Count Min-Sketch: The Heavy Hitters Problem

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Outline

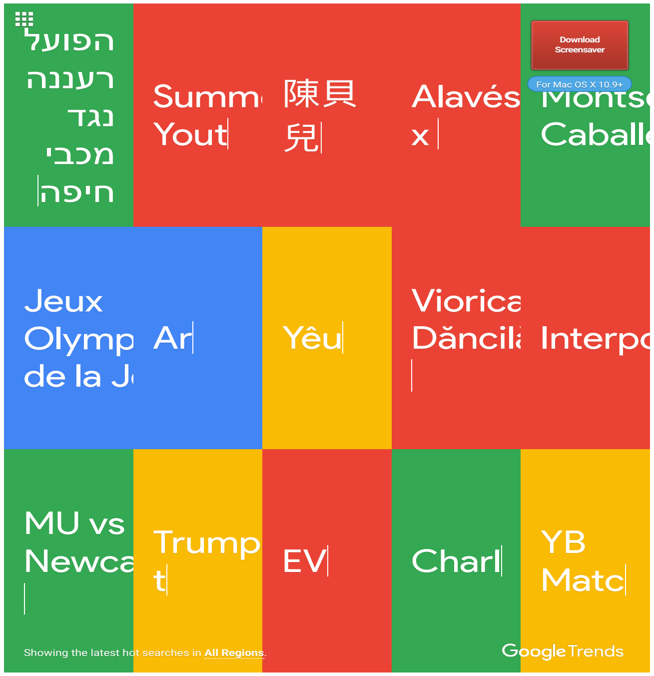
- Review Big Data Streaming Model
  - Bloom Filters
- Application: The Heavy Hitters Problem
  - (Detecting Viral Google Searches)
- Streaming Data Structure: Count Min-Sketch

Big Data

- **Problem.** Too much data to fit in memory (e.g., who can store the internet graph?

Big Data

- **Problem.** Alternatively, maybe we *could* store our data, but it would take too long to process it, and we want a real time (or near real time) application.



Streaming Model

- **Solution.** In the **streaming model** of computation, we process the data one piece at a time, with limited memory.

- **Equivalently:** we develop algorithms that run in a ***single*** **left to right pass over an array, with a small amount of auxiliary storage.**

Football	Duke	Politics	News	…	Weather
0	1	2	3	…	T

Auxiliary Storage of size n
(n << T)

Bloom Filter

- We have already seen how to construct a *bloom filter,* a form of *lossy compression* (as opposed to lossless compression, e.g., Huffman).

- Answers *membership queries*; i.e., “Have I seen element x before in the stream?”

- Applications include:
  - Web browser checking for known malicious urls
  - Checking for “one hit wonders” in web caching (remember consistent hashing?)

Bloom Filter

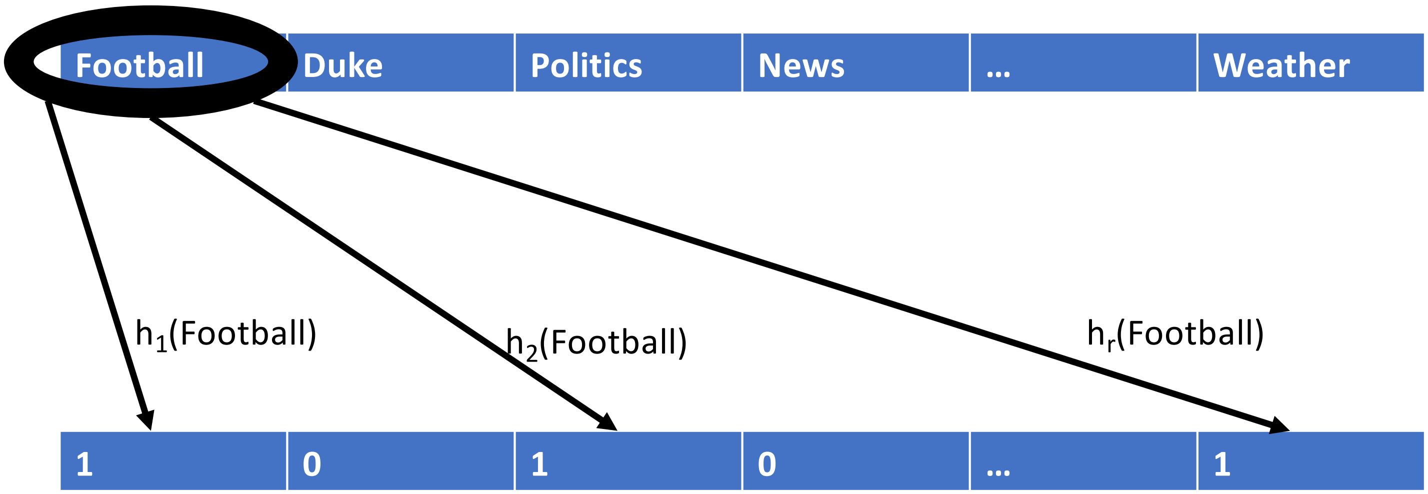
- Our auxiliary storage is just a hash table of size n. Initialize all values to 0.

- We also use r independent hash functions $h_1, \ldots, h_r$.

- Whenever we see an element x in the stream, set $h_1(x) = \ldots = h_r(x) = 1$.

- To check whether we have seen an element y:
  - If $h_1(y) = \ldots = h_r(y) = 1$, return True.
  - Else, return False.

Bloom Filter



Bloom Filter

- Guarantees:
  - If we *have* seen x, we always correctly output True.
  - If we *have not* seen x, we correctly output False with high probability.

- What if we want to remember more than just whether we have seen x?

- How about “How many times have we seen x?”

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Heavy Hitters Problem

- In particular, suppose we want to construct an algorithm for detecting viral google searches.
- There are a few billion google searches every day, and we'll say that a search is viral if it constitutes a constant fraction of those searches (e.g., 1%).
- Can we detect these viral google searches with a *single* pass over the stream of searches?

Heavy Hitters Problem

- We can formalize this as the **heavy hitters problem**.
- We are given a stream of length T and a parameter k.
  - Think of T >> k.
- In a single pass over the stream, we want to find any elements that appear at least T/k times.

Heavy Hitters Problem

- Bloom filters gets us part of the way there.
- In particular, if we had k=T, the heavy hitters problem is the membership problem.
- Thus, the heavy hitters problem is *at least* as hard (computationally, more on reductions later in the course) as the membership problem.
- Since we only had a correct algorithm with high probability for membership, we shouldn’t expect an exact answer here.

Heavy Hitters Problem

- Thus, we consider the **ϵ-approximate heavy hitters problem.** Still given a stream of length T and a parameter k (T >> k), but we are also given an "error tolerance" parameter ϵ.

- In a single pass over the stream using just O(1/ϵ) auxiliary storage, we want to output a list L of elements such that:
  - If x occurs at least T/k times in the stream, then x is in L.
  - If x is in L, then with high probability, x occurs at least T/k - ϵT times in the stream.
  - (e.g., if ϵ = 1/(2k), then we get O(k) storage and should satisfy: if x is in L, with high probability, x occurs at least T/2k times in the stream).

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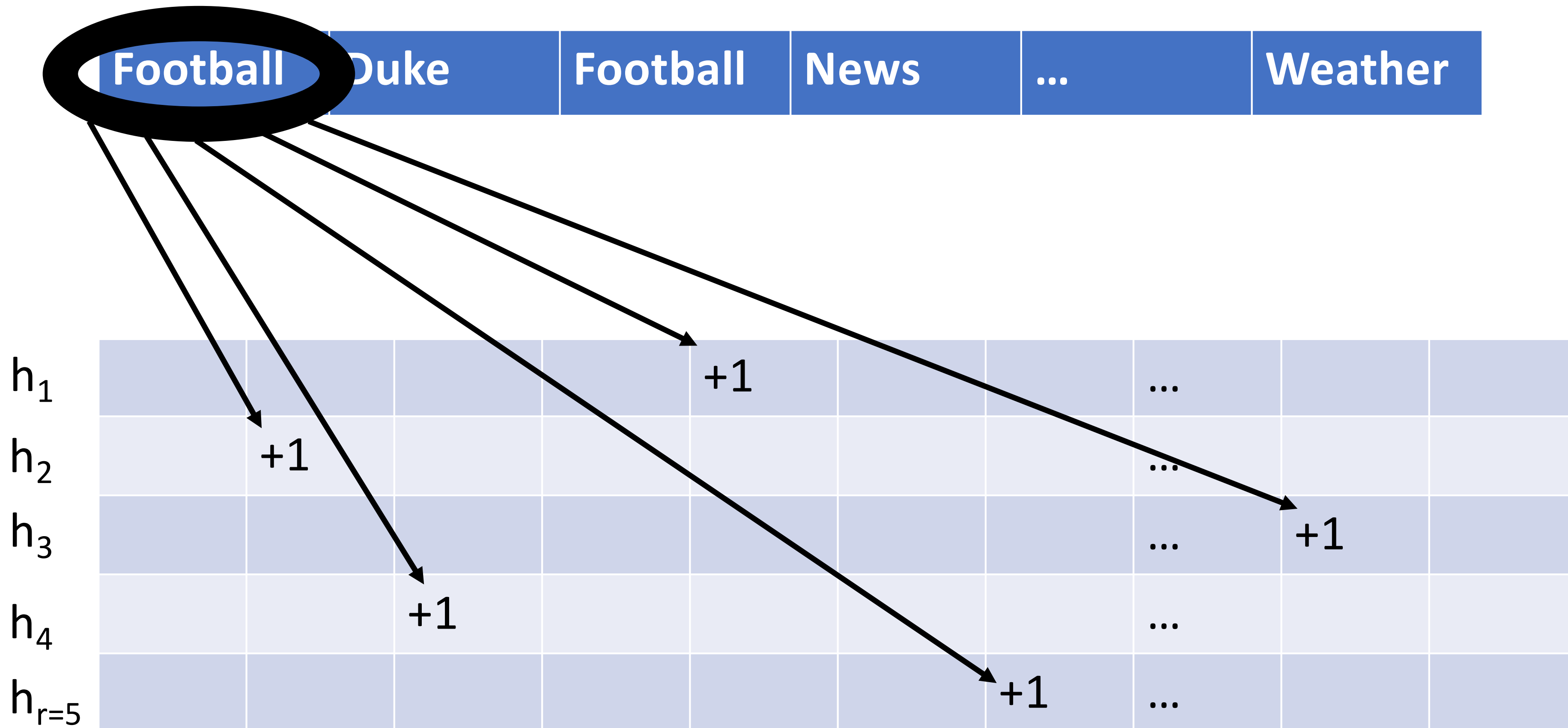
Count Min-Sketch

- **Big Idea.** Just build **a bloom filter that can count**.

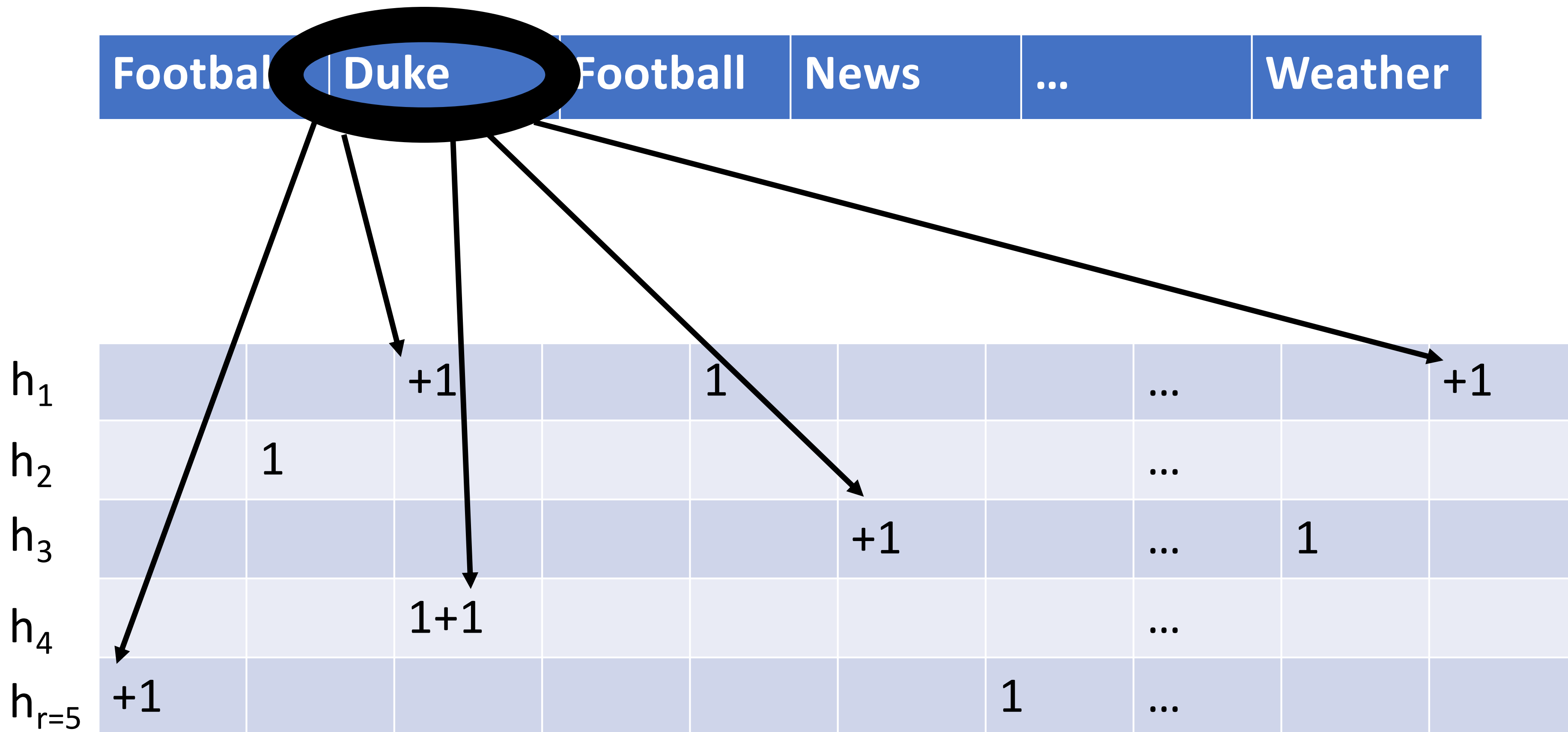
- Our auxiliary storage consists of *r* hash tables, each of size *n* and initialized to 0's, with corresponding *r* independent hash functions h_1, ..., h_r.

- Whenever we see an element x in the stream:
  - For all i=1 to i=r: $\{h_i(x) = h_i(x) + 1\}$
  - if $\min_i h_i(x) \geq T/k$, add x to L.

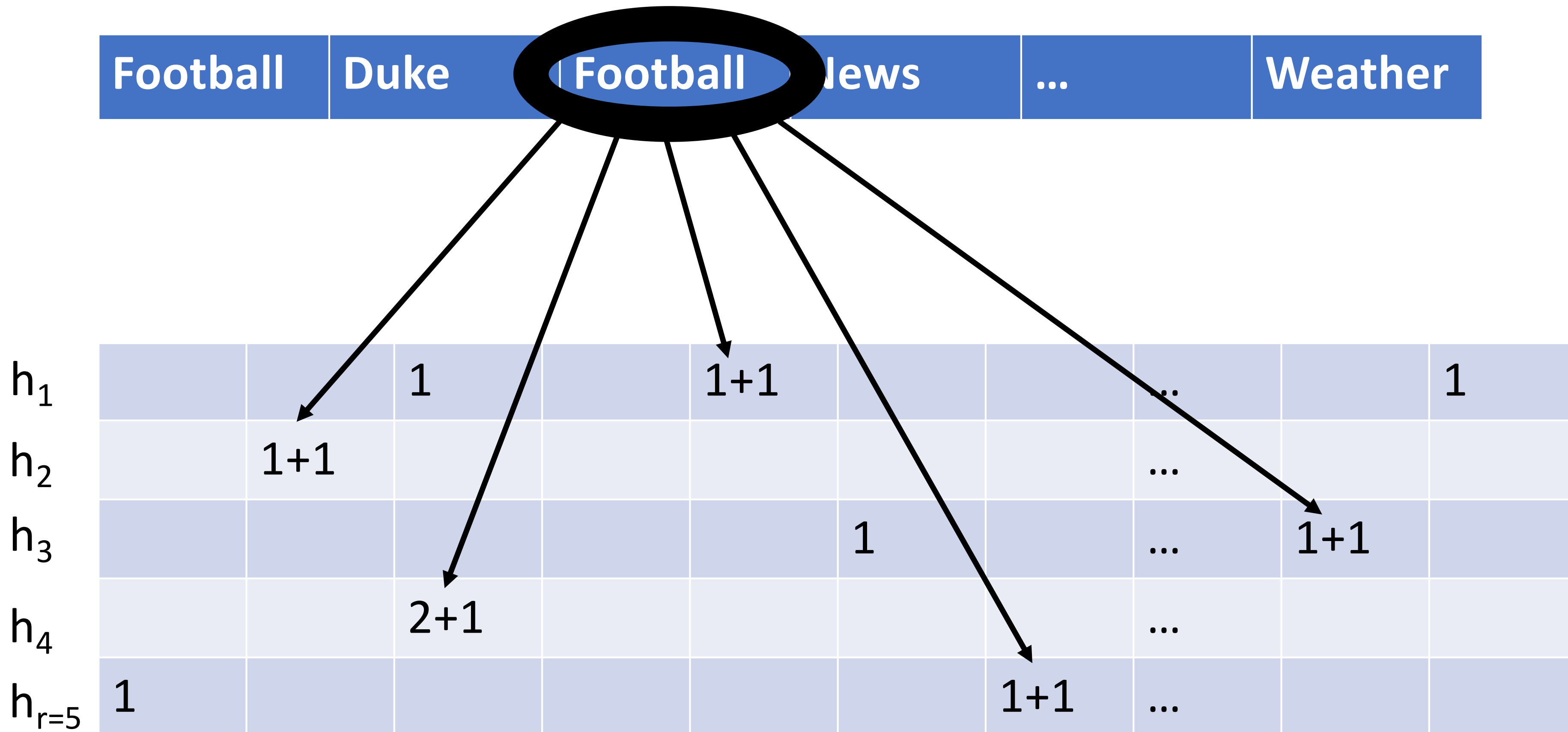
Count Min-Sketch



Count Min-Sketch



Count Min-Sketch



Count Min-Sketch

- Note that we occasionally *overestimate* frequencies, but we **never *underestimate* frequencies**.
- So it is easy to satisfy the first part of the heavy hitter's problem: "If x occurs at least T/k times in the stream, then x is in L."
- **Problem.** We need to argue that it is **unlikely we overestimate so badly** that we violate the other part: "If x is in L, then with high probability, x occurs at least T/k - ϵT times in the stream."

Count Min-Sketch

- Let f_x be the **frequency (# of times in stream) of element *x***.
- Let $\widehat{f_x}[1], \dots, \widehat{f_x}[r]$ be our **estimated frequencies,** that is, $\widehat{f_x}[i] = h_i(x)$ at the end of our pass through the stream.
- Let $I_{x,y}[i]$ be an **indicator random variable**:
  - $I_{x,y}[i]$ = 1 if $h_i(x) = h_i(y)$, and 0 otherwise.
- What is $\mathbb{E}\left[\widehat{f_x}[i]\right]$?

Count Min-Sketch

- We make the assumption of *universal hashing*:
  - For all $x \neq y, \Pr\big(h(x) = h(y)\big) \leq \frac{1}{n}$.

Hence
$$\mathbb{E}\left[\widehat{f_x}[i]\right] = f_x + \mathbb{E}\left[\sum_{y \neq x} f_y \times I_{x,y}[i]\right]$$

$$= f_x + \sum_{y \neq x} f_y \, \mathbb{E}[\, I_{x,y}[i]]$$

$$= f_x + \sum_{y \neq \mathrm{x}} \frac{f_y}{n} \leq \; f_x + \frac{T}{n}$$

Count Min-Sketch

- Recall we want to use O(1/ϵ) storage:
- Set the size of each hash table to n = 3/ϵ.
- Let ϵ = 1/(2k).
- Then

$$\mathbb{E}\left[\widehat{f_x}[i]\right] \leq f_x + \frac{T}{n}$$

$$\leq f_x + \epsilon \frac{T}{3}$$

$$= f_x + \frac{T}{6k}.$$

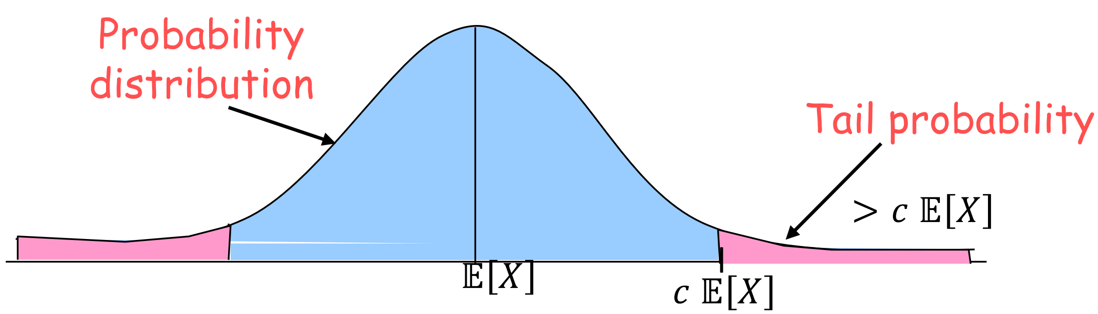
Count Min-Sketch

- Recall we want to use O(1/ϵ) storage:
- Set the size of each hash table n = 3/ϵ and let ϵ = 1/(2k).
- Then we have shown:

$$\mathbb{E}\left[\widehat{f_x}[i]\right] \leq f_x + \frac{T}{6k}.$$

- To bound the probability that we get a large overestimate, we can use Markov's inequality: For any constant c > 1 and random variable X,

General bounds on *tail probability* of a random variable (that is, probability that a random variable deviates far from its expectation)



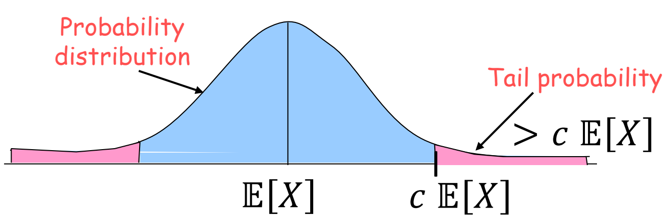
Count Min-Sketch

- Recall we want to use $O(1/\epsilon)$ storage: Set the size of each hash table n = $3/\epsilon$ and let $\epsilon = 1/(2k)$.
- Then we have shown:

$$\mathbb{E}\left[\widehat{f_x}[i]\right] \leq f_x + \frac{T}{6k}.$$

- To bound the probability that we get a large overestimate, we can use Markov's inequality: For any constant c > 1 and random variable X, $\Pr(X > c\,\mathbb{E}[X]) \leq \frac{1}{c}$.

General bounds on *tail probability* of a random variable (that is, probability that a random variable deviates far from its expectation)



So we have show:

For c = 3/2, $\Pr\left(\widehat{f_x}[i] > \frac{3}{2}\,\mathbb{E}\left[\widehat{f_x}[i]\right] = \frac{3}{2} f_x + \frac{T}{4k}\right) \leq \frac{2}{3}$.

Count Min-Sketch

- We have shown: $\Pr\left(\widehat{f_x}[i] > \frac{3}{2}\,\mathbb{E}\left[\widehat{f_x}[i]\right] = \frac{3}{2}f_x + \frac{T}{4k}\right) \leq \frac{2}{3}$.

- Recall however, that we output the *minimum* estimate.
- Since the *r* hash functions are chosen *independently*:

$$\Pr\left(\min_i \widehat{f_x}[i] > \frac{3}{2}\,\mathbb{E}\left[\widehat{f_x}[i]\right]\right) = \Pr\left(\forall i,\ \widehat{f_x}[i] > \frac{3}{2}\,\mathbb{E}\left[\widehat{f_x}[i]\right]\right)$$

$$= \prod_{i=1}^{r} \Pr\left(\widehat{f_x}[i] > \frac{3}{2}\,\mathbb{E}\left[\widehat{f_x}[i]\right]\right)$$

$$\leq \left(\frac{2}{3}\right)^r$$

Count Min-Sketch

- Recall that the Heavy Hitter's Problem for ϵ = 1/(2k) is:
  - we get O(k) storage and should satisfy:
  - if x is in L, with high probability, x occurs at least T/(2k) times in the stream.

- Consider some x with $f_x < \frac{\mathrm{T}}{2\mathrm{k}}$. We have shown that

$$\Pr\left(\min_i \widehat{f_x}[i] > \frac{3T}{4k} + \frac{T}{4k} = \frac{T}{k}\right) \leq \left(\frac{2}{3}\right)^r.$$

- So if x is in L, then it occurs at least T/(2k) times in the stream with probability at least:
  1- $\left(\frac{2}{3}\right)^r$.

- So if we want an error with probability at most 2% (say), we just need to use $r = \lceil \log_{3/2}(50) \rceil = 10$ independent hash functions.

Count Min-Sketch

- In summary, we can use $20/\epsilon = O(1/\epsilon)$ space to:
  - find *all* elements that appear at least T/k times in the stream, and
  - output elements that appear less than T/(2k) times in stream with probability at most 2%.
  - And in practice, even fewer hash functions often suffice for good performance.
- Note that we can do all of this with just a *single* linear scan over the stream (and only constant time operations per element), and just $O(1/\epsilon)$ storage.
  - The amount of auxiliary storage we use is *completely* independent of T!

Count Min-Sketch

- **Food for Thought.** What if you didn't know T beforehand?
  - Maybe this is just a real time application, and you want to maintain a list of any elements that are heavy hitters among what you have seen so far.