# Topic 18: Basic Graph Algorithms

(CLRS Appendix B.4-B.5, 22)

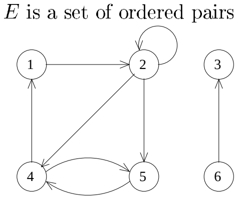
Fall 2001

## 1 Graph Problems

- During the next couple of weeks we will discuss graph algorithms.
- We start with a review of the basic definitions and a few fundamental graph algorithms.

### 1.1 Definitions

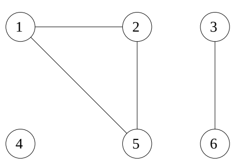
- A graph $G = (V, E)$ consists of a finite set of *vertices* $V$ and a finite set of *edges* $E$.
  - *Directed graph (DAG)*: $E$ is a set of ordered pairs of vertices $(u, v)$ where $u, v \in V$



V = {1, 2, 3, 4, 5, 6}

E = {(1,2), (2,2), (2,4), (2,5)
(4,1), (4,5), (5,4), (6,3)}

  - *Undirected graph*: $E$ is a set of unordered pairs of vertices $\{u, v\}$ where $u, v \in V$



V = {1, 2, 3, 4, 5, 6}

E = {{1,2}, {1,5}, {2,5}, {3,6}}

- Edge $(u, v)$ is *incident* to $u$ and $v$
- *Degree* of vertex in undirected graph is the number of edges incident to it.
- *In (out) degree* of a vertex in directed graph is the number of edges entering (leaving) it.
- A *path* from $u_1$ to $u_2$ is a sequence of vertices $\langle u_1 = v_0, v_1, v_2, \cdots, v_k = u_2\rangle$ such that $(v_i, v_{i+1}) \in E$ (or $\{v_i, v_{i+1}\} \in E$)



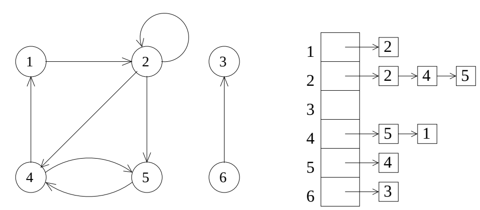
  - We say that $u_2$ is *reachable* from $u_1$
  - The *length* of the path is $k$
  - It is a *cycle* if $v_0 = v_k$
- An undirected graph is *connected* if every pair of vertices are connected by a path
  - The *connected components* are the equivalence classes of the vertices under the "reachability" relation. (All connected pair of vertices are in the same connected component).
- A directed graph is *strongly connected* if every pair of vertices are reachable from each other
  - The *strongly connected components* are the equivalence classes of the vertices under the "mutual reachability" relation.
  - In the DAG pictured earlier, there are three strongly connected components. The subgraph induced by vertices $\{1, 2, 4, 5\}$ is strongly connected and it forms a strongly connected component. The other two strongly connected components consist of the single sets $\{3\}$ and $\{6\}$.

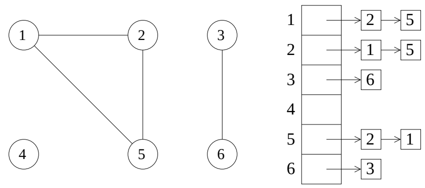
- Graphs appear all over the place in all kinds of applications, e.g:
  - Trees ($|E| = |V| - 1$)
  - Connectivity/dependencies (house building plans, WWW-page connections, ...)
- Often the edges $(u, v)$ in a graph have weights $w(u, v)$, e.g.
  - Road networks (distances)
  - Cable networks (capacity)

### 1.2 Representation

- *Adjacency-list* representation:
  - Array of $|V|$ list of edges incident to each vertex.

    Examples:



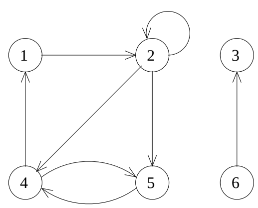



- Note: For undirected graphs, every edge is stored twice. Hence, space is $O(|V| + 2|E|) = O(|V| + |E|)$.
- If graph is weighted, a weight is stored with each edge.

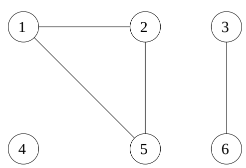
- *Adjacency-matrix* representation:
  - $|V| \times |V|$ matrix $A$ where

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Examples:




| i \ j | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 1 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 0 | 0 | 0 | 1 | 0 |
| 5 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 1 | 0 | 0 | 0 |




| i \ j | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 2 | 1 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 1 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 1 | 0 | 0 | 0 |

  - Note: For undirected graphs, the adjacency matrix is symmetric along the main diagonal (i.e., $A^T = A$).
  - If graph is weighted, weights are stored in the adjacency matrix instead of 1s.

- Comparison of matrix and list representation:



| Adjacency list | Adjacency matrix |
|---|---|
| $O(\lvert V\rvert + \lvert E\rvert)$ space | $O(\lvert V\rvert^2)$ space |
| Good if graph *sparse* ($\lvert E\rvert << \lvert V\rvert^2$) | Good if graph *dense* ($\lvert E\rvert \approx \lvert V\rvert^2$) |
| No quick access to $(u,v)$ | $O(1)$ access to $(u,v)$ |

- We will use adjacency list representation unless stated otherwise ($O(|V| + |E|)$ space).

# 2 Graph traversal

- There are two standard (and simple) ways of traversing all vertices/edges in a graph in a systematic way
  - Breadth-first
  - Depth-first
- We can use them in many fundamental algorithms, e.g., finding cycles, connected components, ...

## 2.1 Breadth-first search (BFS)

- Main idea:
  - Start by visiting some source vertex $s$.
  - Then visit all vertices at distance 1,
  - Then visit all vertices at distance 2,
  - Then visit all vertices at distance 3,

    ⋮
- BFS corresponds to computing *shortest path* distance (in terms of the number of edges) from $s$ to all other vertices.
- To control progress of our BFS algorithm, we think about *coloring* each vertex
  - *White* before we start,
  - *Gray* after we visit the vertex but before we have visited all its adjacent vertices,
  - *Black* after we have visited the vertex and all its adjacent vertices (all adjacent vertices are gray).
- We use a FIFO queue $Q$ to hold all gray vertices—vertices we have seen but are still not done with.
- We remember from which vertex a given vertex $v$ is colored gray (visit[$v$]).

- Algorithm:

```
BFS(s)
    color[s] = gray
    d[s] = 0
    ENQUEUE(Q, s)
    WHILE Q not empty DO
        DEQUEUE(Q, u)
        FOR (u, v) ∈ E DO
            IF color[v] = white THEN
                color[v] = gray
                d[v] = d[u] + 1
                visit[v] = u
                ENQUEUE(Q, v)
            FI
            color[u] = black
    OD
```

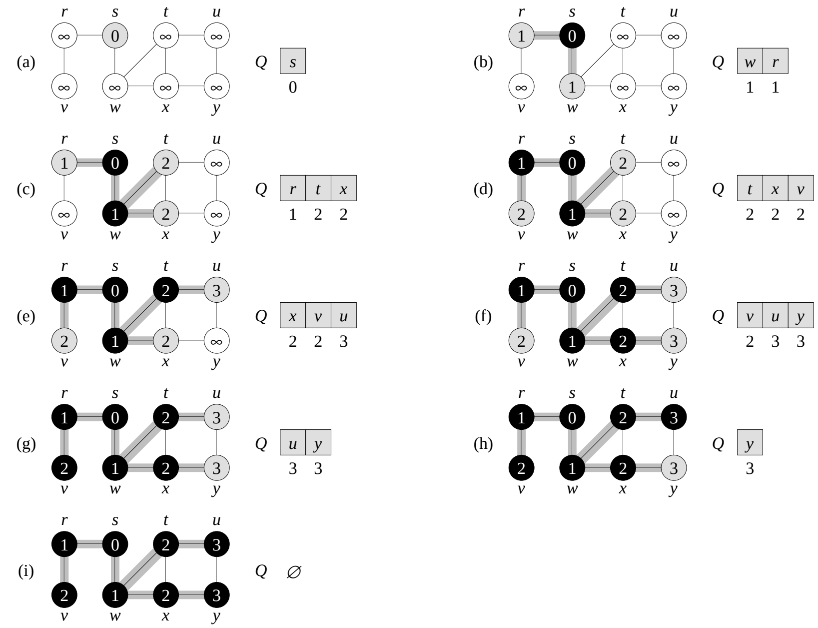
- Algorithm runs in $O(|V| + |E|)$ time
- Note:
  - The edges $(\text{visit}[v], v)$, for all $v \in V$ form a tree called the *BFS-tree*.
  - $d[v]$ contains length of shortest path (in terms of the number of edges) from $s$ to $v$.
  - We can use the visit array to find the shortest path from $s$ to any given vertex $v$, by tracing the path backwards from $v$: $v$, visit$[v]$, visit[visit$[v]$], $\ldots$ .
- If graph is not connected we have to try to start the traversal at all nodes.

```
FOR each vertex u ∈ V DO
    IF color[u] = white THEN BFS(u)
OD
```

  - Note: We can use algorithm to compute connected components in $O(|V| + |E|)$ time.



- BFS Example:




## 2.2 Depth-first search (DFS)

- If we use a stack instead of a FIFO queue $Q$, we get another traversal order: depth-first search
  - We explore "as deeply as possible".
  - Backtrack until we find unexplored adjacent vertex,
  - Explore as deeply as possible,

    $\vdots$
- Often we are interested in "discovery time" and "finish time" of vertex $u$
  - *Discovery time* ($d[u]$): indicates at what "time" vertex $u$ is first visited.
  - *Finish time* ($f[u]$): indicates at what "time" all adjacent vertices of vertex $u$ have been visited.
- Instead of using a stack in a DFS algorithms, we can write a recursive procedure

– We will color a vertex gray when we first meet it and black when we finish processing all adjacent vertices.

- Algorithm:

```
DFS(u)
    color[u] = gray
    d[u] = time
    time = time + 1
    FOR (u,v) ∈ E DO
        IF color[v] = white THEN
            visit[v] = u
            DFS(v)
        FI
    OD
    color[u] = black
    f[u] = time
    time = time + 1
```

- Algorithm runs in $O(|V| + |E|)$ time
- As before we can extend algorithm to unconnected graphs and we can use it to find connected components in $O(|V| + |E|)$ time.

```
FOR each vertex u ∈ V DO
    IF color[u] = white THEN DFS(u)
OD
```

- As previously, the edges (visit$[v], v$), for all $v \in V$ form a tree called the *DFS-tree*.



# DFS: How it works

- Initialize all vertices to white
- Reset global counter
- Check each vertex; visit each *white* vertex using DFS
- Each call to DFS$(u)$ roots a new tree of depth-first forest at vertex $u$
- Vertex is *gray* if it has been discovered, but not all its edges have been explored!
- *gray* edges always form a linear chain!
- Vertex is *black* after all its edges are explored
- When DFS returns, every vertex $u$ is assigned:
  1. a discovery time $d[u]$, and
  2. a finishing time $f[u]$

## DFS: Running time

Running time $O(|V|^2)$, because
  **DFS** called once per vertex
  Each loop over $Adj$ runs $< |V|$ times.
But... can we show a better bound?

- **Amortized bookkeeping:** charge exploration of edge *to* the edge:

  Charge **DFS** loop body to edge (runs once per edge if directed graph, twice if undirected)

  Charge rest of **DFS** to vertex (runs once per vertex)

- Time $= O(|V| + |E|)$, which is *linear time*

  $O(|V| + |E|)$ *is considered linear time for graph because it is linear in size of adjacency-list representation!*



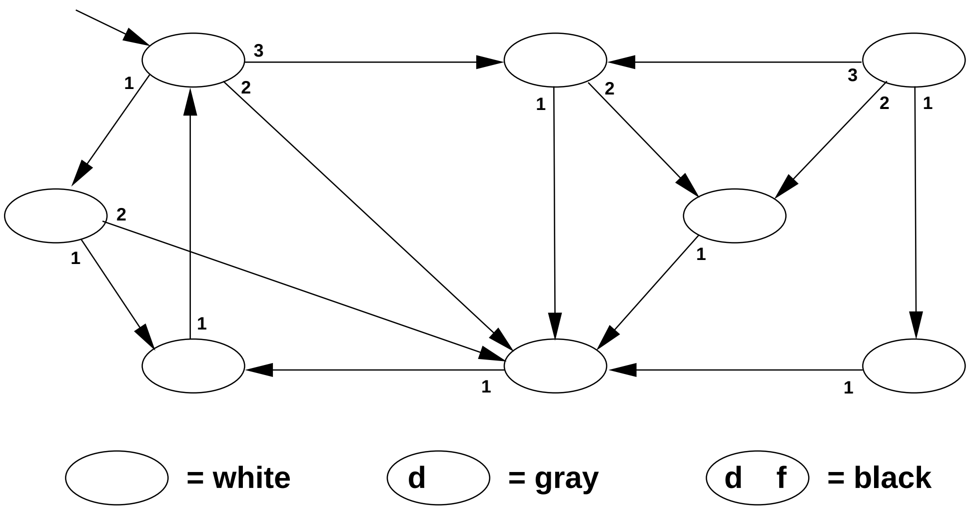
## DFS Timestamping

The procedure **DFS** records:

- discovery time of vertex $u$ in $d[u]$
- finishing time of vertex $u$ in $f[u]$

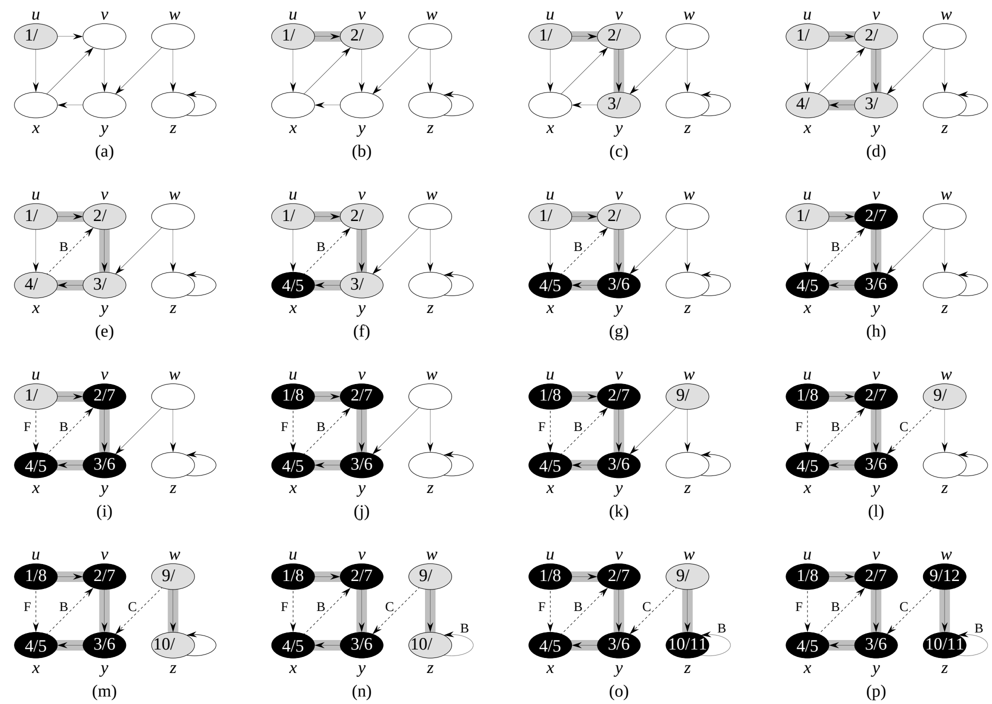
For every vertex $u$,
  $d[u] < f[u]$ .

# DFS Example






# DFS Example




Inside each node above,

- each gray vertex is labeled by its discovery time, and
- each black vertex is labeled by both its discovery time and its finish time.

## DFS: Structure of colored vertices

Vertex $u$ is:

- *white* before time $d[u]$
- *gray* between time $d[u]$ and time $f[u]$
- *black* thereafter.

Also notice structure throughout algorithm:

- *gray* vertices form a linear chain.
  - stack of recursive calls
    *(things started but not yet finished)*



## DFS: parenthesis theorem

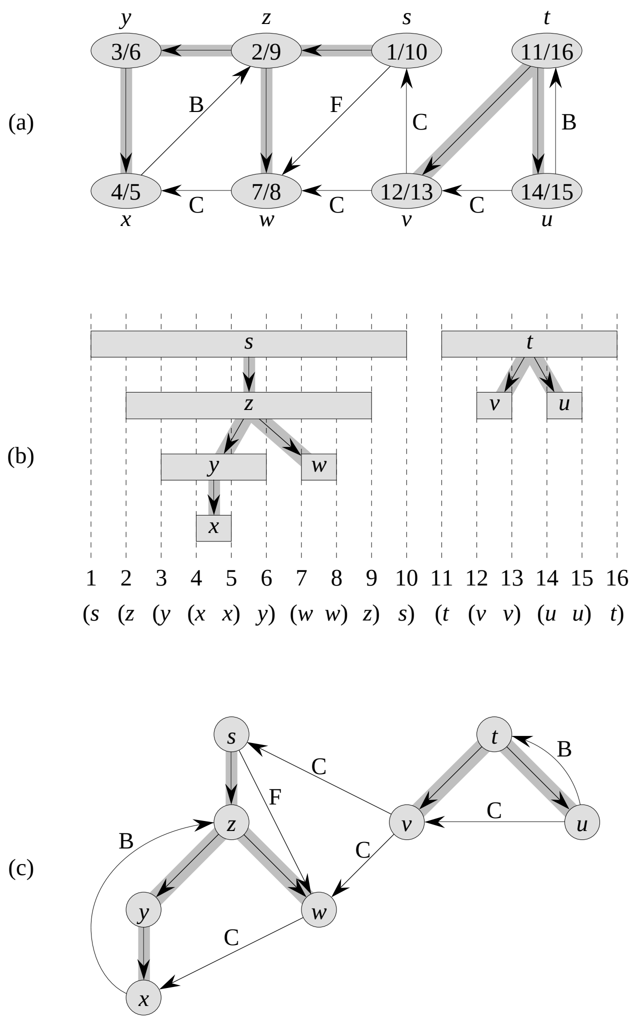
Discovery, finish times have
**parenthesis structure**.

- represent discovery of $u$ with left parenthesis "($u$"
- represent finishing $u$ by right parenthesis "$u$)"
- history of discoveries and finishings makes a well-formed expression! (Parentheses are properly nested.)
- If $v$ is a descendant of $u$ in the DFS tree, then

$$d[u] < d[v] < f[v] < f[u].$$

Proof in CLRS (omitted here); intuition:
Intervals either disjoint or
enclosed, but never (otherwise) overlap
We'll just look at example.

# DFS and Parenthesization






# Edge Classification

**Tree** edge: ($gray$ to $white$)
encounter new ($white$) vertex
Form spanning forest (no cycles)

**Back** edge: ($gray$ to $gray$)
from descendant to ancestor

**Forward** edge: ($gray$ to $black$)
nontree, from ancestor to descendant

**Cross** edge: ($gray$ to $black$)
remainder — between trees or subtrees
(if same tree, can't go anc/desc, or desc/anc)

## DFS: edge classification

Notes:

- ancestor/descendant is with respect to **tree** edges
- **tree** and **back** edges are important;
- most algorithms don't distinguish between **forward** and **cross** edges

Exercise:

- How to distinguish forward, cross edges in DFS? (Hint: look at discovery times.)

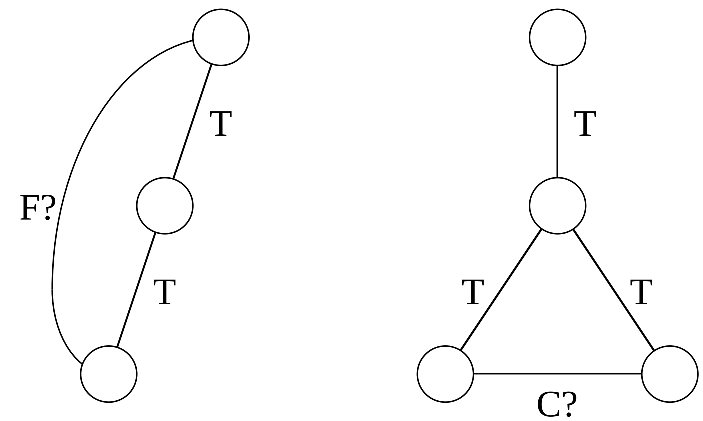


## DFS: Lemma

**Theorem 22.10:**

In a depth-first search of an *undirected graph* $G$, every edge of $G$ is either a tree edge or a back edge.

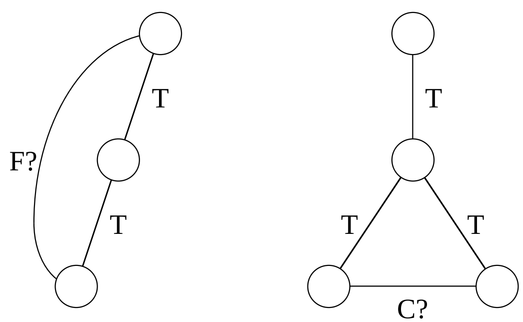
*Sketch of proof:*



# DFS: Lemma

## Theorem 22.10:

*Proof:*



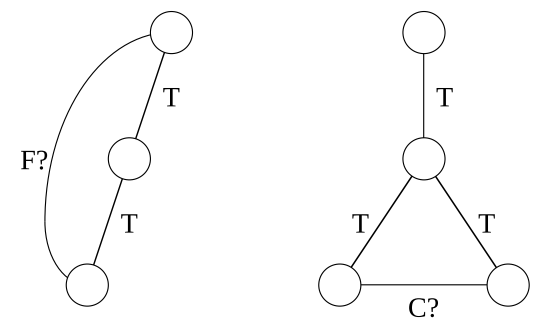

▷ Suppose there's a forward edge $F$? (at left)
But $F$? edge must actually be $B$ because we must finish processing bottom vertex before resuming with top vertex.



# DFS: Lemma

## Theorem 22.10:

*Proof:*




▷ Suppose there's a cross edge $C$? between subtrees (at right)
$C$? edge can't be Cross edge:
It must be explored from its first endpoint to be explored, in which case the other endpoint isn't yet explored, and the edge becomes a T edge instead of a C edge.
The search continues beyond the other endpoint, and the T edge coming out of the other endpoint changes to a B edge.

## Exercise

Can use DFS to find cycles in undirected graphs!

An undirected graph is acyclic (i.e., a forest) iff a DFS yields no back edges.

- Proof that acyclic $\Rightarrow$ no back edge:
  trivial (back edge $\Rightarrow$ cycle)

- Proof that no back edges $\Rightarrow$ acyclic:
  No back edges $\Rightarrow$ only tree edges (by above lemma)
  $\Rightarrow$ forest $\Rightarrow$ acyclic



## Exercise

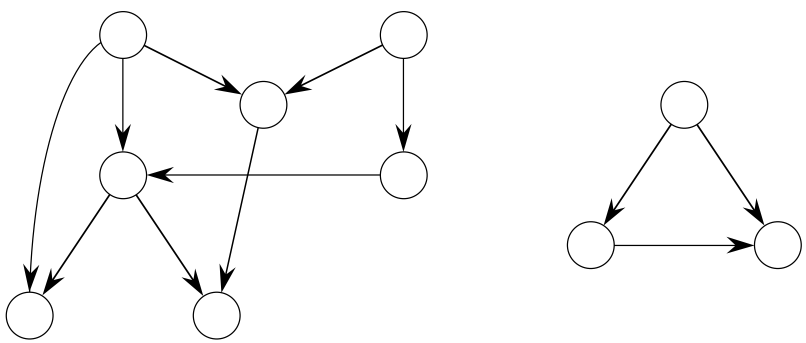
We can thus run **DFS**: if find a back edge, then we can stop and report that there's a cycle

- Time $O(|V|)$, [not $O(|V| + |E|)$ !]

  If ever see $|V|$ distinct edges, must have seen a back edge, because in acyclic (undirected) forest, $|E| \leq |V| - 1$.

## Directed Acyclic Graphs (DAGs)

- No *directed* cycles
  example:



- Used in many applications to indicate precedences among events
- Example: parallel code execution
  - Topological Sort (induce a total ordering)



## DAG: Theorem

Theorem: A directed graph $G$ is acyclic
iff a DFS yields no back edges.
$\Rightarrow$: back edge $\Rightarrow$ cycle

$\Leftarrow$: Contrapositive: cycle $\Rightarrow$ back edge

Suppose $G$ has a cycle. Let $v$ have lowest discovery # on cycle, and let $u$ be predecessor on cycle.

$$\begin{array}{ccc} u & \longrightarrow & v \\ \nwarrow & \cdots & \swarrow \end{array}$$

*($v$ is first vertex visited)*

When $v$ discovered, whole cycle is *white*.

Must visit everything reachable on a *white* path from $v$ before returning from **DFS**$(v)$.
Thus $(u, v)$ is a back edge. $\square$

- $O(|V| + |E|)$ time [Why not $O(|V|)$ as before?]

## Topological Sort

**Topological Sort** of a dag $G = (V, E)$ is a

- Linear ordering of all vertices of a dag

such that

- If $G$ contains an edge $(u, v)$, then $u$ appears before $v$ in the ordering.

If the graph has a cycle, then no linear ordering is possible!



## Topological Sort: pseudocode

The following algorithm topologically sorts a DAG:

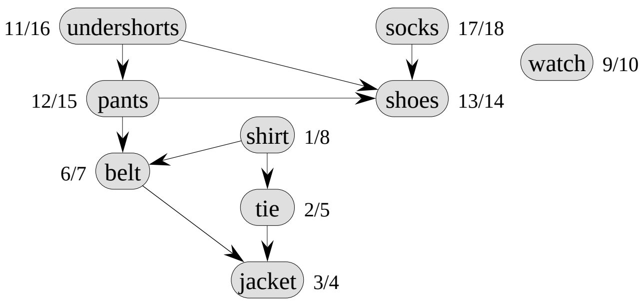
TOPOLOGICAL-SORT($G$)
1 call DFS($G$) to compute finishing times $f[v]$ for each vertex $v$
2 as each vertex is finished, insert it onto the front of a linked list
3 **return** the linked list of vertices

At end, linked list comprises total ordering!

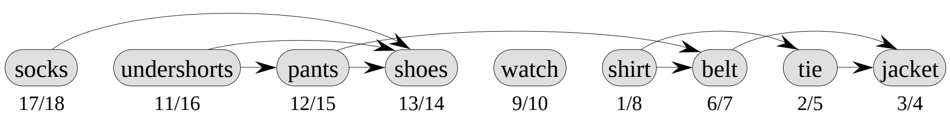
# Topological Sort: Example

Example: precedence relations (don $x$ before $y$)
Intuition: Can “schedule” task only when all of its follow-on tasks have been scheduled. The task is scheduled earlier than its follow-on tasks.

(a)




(b)






# Topological Sort: running time

Running Time:

- depth-first search: takes $O(|V| + |E|)$ time
- insert each of the $|V|$ vertices onto the front of the linked list: takes $O(1)$

We can perform a topological sort in time $O(|V| + |E|)$.

# Topological Sort: correctness

Correctness proof for TOPOLOGICAL-SORT$(G)$

Claim: $(u, v) \in E \Rightarrow f[u] > f[v]$

When $(u, v)$ explored, $u$ is *gray*

If $v =$*gray*
$\Rightarrow (u, v) =$ backedge (cycle, contradiction).

If $v =$*white*
$\Rightarrow v$ becomes descendant of $u$
$\Rightarrow f[v] < f[u]$

If $v =$*black*
$\Rightarrow f[v] < f[u]$



# Alternative algorithm for Topological Sort

Count the in-degree of each vertex. Then repeat the following until there are no more vertices: Remove a vertex with in-degree 0, remove all its outgoing edges, and update the in-degrees of the neighboring vertices.

```
FOR all vertices v DO
    degree[v] = 0
OD
FOR all edges (u,v) ∈ E DO
    degree[v] = degree[v] + 1
    IF degree[v] = 0 THEN ENQUEUE(Q,v)
OD
i = 0
WHILE Q ≠ ∅ DO
    DEQUEUE(Q,u)
    Topsort(u) = i
    i = i + 1
    FOR all edges (u,v) ∈ E DO
        degree[v] =degree[v] − 1
        IF degree[v] = 0 THEN ENQUEUE(Q,v)
    OD
OD
```

## Strongly Connected Components (SCC)

**A strongly connected component** of a directed graph $G = (V, E)$ is:

a maximal set of vertices $U \subseteq V$ such that for every pair of vertices $u$ and $v$ in $U$, we have both

- $u \to \cdots \to v$
  and
- $v \to \cdots \to u$

That is, $u$ and $v$ are reachable from each other!

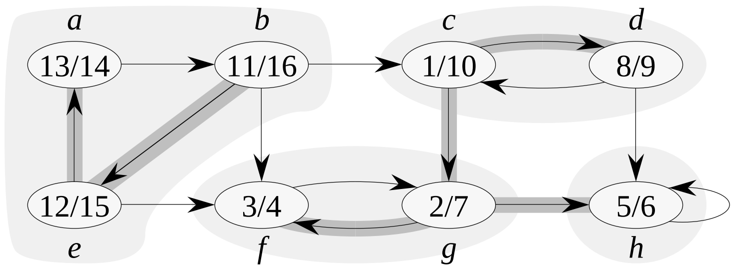
*in other words ...*

- $u$ **R** $v$ if $u$ and $v$ lie on a common cycle.
- **R** is an equivalence relation (r,s,t).
- strongly connected components are a partition of graph $G$ under **R**.

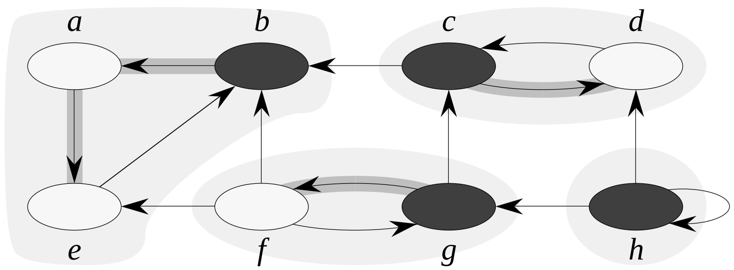


## SCC: examples

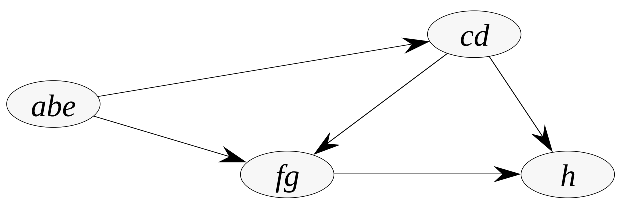
(a)




(b)




(c)



## SCC: Pseudocode

(CLRS §22.5)

To compute SCC of directed graph $G = (V, E)$, use two DFS's, one on $G$ and one on $G^{\mathrm{T}}$ ($G$, with edges swapped):

STRONGLY-CONNECTED-COMPONENTS($G$)
1 call DFS($G$) to compute finishing times $f[u]$ for each vertex $u$
2 compute $G^{\mathrm{T}}$
3 call DFS($G^{\mathrm{T}}$), but in the main loop of DFS, consider the vertices in order of decreasing $f[u]$ (as computed in line 1)
4 output vertices of each tree in the depth-first forest of step 3 as a separate SCC

Intuition: explore latest-finished vertices first
Running time $\Theta(V + E)$ [Why?]

- Strongly-Connected-Components can be found in linear time.



## SCC: Lemmas and Theorems

Lemma 22.13

- Let $C$ and $C'$ be two strongly connected components in directed graph $G$. Let $u, v \in C$ and $u', v' \in C'$. If there is a path in $G$ from $u$ to $u'$, then there cannot be a path in $G$ from $v'$ to $v$.

Lemma 22.14

- Let $C$ and $C'$ be two strongly connected components in directed graph $G$. Suppose there is an edge $(u, v)$ in $G$, where $u \in C$ and $v \in C'$.
  Then $f(C) > f(C')$.

Corollary 22.15

- Let $C$ and $C'$ be two strongly connected components in directed graph $G$. Suppose there is an edge $(u, v)$ in $G^{\mathrm{T}}$, where $u \in C$ and $v \in C'$.
  Then $f(C) < f(C')$.

## SCC: Lemmas and Theorems

Theorem 22.16

- STRONGLY-CONNECTED-COMPONENTS(G) correctly computes the strongly connected components of a directed graph $G$.

See CLRS §22.5 for proofs and further explanations.