Due Date: March 7, 2001

Problem 1: Let M be a 1-tape Turing machine. Let $R(n)$ denote the maximum length of a crossing sequence (between the boundary of any two cells) on any string of length n. Show that if

$$\lim_{n\to\infty} \frac{R(n)}{\log n} = 0,$$

then $L(M)$ is a regular language.
(**Hint:** *If $R(n)$ is bounded, then the proof follows from the argument given in the class. If $R(n)$ is not bounded, then argue that for certain input strings, there are no two distinct indices i and j such that $C(i,x) = C(j,x)$.*)

Problem 2:

(i) Let M be a 1-tape deterministic TM that halts one every string in time $t(n)$. Set $\mu = \sqrt{t(n)}$. Let x be a string of length n. Suppose you are given x, the description of M, and $C_{i+j\mu}(x)$ for every $j \le \mu$; here $i \le \mu$ is some integer. Show that whether $x \in L(M)$ can be determined using $O(\mu + \sum_{j\le\mu} |C_{i+j\mu}(x)|)$ space.

(ii) Let $t(n) \ge n^2$. Show that $\mathsf{DTIME}_1(t(n)) \subseteq \mathsf{DSPACE}\left(\sqrt{t(n)}\right)$.
(**Hint:** *Show that there exists an integer $i \le \mu$ such that $\sum_{d=0}^{\mu} |C_{i+d\mu}(x)| \le \mu$. Then use (i).*)

Problem 3: Show that if $\mathsf{DTIME}(2^{cn}) \subset \mathsf{NP}$ for some c, then

$$\mathsf{NP} = \mathsf{PSPACE} = \mathsf{EXPTIME}.$$

Problem 4: Given a directed graph G and two vertices u, v, the *graph reachability* problem is to determine whether there exists a path from u to v in G. Show that the graph reachability problem is NL-complete under the $\le_M^{\mathsf{LOG}}$-reduction.
(**Hint:** *Show that the configuration graph of a $\mathsf{LOGSPACE}$ nondeterministic Turing machine can be constructed by a DTM using only $O(\log n)$ space. Recall that one does not take the output size into account.*)

Problem 5: Let $G = (V, E)$ be a directed graph. A subset $D \subseteq V$ is called a *dominating subset* of G if for every $v \notin D$ there is a $u \in D$ with $(u,v)\ inE$. Show that the following language is NP-Complete:

$$DS = \{(G,k) \mid G \text{ has a dominating set of size} \le k\}.$$