Convex Hull

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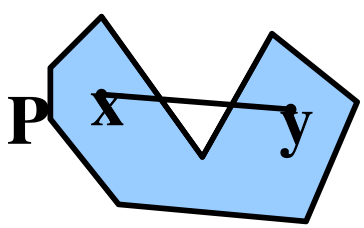
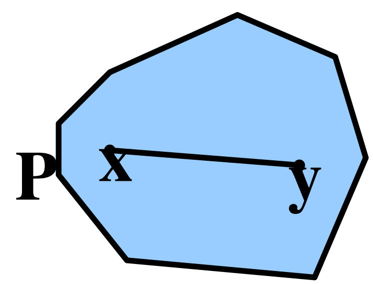
Convex vs. Concave

- A polygon P is convex if for every pair of points x and y in P, the line xy is also in P; otherwise, it is called concave.



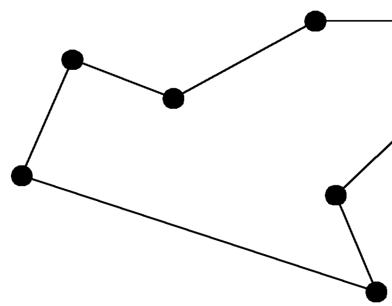
concave



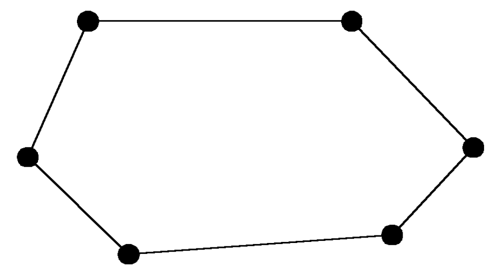
convex

The convex hull problem

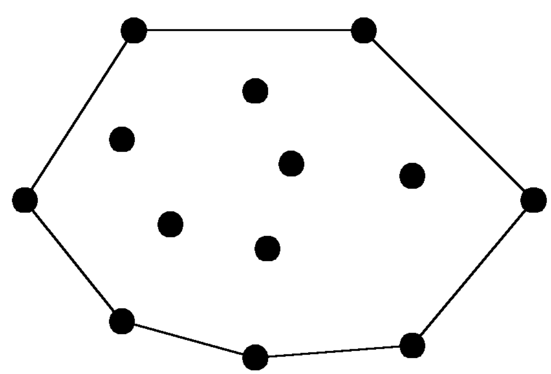
concave polygon:



convex polygon:



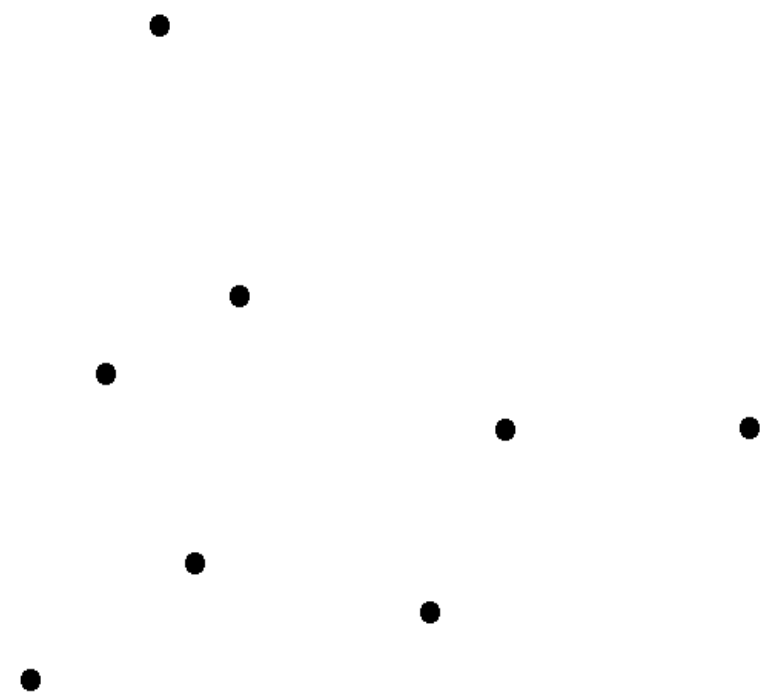
- The convex hull of a set of planar points is the smallest convex polygon containing all of the points.



Graham's Scan

- Start at point guaranteed to be on the hull. (the point with the minimum y value)
- Sort remaining points by polar angles of vertices relative to the first point.
- Go through sorted points, keeping vertices of points that have left turns and dropping points that have right turns.

Graham's Scan



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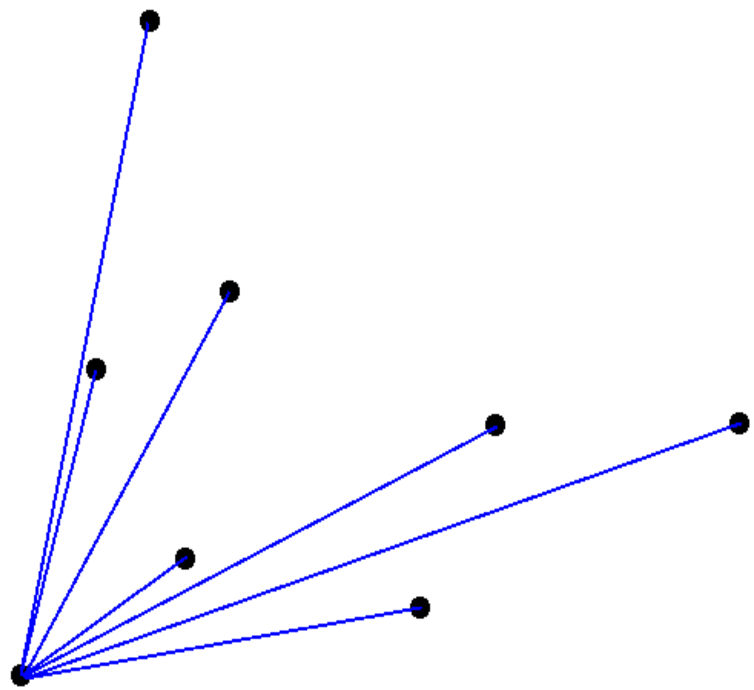

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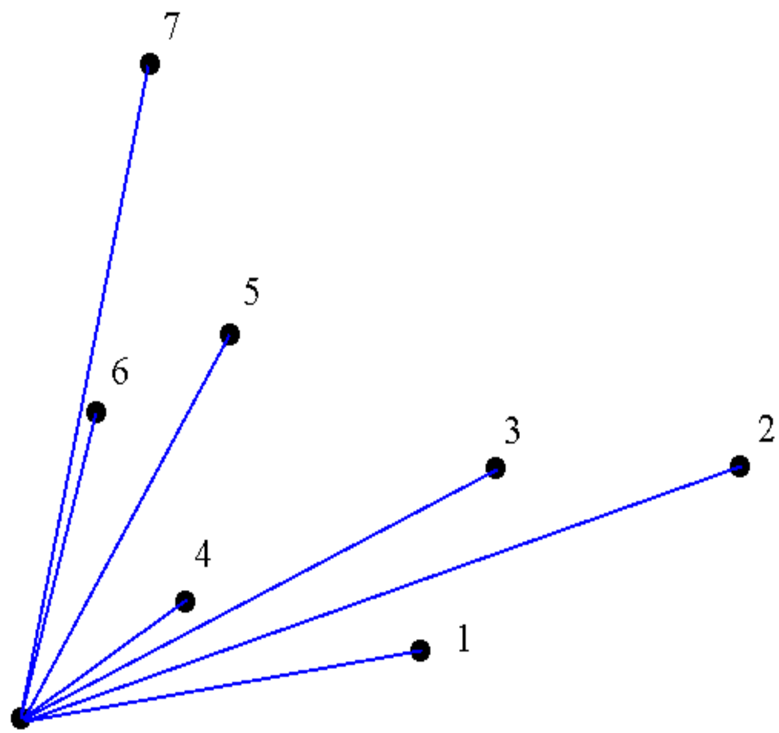

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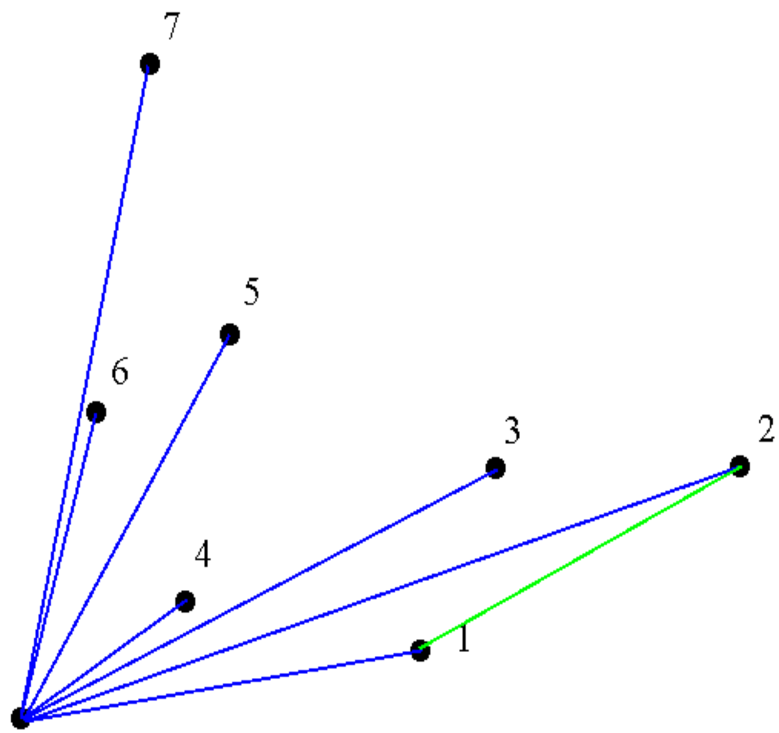

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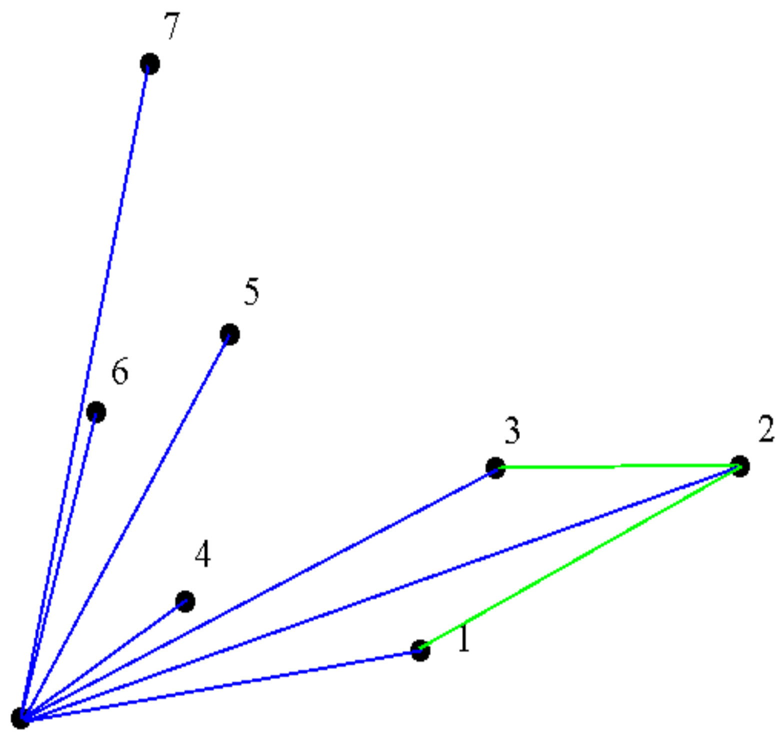

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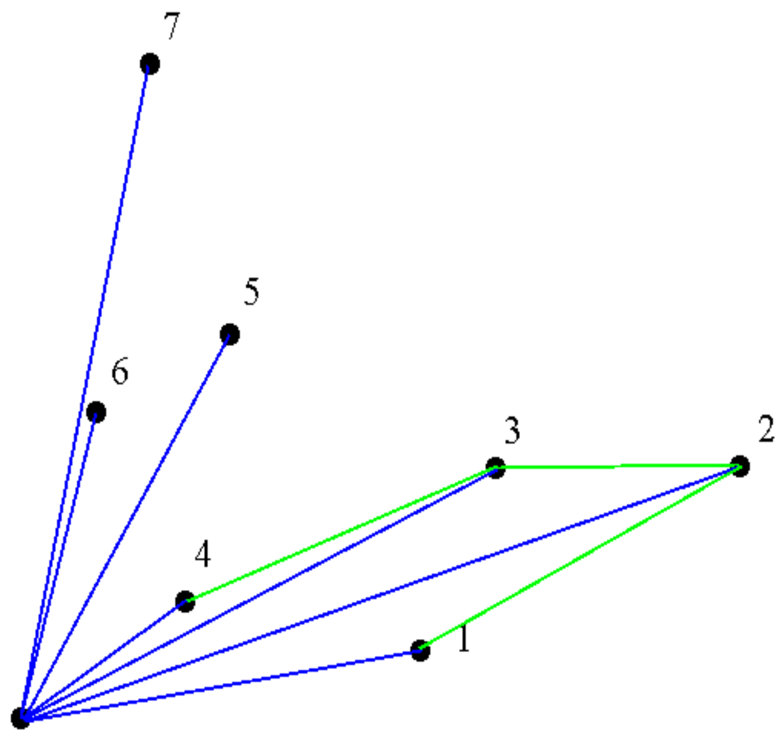

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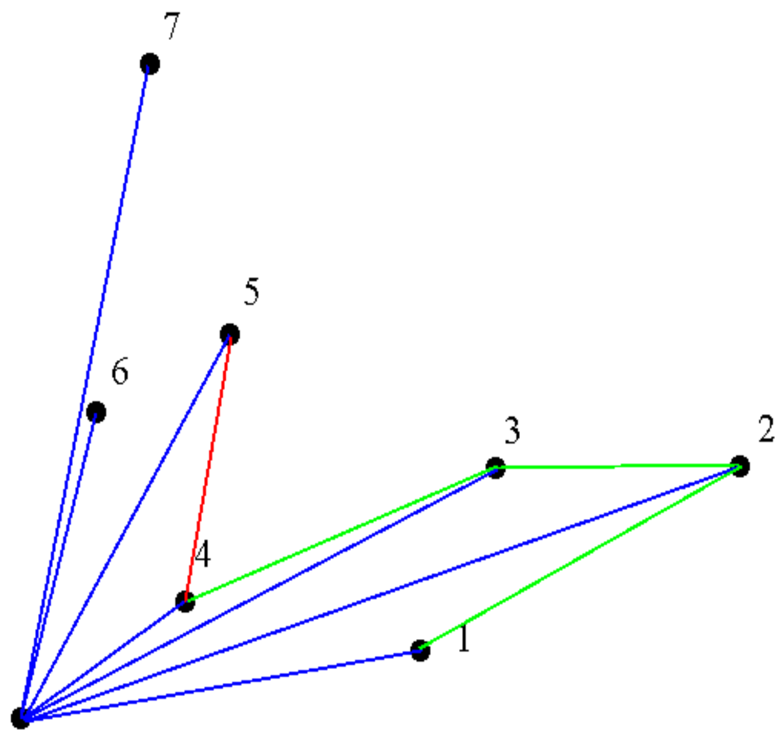

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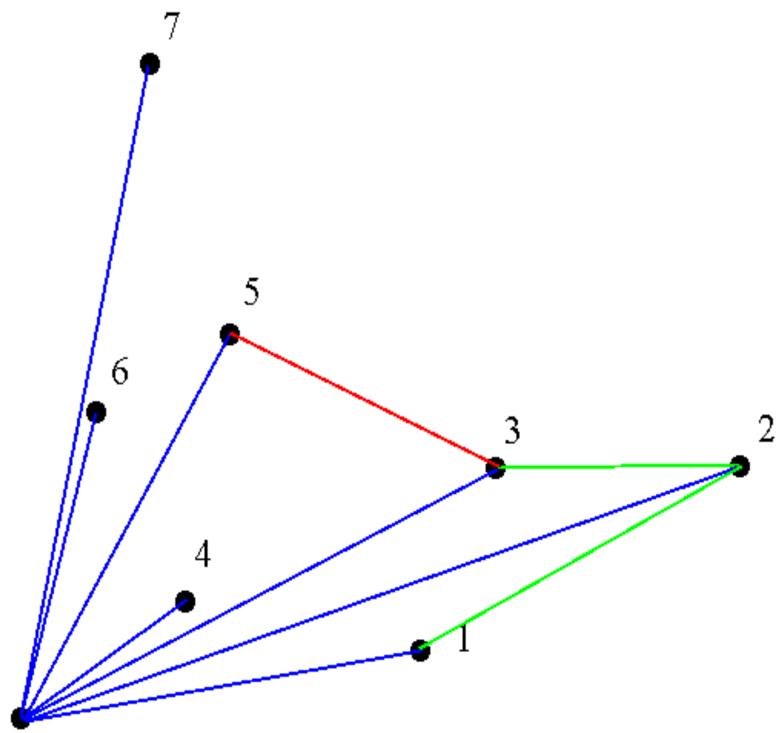

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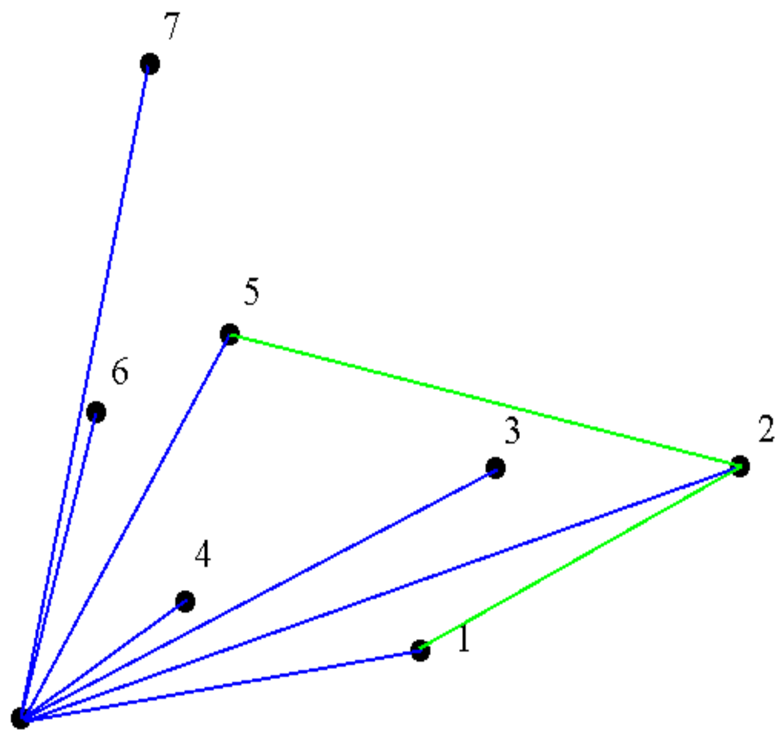

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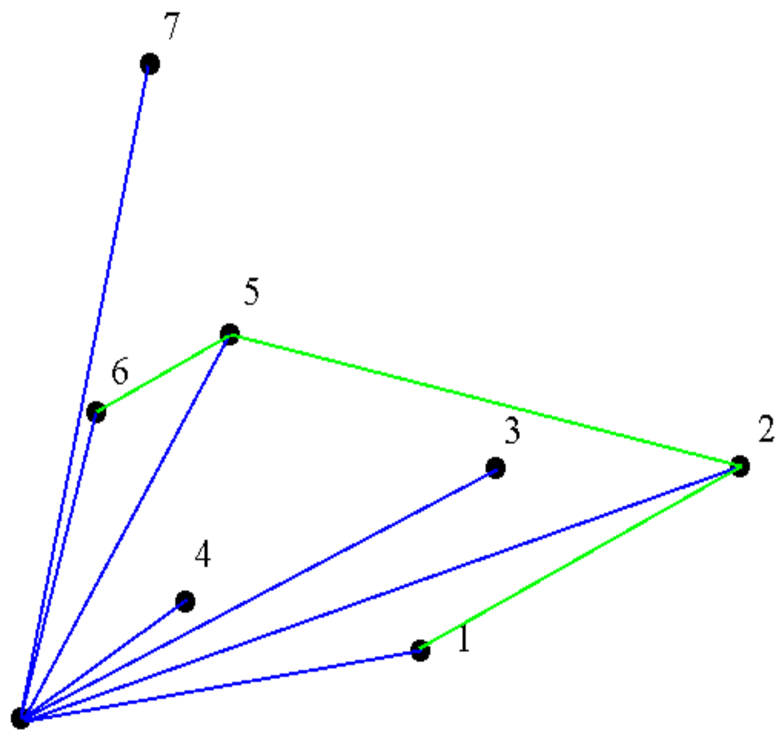

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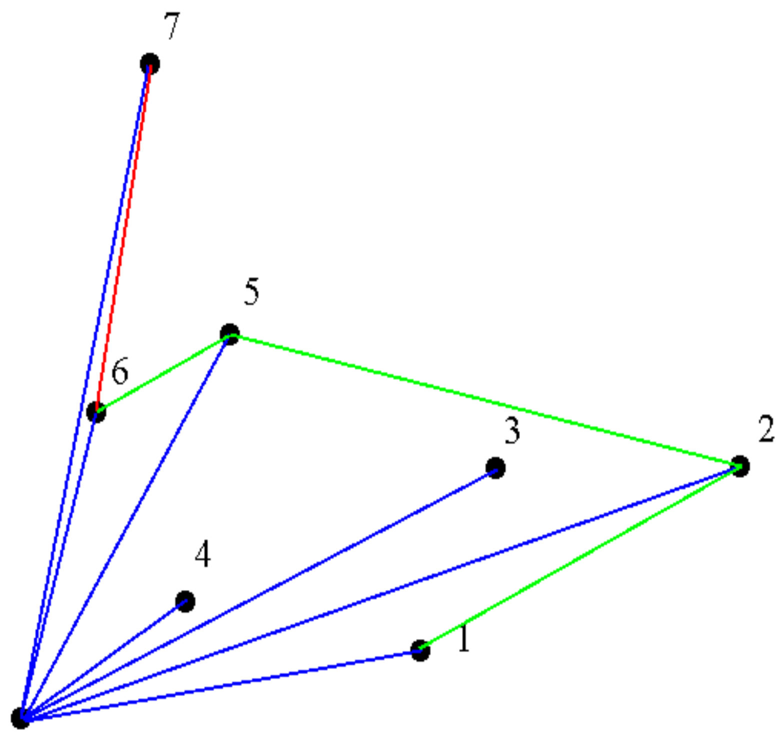

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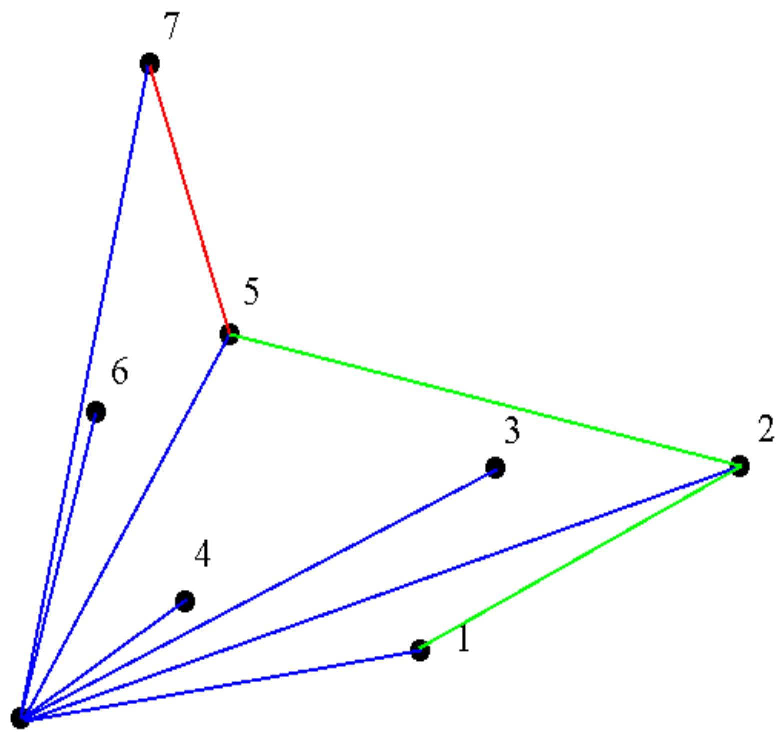

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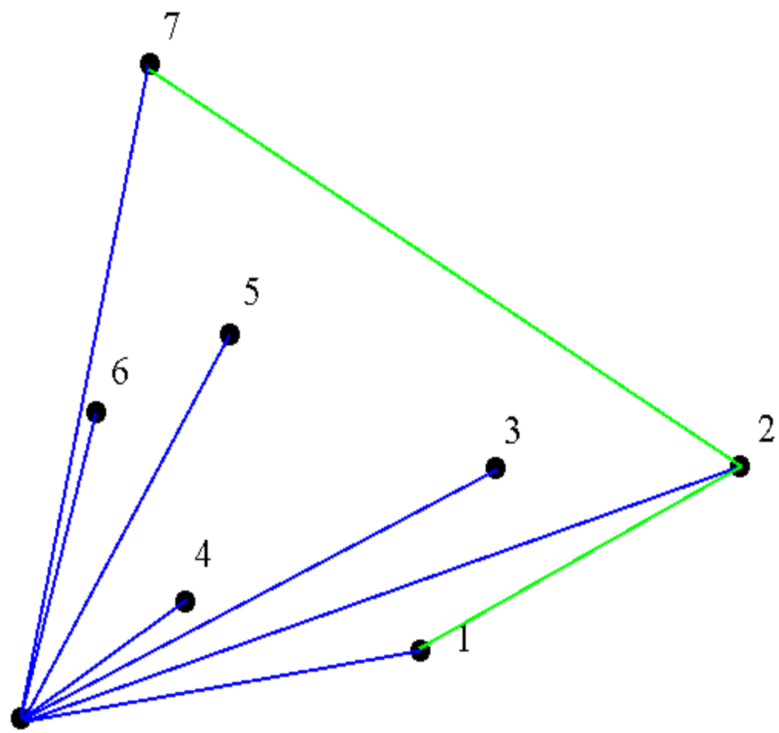

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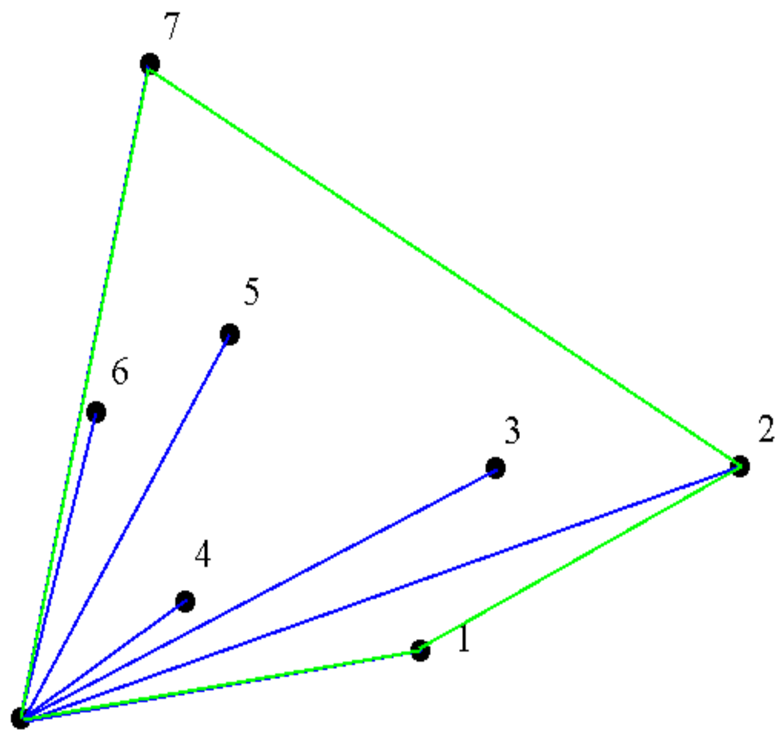


Graham’s Runtime

- Graham’s scan is O(n log n) due to initial sort of angles.

■A more detailed algorithm

GRAHAM-SCAN(Q)

let p_0 be the point in Q with the minimum y-coordinate,
    or the leftmost such point in case of a tie
let $\langle p_1, p_2, \ldots, p_m \rangle$ be the remaining points in Q,
    sorted by polar angle in counterclockwise order around p_0
    (if more than one point has the same angle, remove all but
    the one that is farthest from p_0)
PUSH(p_0, S)
PUSH(p_1, S)
PUSH(p_2, S)
for $i \leftarrow 3$ **to** m
    do while the angle formed by points NEXT-TO-TOP(S), TOP(S),
        and p_i makes a nonleft turn
        do POP(S)
    PUSH(p_i, S)
return S

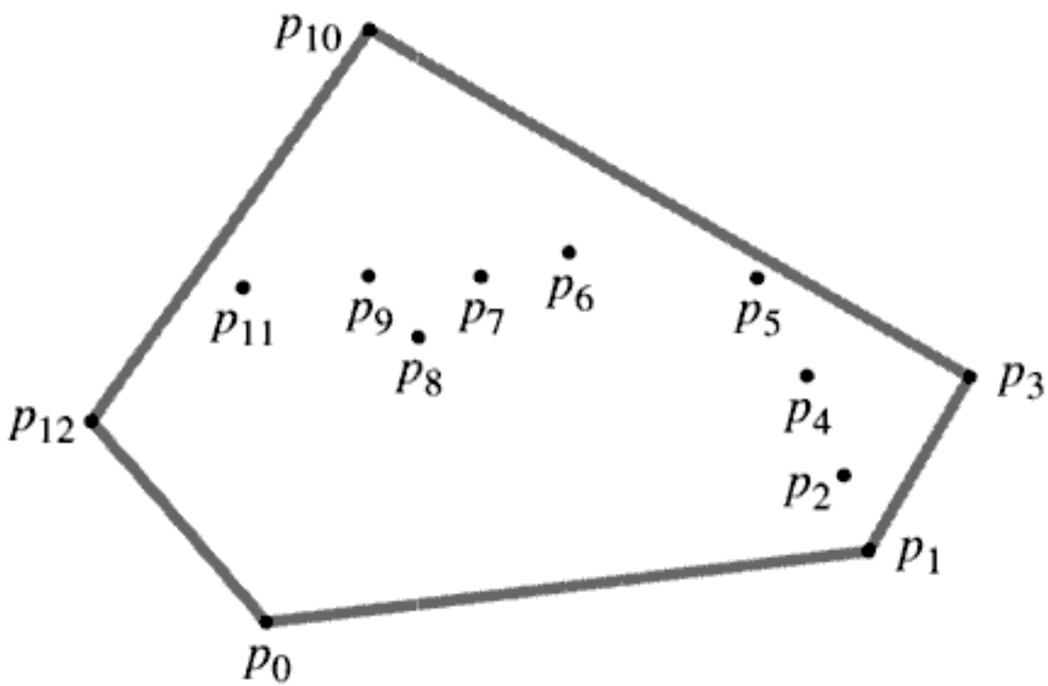
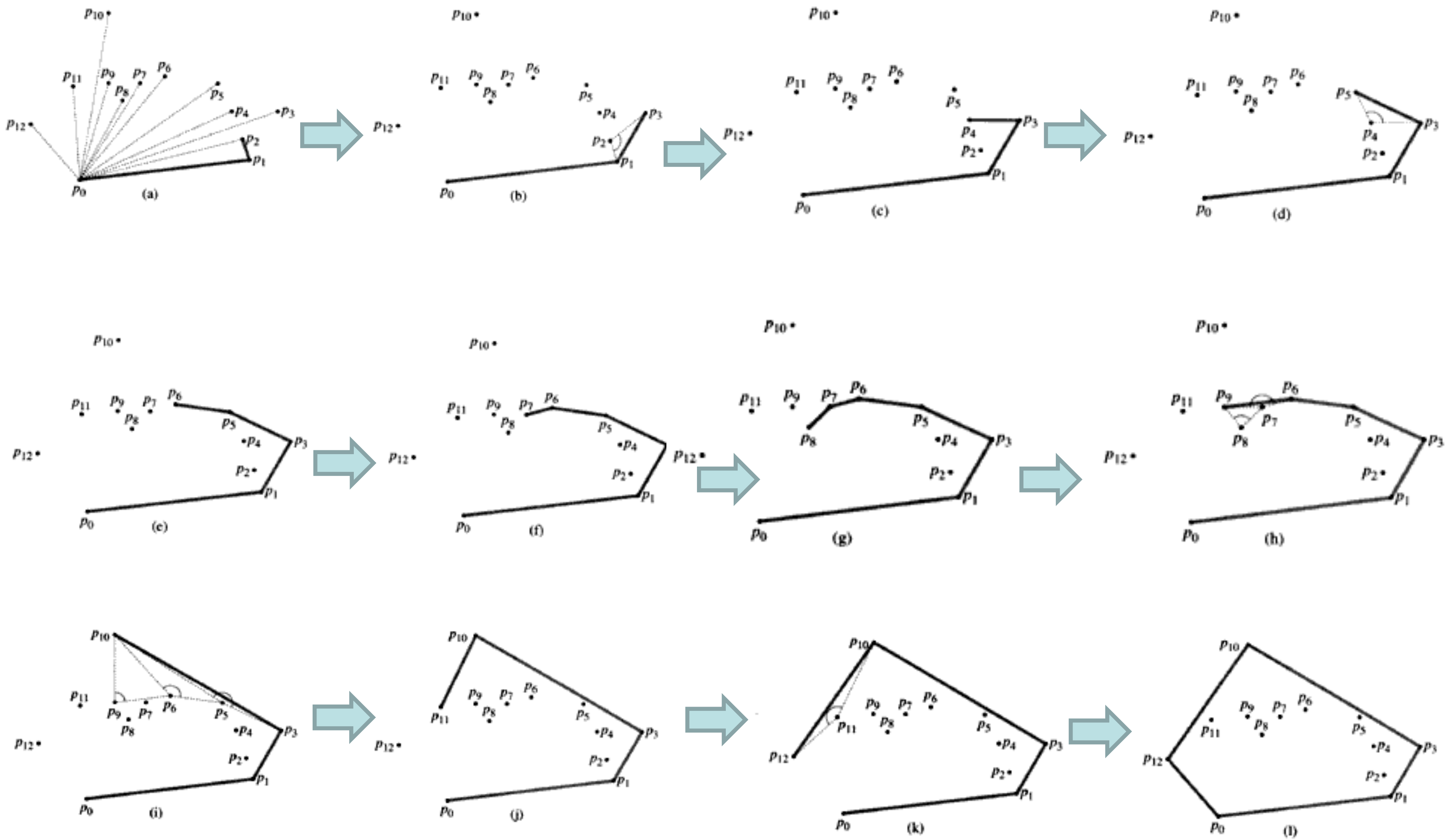


Figure 33.6 A set of points $Q = \{p_0, p_1, \ldots, p_{12}\}$ with its convex hull $\text{CH}(Q)$ in gray.



The execution of GRAHAM-SCAN on the set Q The current convex hull contained in stack S is shown in gray at each step. **(a)** The sequence $\langle p_1, p_2, \ldots, p_{12} \rangle$ of points numbered in order of increasing polar angle relative to p_0, and the initial stack S containing p_0, p_1, and p_2. **(b)–(k)** Stack S after each iteration of the **for** loop of lines 6–9. Dashed lines show nonleft turns, which cause points to be popped from the stack. In part (h), for example, the right turn at angle $\angle p_7 p_8 p_9$ causes p_8 to be popped, and then the right turn at angle $\angle p_6 p_7 p_9$ causes p_7 to be popped. **(l)** The convex hull returned by the procedure.

Convex Hull by Divide-and-Conquer

- First, sort all points by their x coordinate.
  - ($O(n \log n)$ time)
- Then divide and conquer:
  - Find the convex hull of the left half of points.
  - Find the convex hull of the right half of points.
  - Merge the two hulls into one.

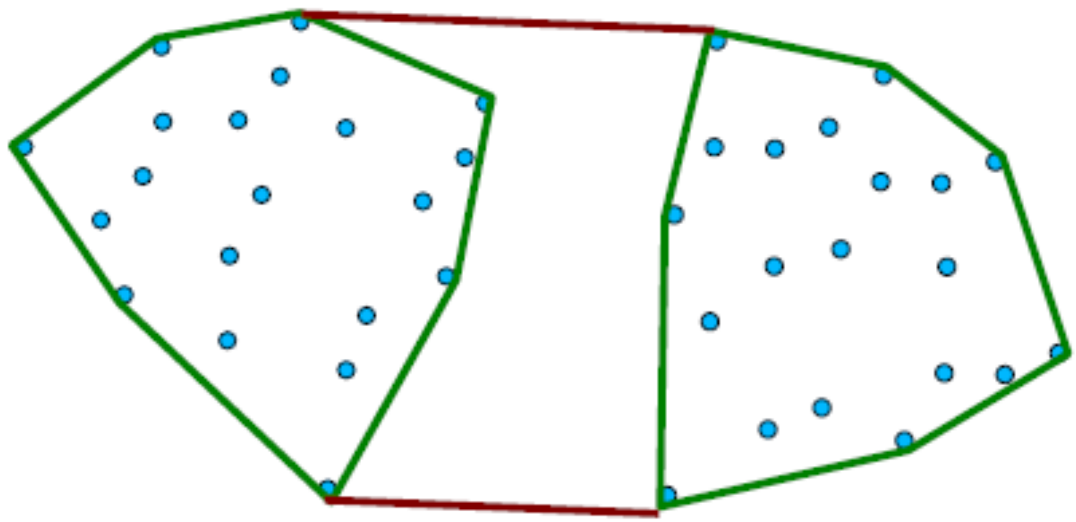
Convex Hull Pseudocode

```
//input: the number of points n, and
//an array of points S, sorted by x coord.
//output: the convex hull of the points in S.

point[] findHullDC(int n, point S[]) {
    if (n > 5) {
        int h = floor(n/2);
        m = n-h;
        point LH[], RH[]; //left and right hulls
        LH = findHullDC(h, S[1..h]);
        RH = findHullDC(m, S[h+1..n]);
        return mergeHulls(LH.size(), RH.size(),
                          LH, HR);
    } else {
        return Hull of S by exhaustive search;
    }
}
```

Merging Hulls

- Big picture:
  - first find the lines that are upper tangent, and lower tangent to the two hulls (the two red lines)

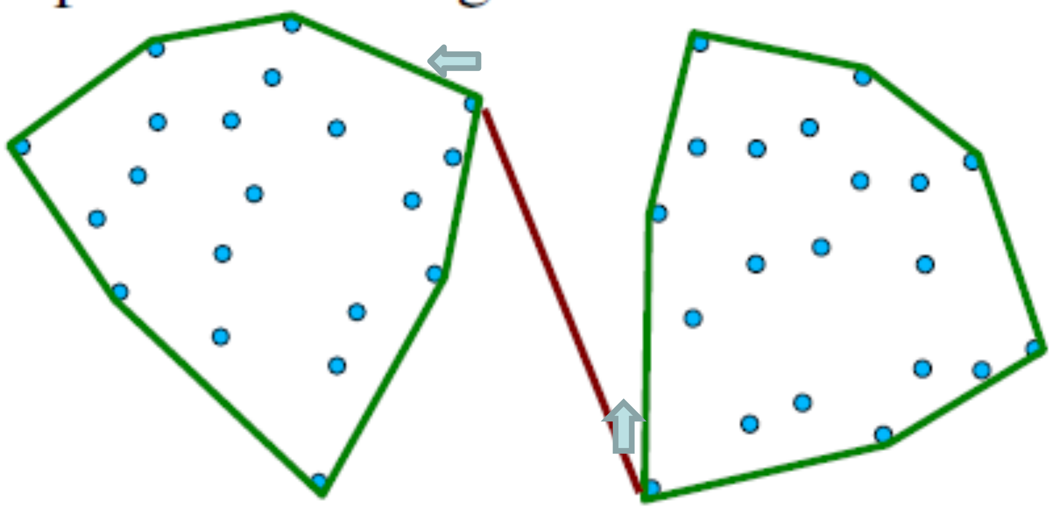


  - Then remove the points that are cut off.

Upper

Finding Tangent Lines

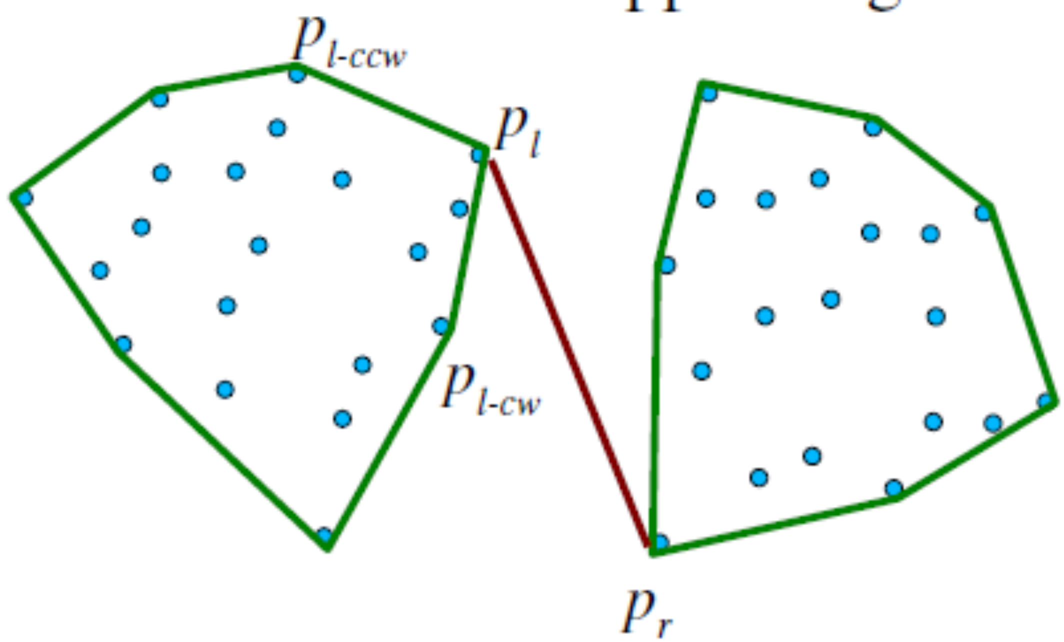
- Start with the rightmost point of the left hull, and the leftmost point of the right hull:



- While the line is not upper tangent to both left and right:
  - While the line is not upper tangent to the left, move to the next point (counter-clockwise).
  - While the line is not upper tangent to the right, move to the next point (clockwise).

Checking Tangentness

- How can we tell if a line is upper tangent to the left hull?

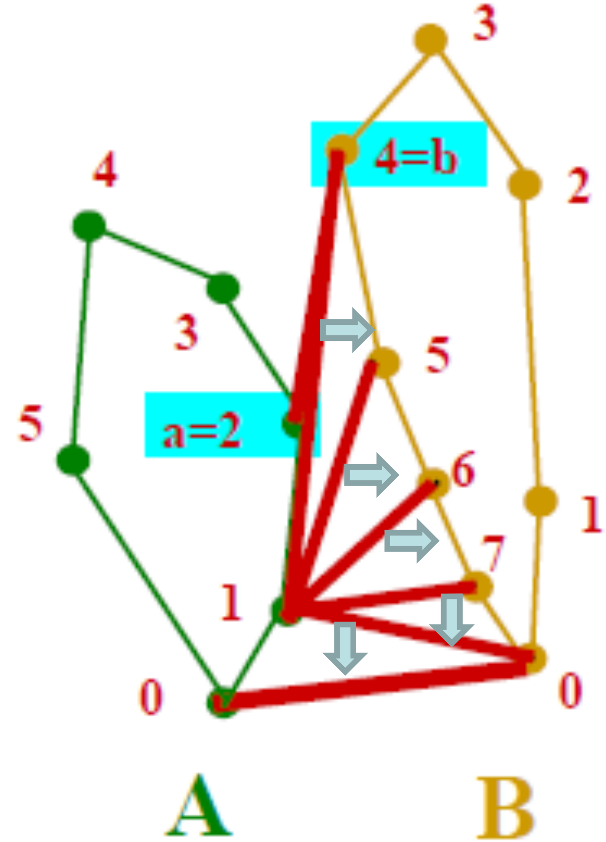


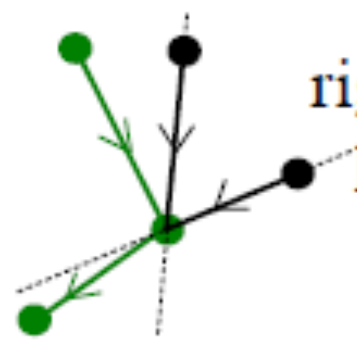
- The pair of line segments $\overline{p_r p_l}$, and $\overline{p_l p_{l\text{-}ccw}}$ should make a CCW turn at p_l.
- The same goes for $\overline{p_r p_l}$ and $\overline{p_l p_{l\text{-}ccw}}$.

Finding the lower tangent in O(*n*) time

```
a = rightmost point of A
b = leftmost point of B
while T=ab not lower tangent to both
      convex hulls of A and B do{
    while T not lower tangent to
     convex hull of A do{
       a=a-1
    }
    while T not lower tangent to
      convex hull of B do{
       b=b+1
     }
}
```

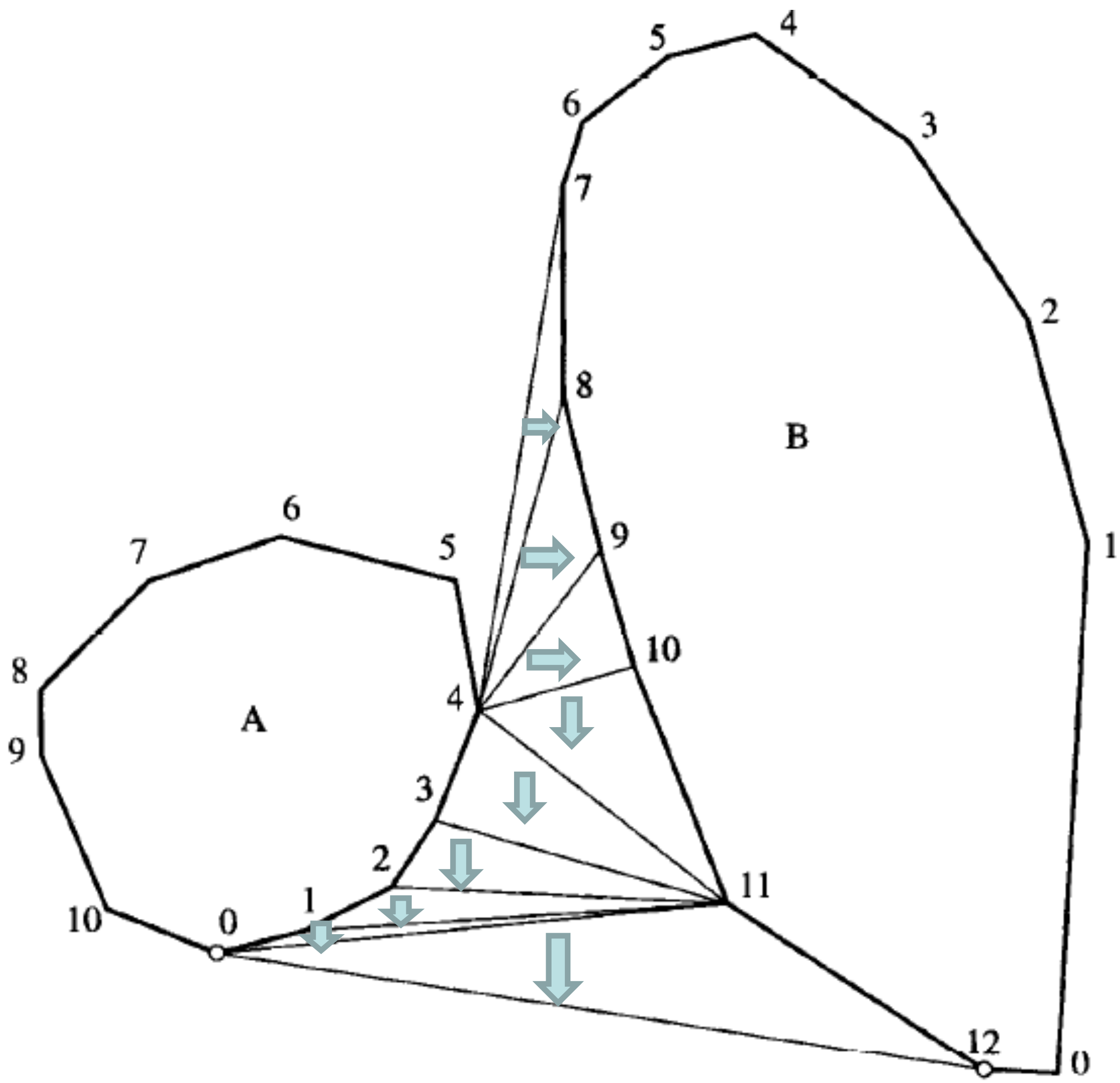
can be checked in constant time





right turn or left turn?

Lower Tangent Example



- •Initially, T=(4, 7) is only a lower tangent for A. The A loop does not execute, but the B loop increments b to 11.
- •But now T=(4, 11) is no longer a lower tangent for A, so the A loop decrements a to 0.
- •T=(0, 11) is not a lower tangent for B, so b is incremented to 12.
- • T=(0, 12) is a lower tangent for both A and B, and T is returned.

Convex Hull: Runtime

• Preprocessing: sort the points by x-coordinate	$O(n \log n)$ just once
• Divide the set of points into two sets **A** and **B**:	$O(1)$
• **A** contains the left $\lfloor n/2 \rfloor$ points,	
• **B** contains the right $\lceil n/2 \rceil$ points	
• Recursively compute the convex hull of **A**	$T(n/2)$
• Recursively compute the convex hull of **B**	$T(n/2)$
• Merge the two convex hulls	$O(n)$

$$T(n) = 2\,T(n/2) + cn$$

$$T(n) = O(n \log n)$$