ is very small. This will prove the existence of a set of $n + 2$ occluders so that each $\chi(\sigma_i)$ is distinct for $j \leq N - m$, which in turn will imply the lower bound. We begin by introducing the following definition.

For a pair of indices $0 \leq a < b \leq N$ and for another pair of indices $0 \leq i < j < m$, we say that the pair (i, j) is *independent with respect to* (a, b) if $\{i + a, j + a\} \cap \{i + b, j + b\} = \emptyset$, which is equivalent to saying that

$$j - i \neq b - a. \tag{3}$$

That is, the intersection points of line ℓ_1 (which contains $\mathcal{O}$) with the segments $s_i\sigma_a, s_j\sigma_a, s_i\sigma_b$, and $s_j\sigma_b$ are all distinct; see Figure 3.

LEMMA 3.1. *For any $0 \leq a < b < N - m$, there exists a subset $\mathcal{I} \subseteq [0..m - 1]$ of size at least $m/2$ so that every pair of indices in $\mathcal{I}$ is independent with respect to the pair (a, b).*

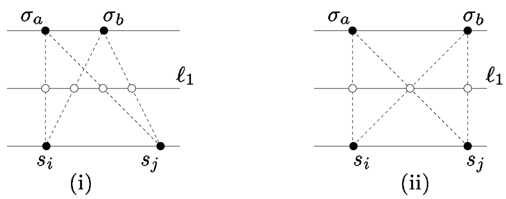

Fig. 3. (i) A pair of independent integers with respect to (a, b); (ii) (i, j) is not an independent pair with respect to (a, b).

PROOF. By (3), for a fixed pair (a, b), each integer $0 \le i < m$ depends (with respect to (a, b)) only on one integer larger than i, namely $i + (b - a)$. We choose

$$\mathcal{I} = \left\{2i(b-a) + r \mid 0 \le i \le \lfloor m/2(b-a) \rfloor, 0 \le r < b - a\right\}.$$

By construction, $|\mathcal{I}| \ge m/2$. Moreover, if $k = 2i(b-a) + r \in \mathcal{I}$, then $k + (b - a) = (2i + 1)(b - a) + r \notin \mathcal{I}$. Hence, every pair of integers in $\mathcal{I}$ is independent with respect to the pair (a, b). □

We now prove the main lemma of this section, which bounds the probability of $\chi(\sigma_a)$ being equal to $\chi(\sigma_b)$.

LEMMA 3.2. *For* $0 \le a < b < N - m$,

$$\Pr[\chi(\sigma_a) = \chi(\sigma_b)] \le \frac{1}{2N^2}.$$

PROOF. Let $\mathcal{I} \subseteq [0..m-1]$ be a subset of pairwise independent indices (with respect to the pair (a, b)) of size at least $m/2$ (Lemma 3.1). Then

$$\begin{aligned}\Pr[\chi(\sigma_a) = \chi(\sigma_b)] &= \Pr\left[\bigcap_{j=1}^{m} \chi_j(\sigma_a) = \chi_j(\sigma_b)\right] \\ &\le \Pr\left[\bigcap_{j \in \mathcal{I}} \chi_j(\sigma_a) = \chi_j(\sigma_b)\right] \\ &= \prod_{j \in \mathcal{I}} \Pr[\chi_j(\sigma_a) = \chi_j(\sigma_b)].\end{aligned}$$

The last equality follows from the fact that every pair of indices in $\mathcal{I}$ is independent with respect to the pair (a, b), thereby implying that the values of $\chi_i(\sigma_a)$, $\chi_i(\sigma_b)$, $\chi_j(\sigma_a)$, $\chi_j(\sigma_b)$ are all independent. Hence,

$$\Pr[\chi_j(\sigma_a) = \chi_j(\sigma_b)] = p^2 + (1-p)^2 = 1 - 2p(1-p) \le 1 - p$$

because $p \le 1/2$. Therefore,

$$\begin{aligned}\Pr[\chi(\sigma_a) = \chi(\sigma_b)] &\le (1-p)^{m/2} \le \exp\left(\frac{-pm}{2}\right) \\ &\le \exp(-4 \ln n) < \frac{1}{n^4} \le \frac{1}{N^2}.\end{aligned}$$

The last inequality follows from (1) and the assumption that $m \le n$. This completes the proof of the lemma. □

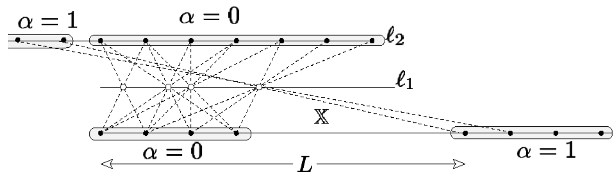

Fig. 4. Handling small values of n.

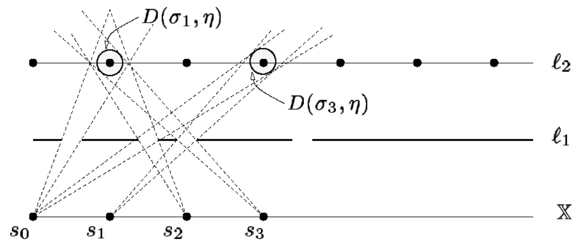

Fig. 5. Shrinking each occluder o_i by δ. All points in $D(\sigma_1, \eta)$ have the same signature as σ_1.

The above lemma implies that

$$\Pr[\exists a \neq b \ \chi(\sigma_a) = \chi(\sigma_b)] \leq \sum_{0 \leq i < j \leq N-m} \Pr[\chi(\sigma_a) = \chi(\sigma_b)] \leq \frac{N(N-1)}{2} \cdot \frac{1}{N^2} \leq \frac{1}{2}.$$

Since $\Pr[|\mathcal{H}| > n] < 1/2$, the probability of $|\mathcal{H}| \leq n$ and $\chi(\sigma_i) \neq \chi(\sigma_j)$, for all $0 \leq i < j \leq N$, is at least 1/4. Hence, there exists a set $\mathcal{O}$ of $n+2 \geq m$ occluders so that $\chi(\sigma_i)$ is unique for $0 \leq i \leq N-m$, thereby implying that $\pi(\mathcal{S}, \mathcal{O}, \ell_2) = \Omega(mn/\log(n))$.

Next, we extend the proof to the case $n < m$. For simplicity, we assume that m is divisible by n. we construct $\mathcal{O} = \{o_0, \ldots, o_{n-1}\}$ as above, with $m = n$, so that $\sigma_0, \ldots, \sigma_u$, for $u = \Omega(n^2/\log(n))$, have distinct signatures. Let $\chi(\sigma_i) = \omega_i$. We choose a sufficiently large even integer $L \gg n^2$ so that the intersection point of ℓ_2 with the line passing through $(L, 0)$ and the left endpoint of o_{n-1}, lies to the left of σ_0. We set $\mathcal{S} = \{(\alpha L + 2i, 0) \mid 0 \leq i \leq n-1, 0 \leq \alpha < m/n\}$, that is, we make m/n copies of n sensors, separated by a large interval L. We now consider the points $\sigma_{-\alpha L + a}$, for $0 \leq a \leq u$ and $0 \leq \alpha \leq m/n$. We divide a signature into m/n blocks. The bits in the αth block of $\chi(\sigma_{-\alpha L+a})$ are the same as that of ω_a and the remaining bits are 0. Hence, we get $\Omega(mn/\log(n))$ distinct signatures. We have thus proved the following:

LEMMA 3.3. *Let $m, n \geq 1$ so that $n \leq 2^{m/16}$. Set $\mathcal{S} = \{(2i, 0) \mid 0 \leq i < m\}$ and $\sigma_i = (2i, 2)$. There exists a set $\mathcal{O}$ of n occluders on the line $y = 1$ so that for $0 \leq i < j \leq cmn/\log(n)$, $\chi(\sigma_i) \neq \chi(\sigma_j)$, where $c > 0$ is a constant.*

It can be shown that even if we replace each point $(x_i, 1) \in \mathcal{H}$ by an interval $[(x_i - \delta, 1), (x_i + \delta, 1)]$, for some $\delta < 1/10$ (i.e., o_i now becomes $[(x_i + \delta, 1), (x_{i+1} - \delta, 1)]$), $\chi(\sigma_a)$ does not change. In fact, there exists a real value $\eta = \eta(\delta) > 0$ so that for all points $p \in D(\sigma_i, \eta)$, the disk of radius η centered at σ_i, $\chi(p) = \chi(\sigma_i)$; see Figure 5. Hence, we can shrink each o_i by δ at both of its endpoints without changing $\Pi(\mathcal{S}, \mathcal{O}, \ell_2)$.

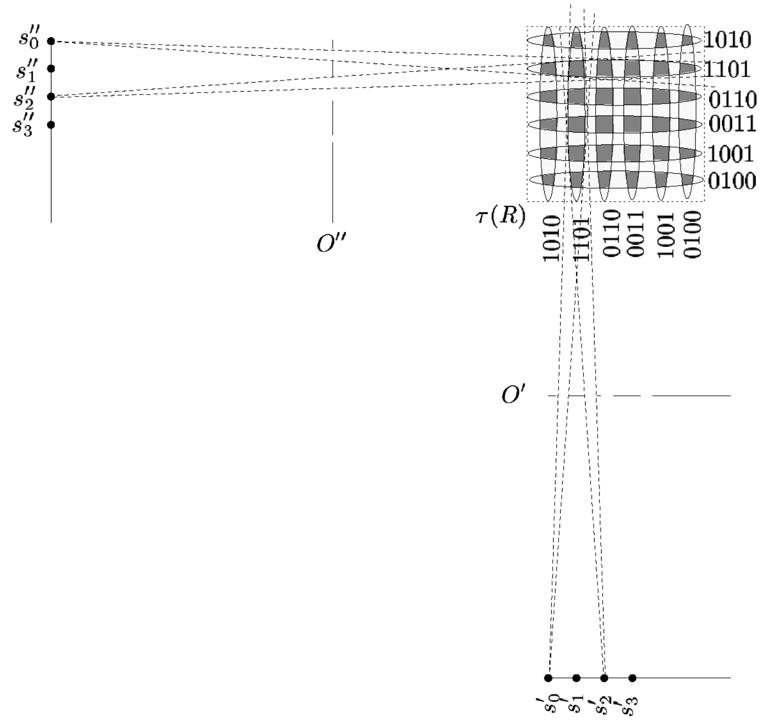

Fig. 6. Lower-bound construction in 2D. The signature of each dark region is distinct.

Putting everything together, we obtain the following.

LEMMA 3.4. *Let $m, n \geq 1$ so that $n \leq 2^{m/16}$. Set $\mathcal{S} = \{(2i, 0) \mid 0 \leq i < m\}$ and $\sigma_i = (2i, 2)$. There exist a set $\mathcal{O}$ of n occluders on the line $y = 1$, each an interval, and a real value $\eta > 0$ so that for $0 \leq i < j \leq u = \Omega(mn/\log(n))$, (i) $\chi(\sigma_i) \neq \chi(\sigma_j)$ and (ii) all points in $D(\sigma_i, \eta)$ have the same signature.*

3.2 Higher Dimensional Object Space

We now show how to extend the 1D construction in Lemma 3.4 to 2D to obtain $\Omega((mn/\log(mn))^2)$ distinct signatures. A similar approach will extend to $d > 2$ as well.

Let $\mathcal{S}$, $\mathcal{O}$, η, and u be the same as in Lemma 3.4. Let $\omega_i = \chi(\sigma_i)$. Then $\omega_i \neq \omega_j$, for all $1 \leq i \neq j \leq u$, and $\chi(p) = \chi(\sigma_i)$ for all points $p \in D(\sigma_i, \eta)$. Let R be the smallest axis-parallel rectangle that contains $D(\sigma_0, \eta), \ldots, D(\sigma_u, \eta)$. Let λ be the length of R, and let τ be the geometric transform (translation in y-direction followed by scaling in y-direction) that maps R to the square $[0, \lambda]^2$. Let $\mathcal{S}' = \{\tau(s_i) \mid 0 \leq i < m\}$ and $\mathcal{O}' = \{\tau(o_j) \mid 1 \leq j \leq n\}$ be the sets of sensors and occluders in the transformed space, and let $E'_i = \tau(D(\sigma_i, \eta))$; E'_i is an ellipse. See Figure 6.

Next, let ϱ be the reflection transform with respect to the line $y = x$, that is, ϱ maps a point (a, b) to the point (b, a). Set $\mathcal{S}'' = \varrho(\mathcal{S}')$, $\mathcal{O}'' = \varrho(\mathcal{O}')$, and $E''_i = \varrho(E'_i)$. Finally, let $\mathcal{S} = \mathcal{S}' \cup \mathcal{S}''$ and $\mathcal{O} = \mathcal{O}' \cup \mathcal{O}''$. By construction, for any point $\xi_{ij} \in E'_i \cap E''_j$, $\chi(\xi_{ij}) = \omega_i \circ \omega_j$, the concatenation of ω_i and ω_j. Since $\omega_i \neq \omega_j$

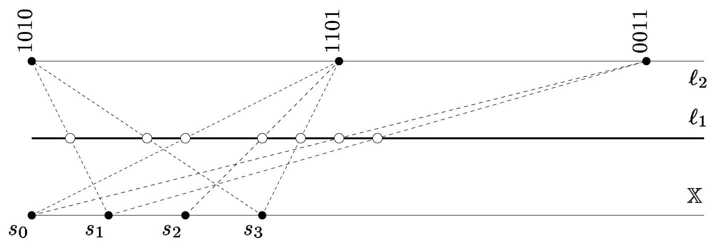

Fig. 7. Realizing a family of signatures.

for $i \neq j$, $\chi(\xi_{ij})$ is distinct for all pairs $1 \leq i, j \leq u$. Consequently,

$$\pi(\mathcal{S}, \mathcal{O}, \mathbb{R}^2) \geq u^2 = \Omega\left(\frac{m^2 n^2}{\log^2 n}\right),$$

thereby proving Theorem 2.1 for $d = 2$. The bound for $d > 2$ can be proved in a similar manner.

4. UPPER BOUNDS

We now prove that the lower bound in the previous section is almost tight. For a pair $s_i \in \mathcal{S}$ and $o_j \in \mathcal{O}$, let C_{ij} denote the cone formed by the set of rays that emanate from s_i and pass through o_j. Let F_{ij} be the *frustum* formed by the closure of the portion of C_{ij} that does not lie in the convex hull of $s_i \cup o_j$, that is,

$$F_{ij} = \text{cl}(C_{ij} \setminus \text{conv}(s_i \cup o_j)).$$

See Figure 1. Note that $\chi_i(p) = 0$, for a point $p \in \Sigma$, if and only if $p \in \bigcup_{i=1}^{n} F_{ij} \cap \Sigma$. Let $\Gamma = \{\gamma_{ij} = F_{ij} \cap \Sigma \mid 1 \leq i \leq m, 1 \leq j \leq n\}$.[2] Since every occluder in $\mathcal{O}$ has constant description complexity, each γ_{ij} is a region of constant description complexity.

We define the *arrangement* of $\mathcal{A}(\Gamma)$ to be the decomposition of Σ into maximal connected regions, called *cells*, so that all points within the same region lie in the same subset of Γ. By a well-known result [Agarwal and Sharir 2000], $\mathcal{A}(\Gamma)$ has $O((mn)^d)$ cells. Moreover, for any pair of points p, q lying in the same cell of $\mathcal{A}(\Gamma)$, $\chi(p) = \chi(q)$. Hence, $\pi(\mathcal{S}, \mathcal{O}, \Sigma)$, the number of signatures induced by $\mathcal{S}$ and $\mathcal{O}$ on Σ, is $O((mn)^d)$. This completes the proof of Theorem 2.2.

Remark. If $\mathbb{P} = \mathbb{R}^2$, then the boundary of C_{ij} consists of two rays, each emanating from s_i, so the bound on $\pi_d(m, n)$ does not depend on the complexity of each occluder and we do not have to assume them to be of constant description complexity. However if $\mathbb{P} = \mathbb{R}^3$, then ∂C_{ij} depends on the complexity o_j, and we need the constant-description-complexity assumption.

5. REALIZING A FAMILY OF SIGNATURES

We now prove Theorem 2.3. Let $\Pi = \{\pi_0, \ldots, \pi_{N-1}\}$ be a set of signatures, where $\pi_i \in \{0, 1\}^m$. We place a family $\mathcal{S} = \{s_0, \ldots, s_{m-1}\}$ of m sensors on the x

[2]We are assuming that the object p does not lie inside any occluder, so we do not care about $\chi_i(p)$ for $p \in o_j$.

axis, where $s_i = (2i, 0)$. The object space is the line $\ell_2 : x = 2$. For $0 \le j \le N-1$, let $\sigma_j = (2jm, 2)$. We will place a set of occluders on the line $\ell_1 : y = 1$ so that $\chi(\sigma_j) = \pi_j$ for $0 \le j < N$. As in the previous section, we construct degenerate occluders so that $\ell_1 \setminus \bigcup \mathcal{O}$ is a set $\mathcal{H} = \{h_0, \ldots, h_{n-1}\}$ of points. We partition $\mathcal{H}$ into N blocks $\mathcal{H}_0, \ldots, \mathcal{H}_{N-1}$, where $\mathcal{H}_j \subset [jm..(j+1)m-1]$. We add the point $(jm+i, 1) \in \mathcal{H}_j$ if $\chi_i(\sigma_j) = 1$, so that σ_j is visible from s_i. By construction, s_i is visible at σ_j if and only if $(jm+i, 1) \in \mathcal{H}_j$. Hence, $\chi(\sigma_j) = \pi_j$. For any point $p \in \ell_2 \setminus \bigcup \sigma_j$, $\chi(p) = 0^m$. This completes the proof of Theorem 2.3.

6. CONCLUSIONS

We proved almost tight bounds on the number of distinct signatures that can be realized by a simple radiation field sensor system that uses reference structure to modulate the radiation field. Our result illustrates the limits on the complexity of signature fields induced by reference structures and binary sensors. Visibility-based sensor systems using the model described here are under development at Duke for human biometric and tracking systems. These systems use differential pyroelectric sensor networks as described in Gopinathan et al. [2003]. Current versions use adaptive optics to shape the sensor visibility, rather than occluders, and use differential sensor pairs rather than sensors that attend to a fixed point in Σ. Despite these differences, the conceptual framework laid in this article for analysis of signature fields is useful in the system design. We plan to extend our framework and to integrate the physical design constraints more closely in our model.

This article raises many more questions than it answers. Not only the number of signatures, but the size and shape distribution of signature cells are equally important, as the goal is to locate and track an object using the signature field. How accurately can one locate or track an object using m sensors and n occluders remains an open question. In practice, one has to assume a lower bound on the size of each occluder and an upper bound on the area in which the occluders are placed. How does the bound on the number of signatures depend on these parameters? For example, assuming that each occluder is a unit disk and that they are packed inside a disk of radius r, how many distinct signatures can one obtain? If other optical elements, such as reflectors or refractors, or multiple object sources are allowed, the problem of bounding the number of signatures becomes significantly harder and no good bounds are known. Our main goal in this area is to design an optimal reference structure, a problem that remains by and large completely open.

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