Introduction to Algorithms

Hashing II
- Universal hashing
- Universality theorem
- Constructing a set of universal hash functions
- Perfect hashing

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A weakness of hashing

**Problem:** For any hash function \( h \), a set of keys exists that can cause the average access time of a hash table to skyrocket.

* An adversary can pick all keys from \( \{k \in U : h(k) = i\} \) for some slot \( i \).

**Idea:** Choose the hash function at random, independently of the keys.

* Even if an adversary can see your code, he or she cannot find a bad set of keys, since he or she doesn’t know exactly which hash function will be chosen.
Universal hashing

**Definition.** Let $U$ be a universe of keys, and let $H$ be a finite collection of hash functions, each mapping $U$ to $\{0, 1, \ldots, m-1\}$. We say $H$ is **universal** if for all $x, y \in U$, where $x \neq y$, we have $|\{h \in H : h(x) = h(y)\}| = |H|/m$.

That is, the chance of a collision between $x$ and $y$ is $1/m$ if we choose $h$ randomly from $H$. 

![Diagram showing the relationship between the size of the hash function set $H$ and the number of collisions]
Universality is good

**Theorem.** Let $h$ be a hash function chosen (uniformly) at random from a universal set $\mathcal{H}$ of hash functions. Suppose $h$ is used to hash $n$ arbitrary keys into the $m$ slots of a table $T$. Then, for a given key $x$, we have

$$E[\#\text{collisions with } x] < n/m.$$
Proof of theorem

Proof. Let $C_x$ be the random variable denoting the total number of collisions of keys in $T$ with $x$, and let

$$c_{xy} = \begin{cases} 
1 & \text{if } h(x) = h(y), \\
0 & \text{otherwise}.
\end{cases}$$

Note: $E[c_{xy}] = 1/m$ and $C_x = \sum_{y \in T - \{x\}} c_{xy}$. 
Proof (continued)

\[ E[C_x] = E \left[ \sum_{y \in T - \{x\}} c_{xy} \right] \]

- Take expectation of both sides.
Proof (continued)

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• Take expectation of both sides.

• Linearity of expectation.
Proof (continued)

\[ E[C_x] = E \left[ \sum_{y \in T - \{x\}} c_{xy} \right] \]

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\[ = \sum_{y \in T - \{x\}} 1/m \]

- Take expectation of both sides.
- Linearity of expectation.
- \( E[c_{xy}] = 1/m \).
Proof (continued)

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\[ = \sum_{y \in T - \{x\}} 1/m \]

\[ = n - 1 \]

- Take expectation of both sides.
- Linearity of expectation.
- \( E[c_{xy}] = 1/m. \)
- Algebra.
Constructing a set of universal hash functions

Let $m$ be prime. Decompose key $k$ into $r + 1$ digits, each with value in the set $\{0, 1, \ldots, m-1\}$. That is, let $k = \langle k_0, k_1, \ldots, k_r \rangle$, where $0 \leq k_i < m$.

**Randomized strategy:**

Pick $a = \langle a_0, a_1, \ldots, a_r \rangle$ where each $a_i$ is chosen randomly from $\{0, 1, \ldots, m-1\}$.

Define $h_a(k) = \sum_{i=0}^{r} a_i k_i \mod m$.  

**Dot product, modulo $m$**

How big is $\mathcal{H} = \{h_a\}$? $|\mathcal{H}| = m^{r+1}$.  

← **REMEMBER THIS!**
Universality of dot-product hash functions

**Theorem.** The set \( \mathcal{H} = \{ h_a \} \) is universal.

**Proof.** Suppose that \( x = \langle x_0, x_1, \ldots, x_r \rangle \) and \( y = \langle y_0, y_1, \ldots, y_r \rangle \) be distinct keys. Thus, they differ in at least one digit position, wlog position 0. For how many \( h_a \in \mathcal{H} \) do \( x \) and \( y \) collide?

We must have \( h_a(x) = h_a(y) \), which implies that

\[
\sum_{i=0}^{r} a_i x_i \equiv \sum_{i=0}^{r} a_i y_i \pmod{m}.
\]
Proof (continued)

Equivalently, we have

$$\sum_{i=0}^{r} a_i (x_i - y_i) \equiv 0 \pmod{m}$$

or

$$a_0 (x_0 - y_0) + \sum_{i=1}^{r} a_i (x_i - y_i) \equiv 0 \pmod{m},$$

which implies that

$$a_0 (x_0 - y_0) \equiv -\sum_{i=1}^{r} a_i (x_i - y_i) \pmod{m}.$$
Fact from number theory

**Theorem.** Let $m$ be prime. For any $z \in \mathbb{Z}_m$ such that $z \neq 0$, there exists a unique $z^{-1} \in \mathbb{Z}_m$ such that

$$z \cdot z^{-1} \equiv 1 \pmod{m}.$$ 

**Example:** $m = 7$.

<table>
<thead>
<tr>
<th>$z$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z^{-1}$</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>
Back to the proof

We have

\[ a_0(x_0 - y_0) \equiv - \sum_{i=1}^{r} a_i(x_i - y_i) \pmod{m}, \]

and since \( x_0 \neq y_0 \), an inverse \( (x_0 - y_0)^{-1} \) must exist, which implies that

\[ a_0 \equiv \left( - \sum_{i=1}^{r} a_i(x_i - y_i) \right) \cdot (x_0 - y_0)^{-1} \pmod{m}. \]

Thus, for any choices of \( a_1, a_2, \ldots, a_r \), exactly one choice of \( a_0 \) causes \( x \) and \( y \) to collide.
**Q.** How many $h_a$'s cause $x$ and $y$ to collide?

**A.** There are $m$ choices for each of $a_1, a_2, \ldots, a_r$, but once these are chosen, exactly one choice for $a_0$ causes $x$ and $y$ to collide, namely

$$a_0 = \left( - \sum_{i=1}^{r} a_i (x_i - y_i) \right) \cdot (x_0 - y_0)^{-1} \mod m.$$  

Thus, the number of $h$'s that cause $x$ and $y$ to collide is $m^r \cdot 1 = m^r \leq |\mathcal{H}|/m$

Since $|\mathcal{H}| = m^r + 1$.  

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Perfect hashing

Given a set of $n$ keys, construct a static hash table of size $m = O(n)$ such that \textsc{search} takes $\Theta(1)$ time in the \textit{worst case}.

\textbf{Idea}: Two-level scheme with universal hashing at both levels.

\textit{No collisions at level 2!}
Collisions at level 2

**Theorem.** Let $\mathcal{H}$ be a class of universal hash functions for a table of size $m = n^2$. Then, if we use a random $h \in \mathcal{H}$ to hash $n$ keys into the table, the expected number of collisions is at most $1/2$.

**Proof.** By the definition of universality, the probability that 2 given keys in the table collide under $h$ is $1/m = 1/n^2$. Since there are $\binom{n}{2}$ pairs of keys that can possibly collide, the expected number of collisions is

$$\binom{n}{2} \cdot \frac{1}{n^2} = \frac{n(n-1)}{2} \cdot \frac{1}{n^2} < \frac{1}{2}.$$
No collisions at level 2

Corollary. The probability of no collisions is at least 1/2.

Proof. Markov’s inequality says that for any nonnegative random variable $X$, we have

\[ \Pr\{X \geq t\} \leq \frac{E[X]}{t}. \]

Applying this inequality with $t = 1$, we find that the probability of 1 or more collisions is at most 1/2.

Thus, just by testing random hash functions in $\mathcal{H}$, we’ll quickly find one that works.
Analysis of storage

For the level-1 hash table $T$, choose $m = n$, and let $n_i$ be random variable for the number of keys that hash to slot $i$ in $T$. By using $n_i^2$ slots for the level-2 hash table $S_i$, the expected total storage required for the two-level scheme is therefore

$$E \left[ \sum_{i=0}^{m-1} \Theta(n_i^2) \right] = \Theta(n),$$

since the analysis is identical to the analysis from recitation of the expected running time of bucket sort. (For a probability bound, apply Markov.)