Spectral Clustering

PPT by Brandon Fain
Outline

• Review
  • Community Detection Problem
  • Conductance
  • Graph Laplacian

• Spectral techniques to find low conductance cuts

• Spectral techniques for clustering and community detection
Motivating Problem: Community Detection

Given a social network, how do you find the strongly connected communities?

Corollary question: How would you suggest friends to a user?
Conductance

• Let $G = (V, E)$ be an undirected graph.
• $S \subseteq V$ denote a cut in the graph.
• Let $\delta(S) := |\{(u, v) \in E: u \in S, v \notin S\}|$.
• Let $Vol(S) = \sum_{i \in S} d_i$, where $d_i$ is the degree of node $i$.
• The **conductance** of $S$ is
  \[
  \phi(S) = \frac{\delta(S)}{\min(Vol(S), Vol(V - S))}.
  \]
• We want to find a low conductance cut: one with many more internal edges than cut edges.
Laplacian Matrix

• The graph Laplacian is defined as
  \[ L = D - A \]
  where \( D \) is the diagonal matrix with \( D_{ii} = d_i \) and \( D_{ij} = 0 \) for \( i \neq j \), and \( A \) is the adjacency matrix.

• Recall \( (Lv)_i = \sum_{j:(i,j)\in E} v_i - v_j \).

• Last time, we observed that the orthogonal eigenvectors corresponding to eigenvalues of 0 told us the connected components of the graph.

• Our intuition was that the eigenvectors for the smallest non-zero eigenvalues should tell us something about low conductance cuts.
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Spectral Algorithm for Low Conductance Cut

- Let $\lambda$ be the smallest non-zero eigenvalue of the graph Laplacian $L$ with corresponding eigenvector $\vec{v}$.
- Sort the vertices $i$ in non-decreasing order of $v_i$. For notational convenience, say that after sorting: $v_1 \leq v_2 \leq \cdots \leq v_n$.
- For $i$ from 1 to $n-1$:
  - $S_i \leftarrow \{1, 2, \ldots, i\}$
  - $C_i \leftarrow \phi(S_i)$
- Return $S_i$ with minimum $C_i$. 
Spectral Algorithm Analysis

• **Efficiency.** Note that the brute force algorithm for the problem considers $2^n$ cuts.
• Clearly, this algorithm considers $O(n)$ cuts.
• You need to calculate conductance at each step (potentially an $\Omega(m)$ calculation). Can you see how to avoid this?

• **Accuracy.** How “correct” is the algorithm?
• This is an NP-Complete problem, so this won’t solve it exactly (i.e., no guarantee of minimum conductance cut). How close do we get?
Spectral Algorithm Analysis

• We will analyze the case of a \textbf{d-regular graph}, that is, one for which every vertex has degree exactly \(d\).
  • (This makes the statement and proof easier, but a similar statement holds for non-regular graphs).
• Then the conductance can be rewritten as
  \[
  \phi(S) = \frac{\delta(S)}{d \cdot \min(|S|, |V - S|)}.
  \]

Suppose w.l.o.g. we just consider cuts where \(|S| \leq |V - S|\). Define
  \[
  \theta(S) = \frac{\delta(S)}{|S|}.
  \]

Then the minimum conductance cut of a graph also minimizes \(\theta(S)\). Call this minimum \(\theta(S)\) value \(\Theta_G\) for a graph \(G\).
Spectral Algorithm Analysis

• **Theorem (Cheeger’s Inequality).** Let $G$ be a $d$-regular connected graph with minimum conductance $\frac{\Theta_G}{d}$. Let $S$ be the cut found by our spectral algorithm. Let $\lambda_2$ be the second smallest eigenvalue of the graph Laplacian of $G$. Then

$$\frac{\lambda_2}{2} \leq \Theta_G \leq \theta(S) \leq \sqrt{2d\lambda_2}.$$

• **Corrolary.**

$$\frac{\theta(S)}{\Theta_G} \leq \frac{\sqrt{2d\lambda_2}}{\Theta_G} \leq \frac{2\sqrt{2d}}{\sqrt{\lambda_2}}$$

This is a fairly pessimistic bound on typical performance in practice.
Spectral Algorithm Analysis

• Proving $\theta(S) \leq \sqrt{2d\lambda_2}$ is difficult, and we don’t have the time.
• Proving $\frac{\lambda_2}{2} \leq \Theta_G \leq \theta(S)$ is relatively easy.
• Note that $\Theta_G \leq \theta(S)$ is by definition, so we really only need to prove $\frac{\lambda_2}{2} \leq \Theta_G$.

• **Proof Sketch.** Recall that for any eigenvector $v$ with eigenvalue $\lambda$, $Lv = \lambda v$. Therefore

$$\frac{v^T L v}{v^T v} = \frac{v^T (\lambda v)}{v^T v} = \lambda$$
• **Proof Sketch** (continued). We have already seen that for a connected graph, the all 1 vector is an eigenvector for eigenvalue 0.

• The second smallest eigenvalue $\lambda_2$ has an eigenvector that is orthogonal to this all 1 vector, and in particular:

$$\lambda_2 = \min_{v: v \cdot \vec{1} = 0} \frac{v^T L v}{v^T v}.$$

• Consider a cut $S$, and define the vector $v_i = 1 - |S|/|V|$ for $i \in S$ and $-|S|/|V|$ otherwise. For every $S$, this vector is orthogonal to $\vec{1}$.

• Furthermore, if you work out the algebra,

$$\frac{v^T L v}{v^T v} = \frac{\delta(S)}{|S| \cdot |V - S|/|V|} = \frac{\delta(S)}{|S| \cdot (1 - |S|/|V|)} \leq 2 \frac{\delta(S)}{|S|}.$$

• Then $\lambda_2$ is at most $2\delta(S)/|S|$.

• Since this holds for any cut, it holds for the minimum cut, so $\lambda_2 \leq 2\Theta_G$. 


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Further Questions

• What if you want to partition your data into more than 2 clusters?

• What if you want to detect the community of an individual, rather than just a good community globally in the graph?

• What if your data isn’t actually a graph to begin with?

• We will conclude with some heuristic spectral approaches for these problems.
More Than 2 Clusters

• Suppose we want to partition the data into $k$ clusters. A common approach is as follows:

• Represent each vertex $i$ as a length $m$ vector, where:
  • The $j$’th component of the vector is the $i$’th entry in the eigenvector of the graph Laplacian corresponding to the $j+1$ smallest eigenvalue.
  • For example, suppose we set $m=2$, and $<5, -1, 3, -2>$ is the eigenvector corresponding to the 2$\text{nd}$ smallest eigenvalue, and $<-1, 5, 0, 0>$ is the eigenvector corresponding to the 3$\text{rd}$ smallest eigenvalue.
  • Then we would represent the first vertex as $<5, -1>$, the second as $<-1, 5>$, the third as $<3, 0>$ and the fourth as $<-2, 0>$.

• Now, run a standard clustering algorithm (e.g., k-means) on these vectors.
Community Detection

• Suppose we have an individual $i$, and we know that she belongs to a community with between $n_1$ and $n_2$ individuals. We want to predict who those individuals are.

• One heuristic is as follows:
  • Represent each individual as a vector according to the eigenvectors corresponding to small (but non-zero) eigenvalues, exactly as in the last slide.
  • Let $d(x,y)$ be a distance function on these vectors (e.g., standard Euclidean distance).
  • For $n$ from $n_1$ to $n_2$:
    • Let $S_i$ be the $n$ individuals with minimum distance to $i$.
    • $C_i \leftarrow \phi(S_i)$
    • Return the $S_i$ with minimum $C_i$
Non Graphical Data

• What if your data wasn’t a graph to begin with? For example, if you wanted to cluster something like:

![Graphs](image)

(a) K-means  
(b) Spectral Clustering

• Just create a graph by setting points that are sufficiently close to one another to be adjacent vertices.
• Then run your favorite spectral analysis.
Summary

• There are deep connections between the eigenvalues and eigenvectors of the graph Laplacian and the connectivity properties of a graph.

• For clustering problems where you care about connectivity, spectral clustering, exploiting these properties, is the standard approach.

• It is useful for minimum conductance cuts and community detection problems on graphs, but it can also be applied to non-graphical data.

• In your last lab homework, you will play around with spectral techniques on an email graph.