# Asymptotics and Recurrence Equations

Analysis of Algorithms

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# Convergent Power Sum

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \leq O(1), \text{ for } 0 < x < 1$$

- A polynomial is asymptotically equal to its leading term as x → ∞

$$\sum_{i=0}^{d} a_i \, x^i = \theta\left(x^d\right)$$

$$\sum_{i=0}^{d} a_i \, x^i = o\left(x^{d+1}\right)$$

$$\sum_{i=0}^{d} a_i \, x^i \sim a_d \, x^d$$

# Sums of Powers

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- for n → ∞

$$\sum_{i=1}^{n} i^{d} \sim \frac{1}{d+1} n^{d+1}$$

- Or equivalently

$$\sum_{i=1}^{n} i^{d} = \frac{1}{d+1} n^{d+1} + o\left(n^{d+1}\right)$$

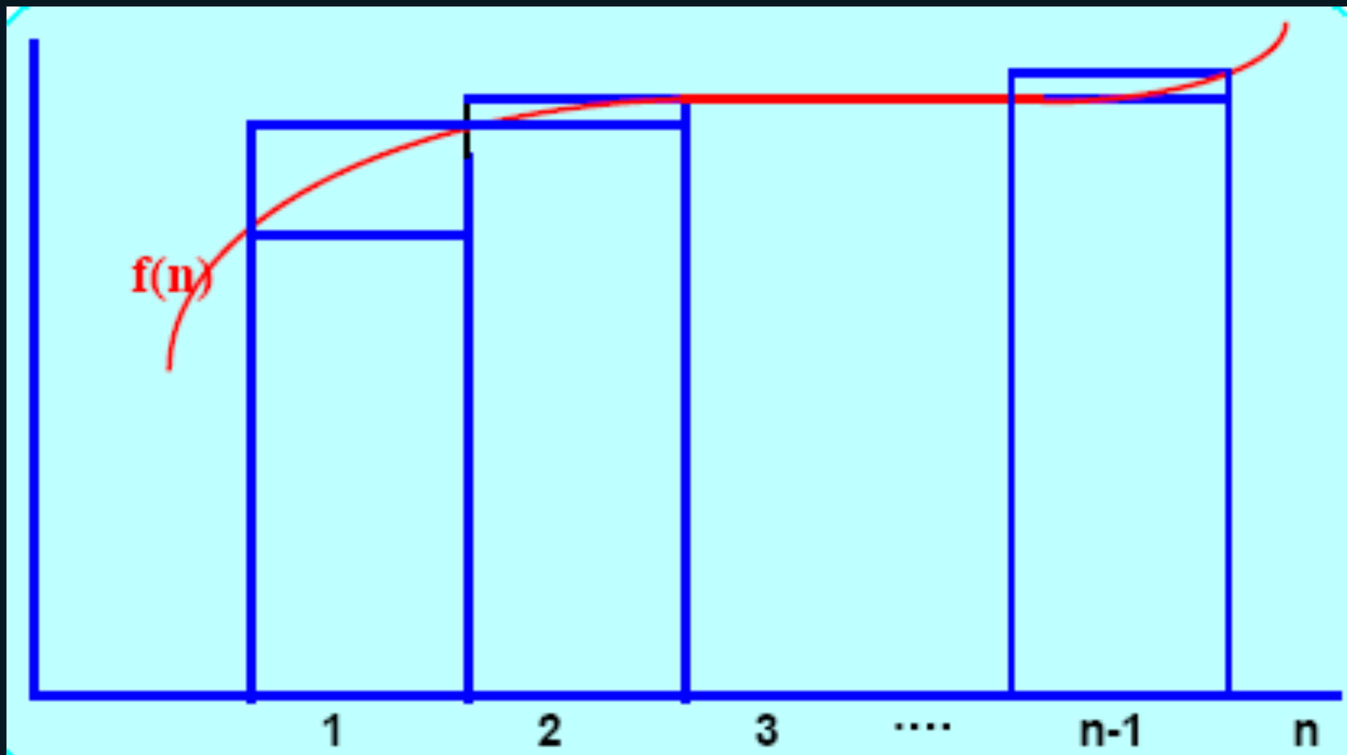
# Examples

$$\begin{cases} \sum_{i=1}^{n} i \sim \frac{n^2}{2} \\ \sum_{i=1}^{n} i^2 \sim \frac{n^3}{3} \end{cases}$$

- 2nd order asymptotic expansion

$$\sum_{i=1}^{n} i^d = \frac{1}{d+1} n^{d+1} + \frac{1}{2} n^d + o\left(n^d\right)$$

# Bounding Sums by Integrals




$$\sum_{k=1}^{n} f(k) \leq \int_{1}^{n+1} f(x)\,dx \leq \sum_{k=1}^{n} f(k+1)$$

# Bounding Sums by Integrals (cont' d)

$$\sum_{k=1}^{n} f(k) \le \int_{1}^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k+1)$$

- So $\displaystyle\int_{1}^{n+1} f(x)\,dx - f(n+1) + f(1) \le \sum_{k=1}^{n} f(k) \le \int_{1}^{n+1} f(x)\,dx$
- Example $\text{if } f(x) = \ln(x)$ then $\displaystyle\int \ln(x)dx = x\ln(x) - x$

# Bounding Sums by Integrals (cont' d)

- So $\sum_{k=1}^{n} \ln k = (n+1)\ln(n+1) - n + \theta(\ln(n))$
- Since $\log(n) = \frac{\ln n}{\ln 2}$
- So $\sum_{k=1}^{n} \log k = (n+1)\log(n+1) - \frac{n}{\ln 2} + \theta(\log n)$

# Other Approximations Derived from Integrals

$$\sum_{k=1}^{n} k \log k = \frac{(n+1)^2}{2} \log (n+1) - \frac{(n+1)^2}{4 \ln 2} + \theta (n \log n)$$

- Harmonic Numbers

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

$$\left( \begin{array}{l} H_n = \ln(n) + \gamma + O\left(\frac{1}{n}\right) \\ \text{Euler's constant } \gamma = .577... \end{array} \right.$$

# Stirling's Approximation for Factorial

- Factorial $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$

$$n! \sim \sqrt{2\pi n}\; n^{n}\, e^{-n} \quad \text{as } n \to \infty$$

- So

$$\log(n!) = n \log n - n \log e + \frac{1}{2} \log(2\pi n) + \theta(1)$$
$$= n \log n - \theta(n)$$

# Some intuition for Stirling's approximation

We can upper bound the log of the factorial:

$$
\begin{aligned}
n! &= (n)(n-1)\ldots(2)(1) \\
&\leq (n)(n)\ldots(n)(n) \\
&= n^n \\
\log(n!) &\leq \log(n^n) \\
&= n\log(n) \\
\log(n!) &= O(n\log n)
\end{aligned}
$$

# Some Intuition for Stirling's Approximation

We can also lower bound the log of the factorial:

$$\begin{aligned}
n! &= (n)(n-1)\dots(2)(1) \\
&\geq (n)(n-1)\dots(n/2) \\
&\geq (n/2)(n/2)\dots(n/2) \\
&= (n/2)^{n/2} \\
\log(n!) &\geq (n/2)\log(n/2) \\
&= (n/2)\log(n) - (n/2)\log(2) \\
\log(n!) &= \Omega(n \log n)
\end{aligned}$$

Combining the above two bounds, we know that $\log(n!) = \Theta(n \log n)$.