Computing on MASSIVE Data

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I/O Crisis! Disk access is 1,000,000 times slower!

Time for rotation \(\approx\) Time for seek.

Amortize search time by transferring large blocks so that Time for rotation \(\approx\) Time for seek \(\approx\) Time to transfer data.

Solution 1: Exploit locality and take advantage of block transfer. *Focus of this talk!*

Solution 2: Parallel disks. (Not our focus today.)
Q. How many h's cause x and y to collide?

A. There are \( m \) choices for each of \( a_1, a_2, \ldots, a_r \), but once these are chosen, exactly one choice for \( a_0 \) causes \( x \) and \( y \) to collide, namely

\[
\text{\`\`} \cdot \cdot \cdot \text{\`\`} \cdot \cdot \cdot \text{\`\`}(x_0, y_0) \cdot \cdot \cdot \text{\`\`}a_i(x_i, y_i) \cdot \cdot \cdot \text{\`\`} \mod m.
\]

Thus, the number of h's that cause \( x \) and \( y \) to collide is

\[
m \times 1 = m = \frac{|H|}{m}.
\]

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L7.15

[Aggarwal & Vitter 88], [Vitter & Shriver 90, 94], …

\[
N = \text{problem data size.}
\]

\[
M = \text{size of internal memory.}
\]

\[
Z = \text{problem output size.}
\]

\[
B = \text{size of disk block.}
\]

Notational convenience (in units of blocks):

\[
n = \frac{N}{B}, \quad m = \frac{M}{B}, \quad z = \frac{Z}{B}.
\]
Proof (completed)

Q. How many ha's cause x and y to collide?

A. There are \( m \) choices for each of \( a_1, a_2, \ldots, a_r \), but once these are chosen, exactly one choice for \( a_0 \) causes \( x \) and \( y \) to collide, namely

\[
\sum_{i=1}^{r} a_i (x_i - y_i) \mod m.
\]

Thus, the number of ha's that cause \( x \) and \( y \) to collide is \( mr = |H|/m \).

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L7.15
Q. How many $h$'s cause $x$ and $y$ to collide?

A. There are $m$ choices for each of $a_1, a_2, \ldots, a_r$, but once these are chosen, exactly one choice for $a_0$ causes $x$ and $y$ to collide, namely

$$h = \sum_{i=1}^{r} a_i(x_i - y_i) \mod m.$$ 

Thus, the number of $h$'s that cause $x$ and $y$ to collide is $mr = |H|/m$. 

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Outline

1. Fundamental I/O Bounds.
   - Merge sort.
   - Distribution sort.
   - Searching.
   - Lower Bounds.

2. Techniques for solving batched geometric problems.
   - Distribution sweeping.
   - Red-blue orthogonal rectangle intersection.
   - Water flow routing and accumulation in terrains.
   - Empirical results (via TPIE programming environment).

3. Online data structures.
   - B-trees, buffer trees, R-trees, etc.
   - Range searching.
Fundamental I/O Bounds (with $D = 1$ disk)

- Batched problems [AV88], [VS90], [VS94]:
  - Scanning (touch problem): $\Theta\left(\frac{N}{B}\right) = \Theta(n)$
  - Sorting:
    $$\Theta\left(\frac{N \log \frac{N}{B}}{B \log \frac{M}{B}}\right) = \Theta\left(\frac{N}{B} \log_{M/B} \frac{N}{B}\right) = \Theta(n \log_m n)$$
  - Permuting: $\Theta(\min\{N, n \log_m n\})$

- For other problems [CGGTVV95], [AKL95], …
  - Graph problems $\propto$ Permutation
  - Computational Geometry $\propto$ Sorting

- Online problems:
  - Searching and Querying: $\Theta\left(\log_B N + \frac{Z}{B}\right) = \Theta(\log_B N + z)$

Notes:

$N$ = problem data size.
$M$ = size of internal memory.
$Z$ = problem output size.
$B$ = size of disk block.
$n = \frac{N}{B}$, $m = \frac{M}{B}$, $z = \frac{Z}{B}$. 
Form \( n/m \) initially sorted runs, each consisting of \( m \) blocks (i.e., one memory load).

Reserve \( R = \Theta(m) \) buffers, each of size \( B \), in internal memory. Repeatedly merge together \( R \) runs at a time

\[
\Rightarrow \# \text{ passes} = \log_R \frac{n}{m} = \log_m n - 1
\]

Each pass uses \( \Theta(n) \) I/Os

\[
\Rightarrow \# \text{ I/Os} = O\left(n \log_m n\right).
\]

\[
N = \text{problem data size,} \quad B = \text{size of disk block,} \quad n = \frac{N}{B}, \quad m = \frac{M}{B}.
\]

\[
\log_m n = \text{depth of recursion}
\]
Select $S = \Theta(m)$ or $\Theta(\sqrt{m})$ partitioning elements that divide the file evenly into buckets. (See [AV88, §5].)

Stream the file through internal memory, forming $S$ subfiles.

Independently sort the $S$ subfiles recursively.

\[ \text{number of levels of recursion is } \log_S n = \log_m n. \]

Each level of recursion uses $O(n)$ I/Os.

\[ \text{# I/Os} = O(n \log_m n). \]

$N =$ problem data size.

$M =$ size of internal memory.

$B =$ size of disk block.

$n = \frac{N}{B}$, $m = \frac{M}{B}$. 
I/O Lower Bound for Permuting

Goal: See how many I/O steps $T$ are needed so that any of the $N!$ permutations of the $N$ items can be realized.

We say that a permutation is realizable if it appears in extended memory in the required order.

Tactic: Determine how much the $t$th I/O step can increase the number of possible realizable permutations.
I/O Lower Bound for Permuting

Permuting problem: Given \( N \) distinct items from \{1, 2, \ldots, N\}, rearrange the \( N \) items into sorted order.

☆ We will show the lower bound that permuting requires
\[ \Omega(\min\{N, n \log_m n\}) \] I/Os.

☆ Typically the min term is \( n \log_m n \).

☆ Permuting is a special case of sorting.

☆ I/O lower bound also applies to sorting. It is based only upon routing considerations, since the order is already known.

☆ For the pathological case when \( N < n \log_m n \), we can show that sorting requires \( \Omega(n \log_m n) \) I/Os in comparison model.

☆ In the RAM model, permutation takes only \( O(N) \) time. But in I/O model, it (and most interesting problems) require sorting complexity (except for pathological case)!

Notes:

\( N = \) problem data size.
\( M = \) size of internal memory.
\( B = \) size of disk block.
\( n = \frac{N}{B}, \quad m = \frac{M}{B} \).
I/O Lower Bound for Permuting

Assumption: the $N$ items to permute are indivisible.

Number of ways to choose $B$ blocks out of $M$ total memory blocks $= \binom{M}{B}$

$\#$ realizable permutations after $t$th I/O

$$\begin{cases} \binom{M}{B} \times (\# \text{ realizable permutations after } (t-1)\text{st I/O}) \\ \text{if block was previously accessed} \end{cases}$$

$$B! \times \binom{M}{B} \times (\# \text{ realizable permutations after } (t-1)\text{st I/O})$$

if this is first access to block

Number of ways to permute $B$ blocks: $B!$

There are $N/B$ blocks initially unaccessed.

$$\# \text{ choices for block accessed in } t\text{th I/O} = \left( \frac{N}{B} + t \right) \leq N^2.$$ 

Notes:

$$\binom{M}{B} = \text{Number of ways to choose } B \text{ memory locations out of } M \text{ total memory}$$

$B! = \text{Number of ways to permute } B \text{ elements of a block}$
Number $T$ of required I/Os for Permuting

$$(B!)^{N/B} \left( \left( \frac{M}{B} \right)^{N^2} \right)^T \geq N!$$

Taking logs and applying Stirling's approximation:

$$\frac{N}{B} \log B! + T \left( \log \left( \frac{M}{B} \right) + \log N \right) = \Omega(\log N!)$$

$$\frac{N}{B} (B \log B) + T \left( B \log \frac{M}{B} + \log N \right) = N \log N$$

$$T \left( B \log \frac{M}{B} + \log N \right) = N \log N - N \log B$$

$$= N \log \frac{N}{B}$$

$$T = \Omega \left( \min \left\{ N, \frac{N \log(N/B)}{B \log(M/B)} \right\} \right)$$

$$= \Omega \left( \min \{ N, n \log_m n \} \right)$$

Notes: $n = \frac{N}{B}, m = \frac{M}{B}$.

Stirling's Bounds for Factorial $N!$: Number of ways to permute $N$ keys:

$N! \sim \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$ and taking logs gives: $\log N! \sim N \log N$

Bounds for Factorial $B!$: Number of ways to permute $B$ elements of a block: $\log B! \sim B \log B$

Number of ways to choose $B$ memory locations out of $M$ total memory:

$$\left( \frac{M}{B} \right) \sim \left( \frac{M}{B} \right)^B$$ and taking logs gives: $\log \left( \frac{M}{B} \right) \sim B \log \frac{M}{B}$
Lower Bound on External Sorting

Theorem 1 External sorting requires $\Omega(n \log_m n)$ I/Os in the comparison I/O model (comparisons only allowed operations in internal memory).

Proof

Consider an input of $B$ records into internal memory. Assuming that we know the order of the records already in internal memory, but not the order of the $B$ newly read records, there are at most $\binom{M}{B} (B!)$ possible orderings of the records in internal memory. If $S$ denotes the number of possible orderings before the input, there exists at least one of the $\binom{M}{B} (B!)$ orderings of records in internal memory, such that the number of total orderings (of the initial $N$ records) consistent with this ordering, is at least $\frac{S}{\binom{M}{B}(B!)}$. The adversary always chooses one such ordering. It follows that after $t$ inputs, the number of possible orderings is at least $\frac{N!}{\binom{M}{B}(B!)^t}$.

The above was under the assumption that we did not know the order of the $B$ records read into internal memory. This is not the case if the $B$ records have been together in internal memory previously, because we always determine the order of the records in internal memory after an input. The number of times we can read $B$ records that have not previously been together in internal memory cannot exceed $\frac{N}{B}$. It follows that after $t$ input operations there are at least $\frac{N!}{\binom{M}{B}(B!)^t}$ orderings consistent with the information obtained from the adversary.
Theorem 1  External sorting requires \( \Omega(n \log_m n) \) I/Os in the comparison I/O model (comparisons only allowed operations in internal memory).

Proof, cont  Consider an input of \( B \) records into internal memory. after \( t \) input operations there are at least \( \frac{N!}{(\frac{M}{B})^t (B!)^B} \) orderings consistent with the information obtained from the adversary.

We want to narrow the possible orderings down to \( 1 \), and the number of I/O-operations needed to do this must therefore be the least \( t \) such that \( \frac{N!}{(\frac{M}{B})^t (B!)^B} \leq 1 \).
Lower Bound on External Sorting

**Theorem 1**  
External sorting requires $\Omega(n \log_m n)$ I/Os in the comparison I/O model (comparisons only allowed operations in internal memory).

Proof, cont  
Consider an input of $B$ records into internal memory.  
After $t$ input operations there are at least $\frac{N!}{(M/B)^t (B!)^B}$ orderings consistent with the information obtained from the adversary.

We want to narrow the possible orderings down to 1, and the number of I/O-operations needed to do this must therefore be the least $t$ such that $\frac{N!}{(M/B)^t (B!)^B} \leq 1$.

Using $\log x! = x \log x$ (Stirlings formula) and $\log \left(\frac{M}{B}\right) = B \log \frac{M}{B}$ we then get the following.

\[
\frac{N!}{(M/B)^t (B!)^B} \leq 1 \quad \Downarrow
\]
\[
(M/B)^t (B!)^{N/B} \geq N! \quad \Downarrow
\]
\[
t \log \left(\frac{M}{B}\right) + \frac{N}{B} \log(B!) \geq \log(N!) \quad \Downarrow
\]
\[
t B \log\left(\frac{M}{B}\right) + \frac{N}{B} B \log B \geq N \log N \quad \Downarrow
\]
\[
t B \log\left(\frac{M}{B}\right) \geq N \log\left(\frac{N}{B}\right) \quad \Downarrow
\]
\[
t \geq \frac{N \log\left(\frac{N}{B}\right)}{B \log\left(\frac{M}{B}\right)} \quad \Downarrow
\]
\[
t \geq n \log_m n
\]
B-Trees [Bayer-McCreight72]

Each node is stored in one disk block.
Leaf nodes store all the data items.
Internal nodes store keys and pointers to guide the searching.
\[ \frac{B}{2} \leq \text{node degree} \leq B - 1, \text{ except for the root.} \]
One I/O per level
\[ \implies O(\log_B N + z) \text{ I/Os for a 1-D range query.} \]

Notes:
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\[ B = \text{size of disk block.} \]
\[ n = \frac{N}{B}, \quad m = \frac{M}{B}, \quad z = \frac{Z}{B}. \]
B-Trees—Updates in $O(\log_B N)$ I/Os

Insertion may cause splits:

Deletions may cause nodes to combine:

Sharing also possible (good packing).

Q. How many $h$'s cause $x$ and $y$ to collide?

A. There are $m$ choices for each of $a_1, a_2, \ldots, a_r$, but once these are chosen, exactly one choice for $a_0$ causes $x$ and $y$ to collide, namely

$$\sum_{i=1}^{r} (x_i \cdot y_i) \mod m.$$ 

Thus, the number of $h$'s that cause $x$ and $y$ to collide is $m \cdot r = |H|/m$. 

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**Batched Dynamic Processing**

Motivation: Sometimes we may want to handle inserts, deletes, and queries in batches. In RAM algorithms, we can use an optimal dynamic data structure and get optimal results.

But NOT in external memory!!!

Example: Sorting by inserting and deleting into a priority queue.

- Insert items one at a time into priority queue.
- Repeat: Find item in priority queue with minimum key. Delete it from priority queue \((delete_{\text{min}}\) operation) and output it.

- \(N\) inserts into priority queue
  \[ \implies O(N \log_B N) \text{ I/Os.} \quad \text{Bad !!!} \]

- We want \(O(n \log_m n)\) I/Os. \quad \text{Good !!!}

Another example: Sweep-line algorithm for finding rectangle intersections.

Notes: \(N =\) problem data size. \(M =\) size of internal memory. \(Z =\) problem output size. \(B =\) size of disk block. \(n = \frac{N}{B}, m = \frac{M}{B}, z = \frac{Z}{B}\).
Main idea: Combine degree-$B$ nodes to form supernodes of degree $m = M/B$. Each supernode gets a buffer of size $M$.

- Insertions are done “lazily”: Items are inserted into buffers. When a buffer runs full, its items are pushed one level down.

- Buffer-emptying of $M$ items takes $O(m)$ I/Os

\[ O(m/M) = O(1/B) \text{ I/Os per item per level} \]

\[ N \text{ items inserted in } O\left(\frac{N}{B} \log_m n\right) = O(n \log_m n) \text{ I/Os.} \]