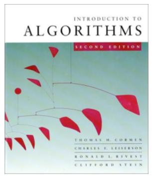
Introduction to Algorithms

6.046J/18.401J



Lecture 25

Prof. Piotr Indyk

Final Exam

- May 19, 2008, 9:00am - 12 pm
- Johnson Ice Rink
- Closed book:
  - two handwritten crib sheets
- Coverage: everything except
  - L17 Hidden Markov Models II
  - L18 Computational Biology
  - L26 Parallel Algorithms
- Quiz review: Fri, 3-5pm, 32-155

Dealing with Hard Problems

- What to do if:
  - Divide and conquer
  - Dynamic programming
  - Greedy
  - Linear Programming/Network Flows
  - …

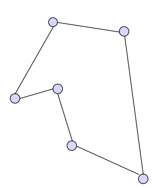
  does not give a polynomial time algorithm?

Dealing with Hard Problems

- Idea I: Ignore the problem
  - Can't do it ! There are thousands of problems for which we do not know polynomial time algorithms
  - For example:
    - Traveling Salesman Problem (TSP)
    - Set Cover

Traveling Salesman Problem

- Traveling Salesman Problem (TSP)
  - Input: undirected graph with lengths on edges
  - Output: shortest cycle that visits each vertex exactly once
- Best known algorithm: $\Omega(2^n)$ time



Set Covering

- Set Cover:
  - Input: subsets $S_1 \ldots S_n$ of X, $\cup_i S_i = X$, $|X|=m$
  - Output: $C \subseteq \{1 \ldots n\}$, such that $\cup_{i \in C} S_i = X$, and $|C|$ minimal
- Vertex cover: special case
  - X = edges
  - S_v = edges incident to vertex v
- Best known algorithm: $\Omega(2^n)$ time

App: bank robbery
- X={plan, shoot, safe, drive, scary}
- Sets:
  - S_{Steve}={plan, safe}
  - S_{Stevie}={shoot, scary, drive}
  - S_{Stevo}={plan, drive}
  - ….

Dealing with Hard Problems, ctd.

- Exponential time algorithms for small inputs
  - E.g., 1.274^n time is not bad for $n < 50$ (such algorithm exists for Vertex Cover)
- Polynomial time algorithms for some inputs (e.g., average-case)
- Polynomial time algorithms for all inputs, but which return approximate solutions

Approximation Algorithms

- An algorithm A is ρ-approximate, if, on any input of size n:
  - The cost C_A of the solution produced by the algorithm, and
  - The cost C_{OPT} of the optimal solution

  are such that $C_A \le \rho\ C_{OPT}$
- We have seen:
  - 2-approximation algorithm for finding a median string (PS7)
- We will see:
  - 2-approximation algorithm for TSP in the plane
  - ln(m)-approximation algorithm for Set Cover

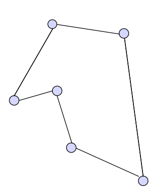
Comments on Approximation

- “ $C_A \le \rho\ C_{OPT}$ “ makes sense only for minimization problems
- For maximization problems, replace by “$C_A \ge 1/\rho\ C_{OPT}$ “
- Additive approximation “$C_A \le \rho + C_{OPT}$ “ also makes sense, although difficult to achieve

TSP

2-approximation for TSP

- Compute MST T
  - An edge between any pair of points
  - Weight = distance between endpoints
- Compute a tree-walk W of T
  - Each edge visited twice
- Convert W into a cycle C using shortcuts



2-approximation: Proof

- Let C_{OPT} be the optimal cycle
- $\mathrm{Cost}(T) \le \mathrm{Cost}(C_{OPT})$
  - Removing an edge from C gives a spanning tree, T is a spanning tree of minimum cost
- $\mathrm{Cost}(W) = 2\ \mathrm{Cost}(T)$
  - Each edge visited twice
- $\mathrm{Cost}(C) \le \mathrm{Cost}(W)$
  - Triangle inequality

$\Rightarrow \mathrm{Cost}(C) \le 2\ \mathrm{Cost}(C_{OPT})$

Set Cover

Approximation for Set Cover

- Input: subsets $S_1 \ldots S_n$ of X, $\cup_i S_i = X$, $|X|=m$
- Output: $C \subseteq \{1 \ldots n\}$, such that $\cup_{i \in C} S_i = X$, and $|C|$ minimal

Greedy algorithm:

- Initialize $C=\emptyset$
- Repeat until all elements are covered:
  - Choose S_i which contains largest number of yet-not-covered elements
  - Add i to C
  - Mark all elements in S_i as covered

Greedy Algorithm: Example

- $X=\{1,2,3,4,5,6\}$
- Sets:
  - $S_1=\{1,2\}$
  - $S_2=\{3,4\}$
  - $S_3=\{5,6\}$
  - $S_4=\{1,3,5\}$
- Algorithm picks C=all sets
- Not optimal!

ln(m)-approximation

- Notation:
  - C_{OPT} = optimal cover
  - $k=|C_{OPT}|$ (we do not know k)
- Fact: At any iteration of the algorithm, there exists S_i which contains at least $1/k$ fraction of yet-not-covered elements
- Proof:
  - C_{OPT} covers the (uncovered) elements using k sets
  - One of those sets must cover $\geq 1/k$ fraction of yet-not-covered elements
- Conclusion: greedy algorithm covers $\geq 1/k$ fraction of yet-not-covered elements in each step

ln(m)-approximation

- Let u_i be the number of yet-not-covered elements at the end of step $i=0,1,2,\ldots$
- We have

$$u_{i+1} \leq u_i (1-1/k)$$
$$u_0=m$$

- Therefore, after $t=k \ln m$ steps, we have
  $u_t \leq u_0 (1-1/k)^t \leq m (1-1/k)^{k \ln m} < m\ 1/e^{\ln m} =1$
- I.e., all elements are covered by the $k \ln m$ sets chosen by greedy algorithm
- Opt size is $k \Rightarrow$ greedy is $\ln(m)$-approximate

Approximation Algorithms

- Very rich area
  - Algorithms use greedy, linear programming, dynamic programming
    - E.g., 1.01-approximate TSP in the plane
  - Sometimes can show that approximating a problem is as hard as finding exact solution !
    - E.g., 0.99 ln(m)-approximate Set Cover