# Line Stabbing Bounds in Three Dimensions*

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## Abstract

Let $S$ be a set of (possibly degenerate) triangles in $\Re^3$ whose interiors are disjoint. A triangulation of $\Re^3$ with respect to $S$, denoted by $T(S)$, is a simplicial complex in which the interior of no tetrahedron intersects any triangle of $S$. The line stabbing number of $T(S)$ is the maximum number of tetrahedra of $T(S)$ intersected by a segment that does not intersect any triangle of $S$. We investigate the line stabbing number of triangulations in several cases—when $S$ is a set of points, when the triangles of $S$ form the boundary of a convex or a nonconvex polyhedron, or when the triangles of $S$ form the boundaries of $k$ disjoint convex polyhedra. We prove almost tight worst-case upper and lower bounds on line stabbing numbers for these cases. We also estimate the number of tetrahedra necessary to guarantee low stabbing number.

## 1 Introduction

### 1.1 Problem Statement

A *triangulation* of $\Re^3$ is a simplicial complex that covers the entire 3-space.[1] Let $S$ be a simplicial complex consisting of $n$ triangles, segments, and/or vertices in $\Re^3$. A *constrained triangulation* of $\Re^3$ with respect to $S$ is a triangulation $T(S)$ of $\Re^3$ in which the interior of any $k$-face ($k = 0, 1, 2$) of $T(S)$ is either disjoint from $S$ or contained in the interior of a $k'$-face of $S$, with $k' \geq k$. For the sake of brevity, we will refer to $T(S)$ as a *triangulation with respect to* $S$. A vertex of $T(S)$ is called a *Steiner point*, if it is not a vertex of $S$.

For a segment $\gamma$ and a triangulation $T$ with respect to $S$, let $\sigma(S, T, \gamma)$ denote the number of simplices of $T$ that $\gamma$ intersects. Define $\sigma(T, S) = \max \sigma(S, T, \gamma)$, where maximum is taken over all segments $\gamma$ that do not intersect $S$; $\sigma(T, S)$ is called the *line stabbing number* of $T(S)$. Finally, define $\sigma(S) = \min \sigma(S, T)$, where minimum is taken over all triangulations with respect to $S$; $\sigma(S)$ is called the *minimum stabbing number* of $S$. In this paper we prove upper and lower bounds on the minimum stabbing numbers of triangulations with respect to $S$ in several cases, including when $S$ is a set of points in $\Re^3$, when the triangles of $S$ form the boundary of a convex or a nonconvex polyhedron, and when the triangles of $S$ form the boundaries of $k$ disjoint convex polyhedra. By an $f(n)$ lower bound, we mean an explicit example of a set of $n$ objects *any* triangulation of which has stabbing number at least $f(n)$. A $g(n)$ upper bound is established by presenting an algorithm that gives a triangulation for any set of points or polyhedra such that the stabbing number is at most $g(n)$. For the lower bounds on stabbing numbers we do not make any assumptions on the size of the triangulation, but for the upper bounds we prove that the size of the triangulation produced by the algorithm is not very large.

### 1.2 Motivation and Previous Results

The line stabbing problem has relevance to several application areas including computer graphics and motion simulation. In computer graphics, for instance, realistic image-rendering methods use the ray tracing technique to compute light intensity at various parts of scene. Given a three dimension scene, mod-

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[1] For the purposes of this paper, a *simplicial complex* is a collection of (possibly unbounded) tetrahedra, triangles, edges, and vertices, in which any two objects are either disjoint or meet along common face (vertex, edge, or triangle).

eled by polyhedral objects, a triangulation having a low line stabbing number could act as a simple, yet efficient, data structure for ray tracing—we just walk through the triangulation complex, visiting only the tetrahedra that are intersected by the directed line that represents the query ray. Indeed, the best data structures for the ray tracing problem in a two dimensional scene utilize exactly this methodology: Hershberger and Suri [13] give a triangulation-based method with the best theoretical performance, while Mitchell, Mount and Suri [15] propose a quad-tree based method with a more practical bent.

Other potential applications of triangulation with low stabbing number are in computer simulations of fluid dynamics or complex motion. Simulating a motion in a complicated and obstacle-filled environment requires visibility-checks to detect collisions. These checks can be made using a ray shooting method: find the first intersection point between a ray and the set of obstacles. Although the motion of the flying object (e.g., a camera) in general may be non-linear, one can approximate it using piecewise linear curves. In simulating fluid dynamics, the finite element method is one of the most popular method: it subdivides the domain into quad-cells; flow values are "measured" at cell vertices, and interpolated at all other points. In some applications, it is desirable to "integrate" the flow along a line. This involves computation over all those cells that are intersected by the line, and a triangulation (or quad cell partitions) that minimizes the number of intersected cells is of obvious interest.

Because of the wide-spread use of triangulations in mesh generation, surface reconstruction, robotics, and computer graphics, the topic of triangulation has attracted a lot of attention within computational geometry. Most of the work to date has, however, concentrated on computing either an arbitrary triangulation or a triangulation that optimizes certain parameters related to shapes of triangles, e.g., angles, edge lengths, heights, volumes. See the survey paper by Bern and Eppstein [3] (and the references therein) for a summary of known results on triangulations. We are not aware of any paper that studies the line stabbing number of 3-dimensional triangulations.

Chazelle and Welzl [9] studied the hyperplane stabbing number of spanning trees of point sets. Specifically, given a set $S$ of $n$ points in $\Re^d$ and a spanning tree $T$ of $S$, the *hyperplane stabbing number* of $T$ is the maximum number of edges of $T$ crossed by a hyperplane. The hyperplane stabbing number of $S$ is the minimum hyperplane stabbing number over all spanning trees of $S$. They proved that the hyperplane stabbing number of $n$ points in $\Re^d$ is $\Theta(n^{1-\frac{1}{d}})$ in the worst case; see also [18]. An analogous result for matchings is also discussed in [9], and a related problem is considered in [1]. Modifying the argument of Chazelle and Welzl, we can prove a lower bound of $\Omega(\sqrt{n})$ on the line stabbing number of a triangulation of the plane with respect to a set of $n$ points. Hershberger and Suri [13] showed that a simple polygon can be triangulated into linear number of triangles, so that the stabbing number of the triangulation is $O(\log n)$. Combining their result with that of Chazelle and Welzl, we can compute a linear-size triangulation of the plane with respect to a set $S$ of $n$ disjoint segments, whose stabbing number is $O(\sqrt{n}\log n)$. If $S$ forms the edges of $k$ disjoint convex polygons, the stabbing number can be improved to $O(\sqrt{k}\log n)$. A result of Dobkin and Kirkpatrick [10, 11] implies that the interior of a convex polyhedron with $n$ vertices can be triangulated with $O(n)$ tetrahedra, whose line stabbing number is $O(\log n)$. See also [6] for some related results.

### 1.3 Summary of Results

Throughout the paper, we assume that the set of points and polyhedra are in general position, namely, no four points are coplanar. This assumption is critical, as discussed below. In our case, the lower bound arguments would often become trivial if degenerate point sets were allowed. In the problems we consider, we triangulate the entire affine 3-space. We define a tetrahedron as the intersection of at most 4 half-spaces, allowing for unbounded intersections.

We first discuss some bounds on stabbing numbers for the case when $S$ forms the boundary of a convex polytope; these bounds follow easily from hierarchical representations of convex polyhedra (Section 2).

Secondly, if $S$ is a set of $n$ points in $\Re^3$, we show that the line stabbing number for a non-Steiner triangulation is $\Theta(n)$. On the other hand, if we allow Steiner points, the lower bound on the stabbing number is $\Omega(\sqrt{n})$ even if the points are in general position. Notice that if points are in degenerate position—all of them are coplanar, the $\Omega(\sqrt{n})$ lower bound follows immediately from the known two-dimensional results. We describe an algorithm for computing a triangulation with respect to $S$ of $O(n)$ size and $O(\sqrt{n}\log n)$ stabbing number.

Next, we consider the case when $S$ forms the boundary of a nonconvex polyhedron. It is known that there are nonconvex polyhedra that cannot be triangulated without using Steiner points [16], and that it is NP-hard to determine whether a polyhedron can be triangulated without adding Steiner points [17]. Therefore we consider only Steiner triangulations in this case. We prove an $\Omega(n)$ lower bound on the stabbing number, and show how to construct a triangulation of $O(n^2)$ size and $O(n\log n)$ stabbing number. A result of Chazelle [5] shows that the size of the triangulation cannot be improved in the worst case. If $S$ has few reflex edges, the bound can be improved, but we omit this improvement from the abstract.

Finally, we consider the case when $S$ forms the faces of $k$ disjoint convex polyhedra. The lower bound result for the general polytope can be modified to obtain an $O(k+\log n)$ lower bound on the stabbing number. We describe an algorithm that computes a triangulation of size $O(nk \log n)$ and $O(k \log^2 n)$ stabbing number. If $k < \sqrt{n}$, the size can be improved: we can compute a triangulation of $O(n \log n + k^3 \log k)$ size and $O(k \log^2 n)$ stabbing number, or of $O(n+k^3 \log k)$ size and $O(k \log^3 n)$ stabbing number.

## 2 Preliminaries

In this section, we discuss some conventions used in our triangulations and review a method for triangulating a convex polytope or its exterior using polyhedral hierarchies to achieve logarithmic line stabbing number.

Given a convex polytope $P$ of $n$ vertices, the *inner hierarchy* of $P$ is a sequence of convex polytopes $P = P_1, P_2, \ldots, P_k$ where $P_{i+1}$ is obtained by removing a constant fraction of constant degree vertices of $P_i$ and taking the convex hull of the remaining vertices. Similarly, the *outer hierarchy* of $P$ is a sequence of convex polytopes $P = P_1, P_2, \ldots, P_k$, where $P_{i+1} \supseteq P_i$ is obtained from $P_i$ by removing a constant fractions of faces, each of constant size, and taking the intersection of the halfplanes defined by remaining faces. We can use the inner and outer hierarchies, respectively, to triangulate the interior and exterior of $P$, as explained below.

First, consider triangulating the interior of a convex polytope using the inner hierarchy. In going from $P_i$ to $P_{i+1}$, we remove an independent set of polyhedral caps of $P_i$, each of constant size; given a convex polytope $P$ and a vertex $q$, the cap of $q$ with respect to $P$ is the closure of $P \setminus CH(V(P) \setminus \{q\})$. We can triangulate each of these caps with $O(1)$ tetrahedra without using any Steiner points. The union of all these triangulations is a non-Steiner triangulation of $P$. We show that it has line stabbing number $O(\log n)$.

Since each of the polytopes $P_i$, $i = 1, 2, \ldots, k$, is a convex polytope, a line $\ell$ intersects it in at most two points. Since the caps at each level $i$ belong to independent vertices, the line $\ell$ intersects at most two caps between $P_i$ and $P_{i+1}$. Each cap has only a constant number of tetrahedra. Since there are $O(\log n)$ levels in the hierarchy, the total number of tetrahedra intersected by $\ell$ is $O(\log n)$. The lower bound can be proved by a direct decision-tree argument. We omit the details. Hence, we obtain Add details...

**Lemma 2.1 ([11])** *The interior of a convex polytope of $n$ vertices can be triangulated into $O(n)$ tetrahedra, without using Steiner points, so that the line stabbing number of the triangulation is $O(\log n)$. Moreover, this bound is tight in the worst case.*

The outer hierarchy of $P$ can be used to triangulate the exterior of $P$. The space between $P_i$ and $P_{i+1}$ can be filled with $O(|P_i| - |P_{i+1}|)$ tetrahedra, where $|P|$ denotes the number of edges, vertices, and faces of $P$. These families of tetrahedra, however, use Steiner points as their vertices, and do not necessarily form a simplicial complex. Some vertices of $P_i$ lie in the interiors of faces of $P_{i+1}$. We can convert this convex decomposition easily into a simplicial complex by turning all faces that have interior points on one of their sides into "thin" convex polytopes and triangulating their interiors using the inner hierarchy. Line stabbing number of this triangulation is $O(\log^2 n)$: any line meets $O(\log n)$ levels of the hierarchy and thus $O(\log n)$ cells of the unrefined decomposition; the second logarithmic factor comes from the inner hierarchies introduced to make the decomposition a triangulation. We omit the details, and summarize the result in the following lemma.

**Lemma 2.2** *The exterior of a convex polytope with $n$ vertices can be triangulated into $O(n)$ tetrahedra, using Steiner points, so that the line stabbing number of the triangulation is $O(\log^2 n)$.*

Using inner and outer hierarchies, Chazelle and Shouraboura [8] showed that, given two convex polytopes $P$ and $Q$, with $P \subseteq Q$ and $|P| + |Q| = n$, the region between $P$ and $Q$ can be triangulated into $O(n)$ simplices. Their procedure can be modified to obtain a linear size triangulation of $Q \setminus P$ whose line stabbing number is $O(\log^3 n)$. On the other hand, by modifying the algorithm due to Bern [4] for triangulating $Q \setminus P$, we can compute a triangulation of $Q \setminus P$ of $O(n \log n)$ size and $O(\log^2 n)$ stabbing number. Omitting all the details, we conclude

**Lemma 2.3** *Given two convex polytopes $P, Q$, with $P \subseteq Q$ and $|P|+|Q| = n$, the region between $P$ and $Q$ can be triangulated into $O(n)$ (resp. $O(n \log n)$) tetrahedra, so that the line stabbing number of the triangulation is $O(\log^3 n)$ (resp. $O(\log^2 n)$).*

## 3 Triangulations for Point Sets

In this section we assume that $S$ is a set of $n$ points in $\Re^3$. We prove sharp bounds on the minimum line stabbing number of $S$ for both Steiner and non-Steiner triangulations. The most interesting result of this section is the $\Omega(\sqrt{n})$ lower bound on the line stabbing number of any Steiner triangulation with respect to a set of $n$ points in $\Re^3$.

## 3.1 Non-Steiner Triangulation

In this subsection we consider non-Steiner triangulations of $S$. For the sake of simplicity, we assume that we want to triangulate the convex hull of $S$. The bounds hold even if we want to triangulate the entire space.

It is easily seen that the convex hull of any set of $n$ points $S$ in $\Re^3$ in general position can be triangulated using $O(n)$ tetrahedra [12]. Describe it for the entire space... A simple algorithm works as follows. Compute the convex hull of $S$. Triangulate the boundary of the convex hull. Pick the bottommost point $b$ of $S$, and join $b$ to each triangle on the convex hull boundary that is not already incident to $b$. This introduces $O(n)$ tetrahedra that together partition the space inside the convex hull. Now, consider the remaining points of $S$ one at a time: each such point lies in an existing tetrahedron; decompose that tetrahedron into four tetrahedra by joining the new points to the four facets of the containing tetrahedron.

As for the lower bound, let $p_1, \ldots, p_{n-2}$ be a set of $n-2$ points lying on the curve $y = 0$, $z = x^2 + 1$. We perturb these points slightly so that the set is no longer degenerate, but the plane determined by any triple still has the points $q_1 = (0, -1, 0)$ and $q_2 = (0, +1, 0)$ on opposite sides. See Figure 1. We prove that any non-Steiner triangulation with respect to the point set $S = \{p_1, \ldots, p_{n-2}, q_1, q_2\}$ has line stabbing number $\Omega(n)$.

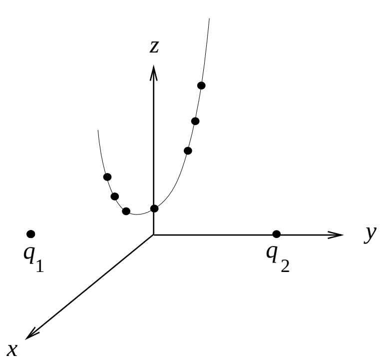



Figure 1: The lower bound for non-Steiner triangulation.

Let $T$ be such a triangulation of $S$. Consider any point $p_i$, for $1 < i < n-2$, and let $p_i^-$ denote the point obtained by translating $p_i$ downward (in the negative $z$-direction) by a suitably small distance. The point $p_i^-$ lies in the convex hull of $S$ and therefore in some tetrahedron of $T$. We claim that $p_i^-$ lies in a tetrahedron $\Delta_i \in T$ whose four vertices are $p_i, p_j, q_1, q_2$, for some $j \neq i$.

To prove the claim, observe that any tetrahedron touching $p_i$ must have $p_i$ as a vertex, as $T$ is a simplicial complex. By construction, $p_i^-$ lies below any tetrahedron defined by four vertices from the set $S \setminus \{q_1, q_2\}$. Finally, $p_i^-$ also lies outside any tetrahedron defined by $\{p_j, p_k, p_\ell, q_1\}$ or $\{p_j, p_k, p_\ell, q_2\}$. Thus, the tetrahedron containing $p_i^-$ is defined by $p_i, q_1, q_2$ and a fourth vertex $p_j$, for $j \neq i$. Repeating this argument for all $i$, where $1 < i < n-2$, we conclude that at least $(n-4)/2$ tetrahedra of $T$ are determined by quadruples of the form $\{q_1, q_2, p_i, p_j\}$. All of these tetrahedra can be stabbed by a line $\ell : y = 0, z = \varepsilon$, for a suitably small $\varepsilon > 0$. Thus, the stabbing number of $T$ is $\Omega(n)$, establishing the following theorem.

**Theorem 3.1** *There exists a set of $n$ points in $\Re^3$ whose minimum stabbing number is $\Omega(n)$. Moreover, for every set of $n$ points in $\Re^3$, there exists a triangulation whose line stabbing number is $O(n)$. If $S$ is in general position, the size of the triangulation is linear. Otherwise, the size is $O(n^2)$ in the worst case.*

## 3.2 Steiner Triangulation

In this subsection, we show that by allowing Steiner points the upper bound on line stabbing number can be reduced to $O(\sqrt{n} \log n)$. We also prove a lower bound of $\Omega(\sqrt{n})$. As mentioned in the introduction, the line stabbing number for two-dimensional point sets is $\Omega(\sqrt{n})$ in the worst case [9]. Then, why doesn't it imply a similar lower bound in three dimensions? This is where the non-degeneracy assumption of no four points being coplanar is crucial. It seems difficult to make lower-bound arguments for a two-dimensional construction *perturbed* to remove degeneracies, since we have no control over the positions of Steiner points. Actually, since a line has co-dimension 2 in 3-space and 1 in the plane, one might think that the line stabbing number should *decrease* as one increases the dimension of the ambient space. The standard lower bound example in two dimensions for $\Omega(\sqrt{n})$ line stabbing number is a $\sqrt{n} \times \sqrt{n}$ grid. A 3-dimensional grid, however, yields a lower bound of only $\Omega(n^{1/3})$.

**The upper bound** Let $S \subseteq \Re^3$ be a set of $n$ points. Without loss of generality, assume that no two points have the same $x$ and $y$ coordinates; otherwise first perform a rotation of the space. Orthogonally project the points onto the plane $\pi : z = 0$. Let $\bar{S}$ denote the projection of $S$. Construct a Steiner triangulation $\bar{T}$ of $\pi$ with respect to $\bar{S}$ with line stabbing number

$O(\sqrt{n}\log n)$ [9, 13]. Lift this triangulation to 3-space: each point of $\bar{S}$ maps to its corresponding point in $S$, and we lift each Steiner point so that it lies in the convex hull of $S$. This results in a triangulated, piecewise linear, $xy$-monotone surface $\Sigma$ in 3-space, with $O(n)$ triangles. We add two more Steiner points at infinity, at $z = \infty$ and $z = -\infty$, and connect each triangle of $\Sigma$ to these points. This is a triangulation $T$ of $\Re^3$. In this triangulation, all tetrahedra are unbounded, and each tetrahedron $\tau \in T$ is uniquely associated with a triangle $t \in \bar{T}$: the triangle $t$ forms the base of $\tau$.

The line stabbing number of $T$ is $O(\sqrt{n}\log n)$, which can be proved as follows. Consider a line $\ell$, and let $H(\ell)$ denote the vertical plane passing through $\ell$. $H(\ell)$ intersects precisely those tetrahedra of $T$ whose associated (base) triangles are intersected by the line $H(\ell) \cap \pi$. Since the line stabbing number of $\bar{T}$ is $O(\sqrt{n}\log n)$, the number of tetrahedra intersected by $\ell$ is also $O(\sqrt{n}\log n)$.

**The lower bound** We exhibit a non-degenerate set of $n$ points $S \subseteq \Re^3$ every triangulation of which, with or without Steiner points, has line stabbing number $\Omega(\sqrt{n})$. The main idea of the lower bound is best demonstrated for a degenerate construction, described below. the construction can be modified to remove the degeneracies, as follows. Randomly perturb each point of S in x and y dimensions, so that its x- and y-coordinates lie in the $\varepsilon \times \varepsilon$ square around $(i, j)$. Now, instead of arguing about points of $\bar{S}$, argue about four corners of this box. Call a box boundary-box if at least one of its corners is outside $\Delta$—claim: only $O(|\mathcal{L}(\Delta)|)$ boundary boxes. For the others, define neighboring boxes (instead of neighboring points) and the proof goes through as before.

Consider the following set of $n$ points:

$$S = \{(i,\, j,\, ij \pm 1/2) \mid i, j = 1, \ldots, k\},$$

where we assume for simplicity that $n = 2k^2$ for an integer $k$. The set $S$ lies in two shifted copies of the hyperbolic paraboloid $\sigma:\ z = xy$. See Figure 2 for an illustration.

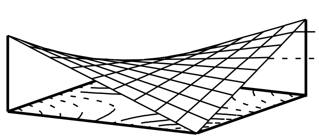


Figure 2: The surface $\sigma : z = xy$ and the lattice lines.

Our proof uses an auxiliary set of points $\bar{S} = \{(i,\, j,\, ij) \mid i, j = 1, \ldots, k\}$, which is the vertical projection of $S$ onto $\sigma$. The points of $\bar{S}$ are arranged in a $k \times k$ "grid" on $\sigma$. The lines of the grid have the form (1) $x = i, z = iy$ and (2) $y = i, z = ix$, for $i = 1, 2, \ldots, k$. Let $\mathcal{L}$ denote this set of "lattice" lines. $\bar{S}$ is exactly the set of pairwise intersections of lines of $\mathcal{L}$.

Let $T$ denote an arbitrary Steiner triangulation of the point set $S$. Consider the point $\bar{p} \in \bar{S}$ with coordinates $(i, j, ij)$. The point $\bar{p}$ has four primary neighbors (north, east, south, west) and four secondary neighbors (north-east, south-east, south-west, north-west). The *north neighbor* of $\bar{p}$ is denoted $\bar{p}_n$, and it has coordinates $(i, j+1, i(j+1))$, while the *north-east neighbor* is $\bar{p}_{ne} = (i+1, j+1, (i+1)(j+1))$. The remaining neighbors are defined similarly. The following lemma states a key property of our construction.

**Lemma 3.2** *Let $\Delta$ be a tetrahedron of $T$, and let $\bar{p} = (i, j, ij)$ be a point of $\bar{P}$ contained in $\Delta$. Then, the north-east and the south-west neighbors of $\bar{p}$ cannot both lie in $\Delta$.*

**Proof:** Our proof is by contradiction. Suppose that both $\bar{p}_{ne}, \bar{p}_{sw}$ lie in $\Delta$. Then, by the convexity of $\Delta$, the midpoint of the segment $\bar{p}_{ne}\bar{p}_{sw}$, namely, $q = (i, j, ij + 1)$, also lies in $\Delta$. Now, observe that $S$ contains a point $p = (i, j, ij + 1/2)$, which is the midpoint of the segment joining $\bar{p} = (i, j, ij)$ and $q = (i, j, ij + 1)$. Thus, $p$ must lie in the interior of $\Delta$ or in the relative interior of one of its faces. Either case contradicts the fact that $\Delta$ is a tetrahedron in a triangulation of $S$. □

We will show that one of the lattice lines (a line of $\mathcal{L}$) stabs $\Omega(\sqrt{n})$ tetrahedra of $T$. Our lower bound proof rests on the following lemma, which says that the "volume" of each tetrahedron in terms of the number of points of $\bar{P}$ contained in it is proportional to the number of lattice lines intersecting it. We introduce some notation to facilitate the proof. Given a tetrahedron $\Delta \in T$, let $\mathcal{L}(\Delta) = \mathcal{L} \cap \Delta$. Define $\bar{S}(\Delta) = \bar{S} \cap \Delta$, and $\mathrm{vol}(\Delta) = |\bar{S}(\Delta)|$. Since every point of $\bar{S}$ lies in some tetrahedron of $T$, we have

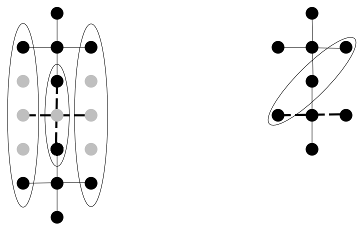
$$\sum_{\Delta \in T} \mathrm{vol}(\Delta) \;\geq\; k^2. \tag{1}$$

Finally, let $m(T)$ denote the stabbing number of $T$. Then,

$$m(T) \;\geq\; \frac{1}{|\mathcal{L}|}\sum_{\Delta \in T} |\mathcal{L}(\Delta)| \;=\; \frac{1}{2k}\sum_{\Delta \in T} |\mathcal{L}(\Delta)|. \tag{2}$$

**Lemma 3.3** *For any tetrahedron $\Delta \in T$, $\mathrm{vol}(\Delta) \leq 4|\mathcal{L}(\Delta)|$.*

**Proof:** By the convexity of $\Delta$, a lattice line $\ell$ intersects $\Delta$ in a line segment. In particular, the set $\bar{S} \cap \ell$,



(a) (b)

Figure 3: Illustration to the proof of Lemma 3.3.

if nonempty, consists of a block of consecutive lattice points. We call $\bar{p} \in \bar{S}(\Delta)$ a *boundary point* if at least one of its primary neighbors (north, east, south, or west) is outside $\Delta$. The number of boundary points is at most $2|\mathcal{L}(\Delta)|$: every line of $\mathcal{L}(\Delta)$ contains at most two of them, one where it enters $\Delta$ and one where it leaves.

We call the non-boundary points of $\bar{S}(\Delta)$ its *kernel points*. We claim that no lattice line has three or more kernel points in $\Delta$, which implies that the number of kernel points is also at most $2|\mathcal{L}(\Delta)|$.

We first argue that kernel points form blocks of consecutive lattice points along every lattice line. Indeed, without loss of generality, suppose that there are two kernel points $(i, j, ij)$ and $(i, j', ij')$ in $\Delta$, where $j' > j + 1$. Then, by convexity, the points $(i, j'', ij'')$ for $j < j'' < j'$, are also kernel points: this follows because the four primary neighbors of $(i, j'', ij'')$ lie on the lines $\{x = i - 1, z = (i - 1)y\}$, $\{x = i, z = iy\}$, and $\{x = i+1, z = (i+1)y\}$, between the corresponding neighbors of $(i, j, ij)$ and $(i, j', ij')$; see Figure 3(a). In particular, if the line $\{x = i, z = iy\}$ contains three or more kernel points, it must contain kernel points of the form $(i, j, ij)$, $(i, j + 1, i(j + 1))$, and $(i, j+2, i(j+2))$ for some $j$ (see Figure 3(b)). We now have the situation that both the north-east and south-west neighbors of $(i, j+1, i(j+1))$, namely, $(i - 1, j, (i - 1)j)$ and $(i + 1, j + 2, (i + 1)(j + 2))$ are contained in $\Delta$, which contradicts the claim in Lemma 3.2. This completes the proof of the lemma. □

By combining the inequality in the preceding lemma with Eqs. (1) and (2), we obtain:

$$m(T) \ \geq \ \frac{1}{2k} \sum_{\Delta \in T} |\mathcal{L}(\Delta)| \ \geq \ \frac{1}{8k} \sum_{\Delta \in T} \mathrm{vol}(\Delta) \ \geq \ k/8.$$

Since $k = \sqrt{n/2}$, the stabbing number of $T$ is $\Omega(\sqrt{n})$. Since $T$ was assumed to be an arbitrary triangulation of $S$, we have proved the following theorem.

**Theorem 3.4** *There exists a set of $n$ points in $\Re^3$, so that the line stabbing number of any triangulation with respect to $S$ is $\Omega(\sqrt{n})$. Moreover, for any set $S$ of $n$ points in $\Re^3$ there is a triangulation with respect to $S$ whose line stabbing number is $O(\sqrt{n} \log n)$.*

# 4 Triangulation for General Polyhedra

As mentioned in the introduction, a general polyhedron cannot always be triangulated without using Steiner points. A well-known example is the Schönhardt polytope [16] (see Figure 4). Thus it makes sense to triangulate the space with respect to a general polyhedron only with Steiner points.

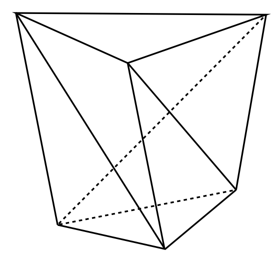


Figure 4: A polyhedron that cannot be triangulated without Steiner points.

**The upper bound** Every polyhedron with $n$ facets or, more generally, every collection of $n$ non-intersecting triangles in $\Re^3$ can be Steiner-triangulated so that the line stabbing number is $O(n \log n)$. Briefly, we construct a decomposition of space by drawing vertical planes through all edges, which partitions the space into convex, polyhedral cells; this step is borrowed from the triangulation algorithm of Chazelle and Palios [7]. An analogous construction for an arbitrary collection of possibly intersecting triangles is described by Aronov and Sharir [2]. Each cell is then triangulated using the inner polyhedral hierarchy (cf. Section 2). It can be shown that a line intersects $O(n)$ cells of the convex decomposition, and thus has stabbing number $O(n \log n)$.

**The lower bound** Our lower bound construction uses a version of the polyhedron $S$ introduced by Chazelle in [5]. See Figure 5. This polyhedron is obtained from the rectangular box of dimensions $[0, n+1] \times [0, n+1] \times [0, n^2+1]$ by removing two families of $n$ narrow notches. Notches of one family start at the bottom of the cube: the top edges of these notches lie along the lines $\{x = i, z = iy - 1/4\}$ for $i = 1, \ldots, n$.

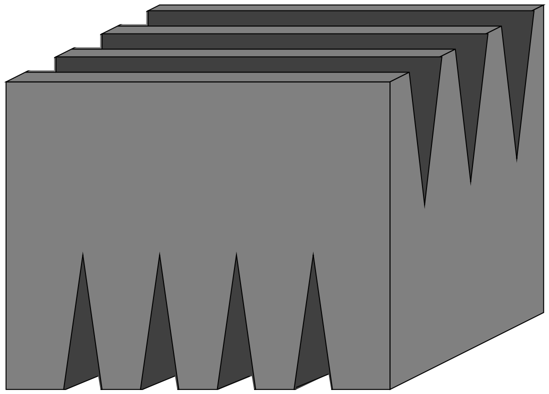


Figure 5: A non-convex polyhedron with $\Omega(n)$ line stabbing number.

The second starts from the top face and is orthogonal to the first family. The bottom edges of the top family of notches lie along lines $\{y = i, z = ix + 1/4\}$, for $i = 1, \ldots, n$. It is shown in [5] that any convex decomposition of this polyhedron requires $\Omega(n^2)$ polyhedra. We show that any triangulation of this polyhedron has line stabbing number $\Omega(n)$. (Again, we defer the details of removing degeneracies from this polyhedron to the full paper.)

Our proof is similar to the one used in the previous section for the case of point sets. We consider the set of points $\bar{S} = \{(i, j, ij) \mid i, j = 1, \ldots, n\}$ lying on the surface $\sigma : z = xy$. We then observe that there are $2n^2$ points $\{(i, j, ij \pm 1/2) \mid i, j = 1, \ldots, n\}$ that are outside the polyhedron and therefore cannot lie in a tetrahedron of any triangulation of $S$. The remainder of the proof is essentially unchanged, except that in the polyhedron case, the grid of points $\bar{S}$ has size $n \times n$, rather than $\sqrt{n/2} \times \sqrt{n/2}$ as in the case of points. Thus, the line stabbing number of any triangulation with respect to $S$ is $\Omega(n)$.

**Theorem 4.1** *There exists a set $S$ of $n$ disjoint triangles in $\Re^3$ forming the boundary of a polyhedron in $\Re^3$, such that any triangulation with respect to $S$ has line stabbing number $\Omega(n)$. Moreover, one can always triangulate the space with respect to a given set of $n$ disjoint triangles into $O(n^2)$ simplices so that the line stabbing number of the triangulation is $O(n \log n)$.*

**Remark.** If $S$ forms the boundary of a polyhedron with $r$ reflex edges, then there exists a triangulation of the *interior* of the polyhedron of $O(nr)$ size and $O(r \log n)$ stabbing number. Check! On the other hand if the triangles of $S$ intersect, there exists a $T(S)$ of $O(n^3)$ size and $O(n \log n)$ stabbing number.

# 5 Triangulation for Sets of Convex Polyhedra

We can interpolate the stabbing number bound from one end of the spectrum, $O(\log^2 n)$ when $S$ forms the boundary of one convex polytope, to the other end, $O(n \log n)$ when $S$ is a set of arbitrary disjoint triangles. Specifically, we show that, if $S$ forms the faces of $k$ disjoint convex polyhedra, we can construct a triangulation $T(S)$ of $O(nk \log n)$ size and $O(k \log^2 n)$ stabbing number. If $k < \sqrt{n}$, we can obtain a slightly better bound on the size of the triangulation: we can compute a triangulation of $O(n + k^3 \log k)$ size and $O(k \log^3 n)$ stabbing number, or of $O((n + k^3) \log n)$ size and $O(k \log^2 n)$ stabbing number. This is close to optimal, because a lower bound of $\Omega(n + k^2)$ on size and $\Omega(k + \log n)$ on the stabbing number can be obtained by modifying the construction described in the previous section. We omit the details of the lower bound. In the remainder of this section, we sketch an algorithm for computing $T(S)$.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be the family of $k$ disjoint convex polyhedra induced by $S$. Let the *silhouette* of $P_i$ with respect to the $z$-axis be the set of points of vertical tangencies. We describe how to triangulate the closure of the exterior $U$ of the polyhedra. Adding triangulations of the interiors and making sure that the inner and outer triangulations agree along polyhedra boundaries is easier.

We begin by computing a kind of a "vertical decomposition" of $U$. More precisely, we add "walls" that are the union of maximal vertical (i.e., parallel to $z$-axis) segments passing through points of silhouettes and contained in $U$. In addition, we draw a plane parallel to $yz$-plane through each $x$-extreme vertex of a polyhedron and thus form at most $2k$ more "walls" consisting of all points of $U$ lying in such planes. It is easily checked that the additional complexity introduced by this construction is $O(nk)$ and that it results in partitioning of $U$ into "cylinders," each of which is a set bounded by one polyhedron on the top, one on the bottom, and by vertical walls elsewhere (top and/or bottom may be missing, in which case the cylinder is not bounded; this case will not be further considered below).

We cut each cylinder by a plane separating the top polyhedron from the bottom one. The cross section $S$ is a simple polygon. We triangulate the top half $T$ of the cylinder separately from the bottom half $B$ and then patch the triangulations together. The top is triangulated as follows: We take a linear-size triangulation of $S$ that guarantees $O(\log n)$ stabbing number [13] and turn each triangle into an infinite vertical prism. We also take the unrefined outer hierarchical decomposition (which is not a proper triangulation) of the ceiling polytope. It has stabbing number $O(\log n)$. We overlay the two partitions, obtaining a convex decomposition of $T$. As each decomposition has low stabbing number and the vertices of the overlay must come from intersection of edges of one decomposition with faces of the other, the size of the overlay is not much larger than the sum of sizes of the decompositions being overlayed. Moreover, the cells of the resulting overlay have constant complexity and thus can be easily triangulated without affecting the stabbing number or total size. One needs to compensate for the fact that we did not take a proper outer triangulation, which introduces layers of arbitrarily thin convex polyhedra between layers of the hierarchy. They do not asymptotically increase the complexity, but they increase the stabbing number by another factor of $\log n$. So we obtain a family of triangulations, each of a top or bottom of a cylinder, each with a stabbing number of $O(\log^2 n)$. A segment cannot pass through more than $4k$ cylinder boundaries, implying stabbing number $O(k \log^2 n)$ overall. It remains to patch the triangulations together, along cylinder boundaries and simple polygons separating cylinder tops from bottoms. The details are highly technical, but the net result is that the last step increases asymptotically neither the stabbing number nor the size of the triangulation.

Thus the stabbing number of the resulting triangulation is $O(k \log^2 n)$. What about its size? After vertical decomposition, the size was $O(nk)$. Outer hierarchy and simple polygon triangulation were linear in the size of the each object. Overlaying raised complexity by a logarithmic factor and final patching phase does not affect asymptotic complexity. Thus the complexity of the resulting triangulation is $O(nk \log n)$. We omit the details.

If $k \le \sqrt{n}$, we can compute a triangulation with fewer simplices, as follows. We separate every pair of polyhedra by a plane and use the resulting planes to build a family of $k$ disjoint polyhedra with a total of $k^2$ facets, each containing one the input polyhedra. We then triangulate the region outside the larger polyhedra using the above method, giving $O(k^3 \log n)$ size and $O(k \log^2 k)$ stabbing number, and triangulate the regions between each larger polyhedron and the smaller one contained within it, using methods described in Section 2. Finally, we patch the triangulations together. Omitting all the details, we obtain a triangulation of $O(n + k^3 \log k)$ size and $O(k \log^3 n)$ stabbing number, or of $O((n + k^3) \log n)$ size and $O(k \log^2 n)$ stabbing number. To summarize:

**Theorem 5.1** *If $S$ forms the boundary of $k$ convex polyhedra, then there exists a triangulation $T(S)$ of $O(nk \log n)$ size and $O(k \log^2 n)$ stabbing number, of $O((n + k^3) \log n)$ size and $O(k \log^2 n)$ stabbing number, or of $O(n + k^3 \log k)$ size and $O(k \log^3 n)$ stabbing number. Moreover, there exists a set of $n$ triangles forming the boundary of $k$ disjoint convex polyhedra such that any triangulation $T(S)$ has size $\Omega(n + k^2)$ and $\Omega(k + \log n)$ stabbing number.*

**Remark.** If the polyhedra in $\mathcal{P}$ are intersecting, we can compute a triangulation with respect to $S$ whose size is $O(nk^2 \log^2 n)$ and whose line stabbing number is $O(k \log^2 n \log k)$. This is close to optimal, as the complexity of the overlay of $k$ polyhedra can be as high as $\Theta(nk^2)$ and the lower bound of $\Omega(k + \log n)$ on the stabbing number applies here.

# 6 Concluding Remarks and Open Problems

We have obtained nearly sharp bounds on the line stabbing number of triangulations in several cases. Although the hyperplane stabbing problem has been studied previously, the line stabbing number has not received much attention. In several applications, however, the line stabbing appears quite naturally, such as in computer graphics. Throughout the paper we assumed that we wanted to triangulate the entire space. Most of the results, however, hold even if we want to triangulate a part of the space, e.g., a simplex containing $S$, or the convex hull of $S$.

Several open problems are suggested by our work; we are currently investigating some of them: (i) minimize the number of Steiner points, (ii) obtain tight bounds in all the cases considered here, (iii) obtain sharp bounds when $S$ forms a terrain, (iv) define *average line stabbing number* and obtain bounds on the average line stabbing number, (v) extend the results to yield lower bounds on ray-shooting in a more general model.

# References


[1] B. Aronov and J. Matoušek. On stabbing triangles by lines in 3-space. Manuscript, 1993.

[2] B. Aronov and M. Sharir, Triangles in space, or building (and analyzing) castles in the air, *Combinatorica* 10(2) (1990), 137–173.

[3] M. Bern and D. Eppstein, Mesh generation and optimal triangulation, in: *Computing in Euclidean Geometry*, (D. Du and F. Hwang, eds.), 1992, World Scientific, Singapore.

[4] M. Bern. Compatible tetrahedralizations. In *Proceedings of the 9th ACM Symposium on Computational Geometry*, pages 281–288, 1993.

[5] B. Chazelle. Convex partitions of polyhedra: A lower bound and worst-case optimal algorithm. *SIAM J. of Computing*, 13:488–508, 1984.

[6] B. Chazelle, H. Edelsbrunner, and L. Guibas. The complexity of cutting complexes. *Discr. Comput. Geom.*, 4:139–181, 1989.

[7] B. Chazelle and L. Palios. Triangulating a nonconvex polytope. *Discr. Comput. Geom.*, 5:505–526, 1990.

[8] B. Chazelle and N. Shouraboura. Bounds on the size of tetrahedralizations. In *Proceedings of the 10th ACM Symposium on Computational Geometry*, pages 231–239, 1994.

[9] B. Chazelle and E. Welzl. Quasi-optimal range searching in spaces with finite VC-dimension. *Discr. Comput. Geom.*, 4:467–489, 1989.

[10] D. P. Dobkin and D. G. Kirkpatrick. Fast detection of polyhedral intersections. In *Proceedings of ICALP '82*, pages 154–165. Springer-Verlag, 1982. Lecture Notes in Computer Science 140.

[11] D. Dobkin and D. Kirkpatrick. A linear algorithm for determining the separation of convex polyhedra. *J. of Algorithms*, 6:381–392, 1985.

[12] H. Edelsbrunner. Triangulations. Course Notes CS 497. University of Illinois, Urbana-Champaign, 1991.

[13] J. Hershberger and S. Suri. A pedestrian approach to ray-shooting: shoot a ray, take a walk. In *Proceedings of the Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 54–63, 1993.

[14] J. Matoušek. Efficient partition trees. *Discrete Comput. Geom.*, 8:315–334, 1992.

[15] D. Mount, J. Mitchell, and S. Suri. Query-sensitive ray shooting, *Proc. 10th ACM Symp. Computational Geometry* 1994, 359–368.

[16] J. O'Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1987.

[17] J. Ruppert and R. Seidel. On the difficulty of tetrahedralizing 3-dimensional non-convex polyhedra. In *Proceedings of the 5th ACM Symposium on Computational Geometry*, pages 380–392, 1989.

[18] E. Welzl. On spanning trees with low crossing numbers. Tech. Rept. B92–02, Free University, Berlin, 1992.