# Probability Theory Overview and Analysis of Randomized Algorithms

Analysis of Algorithms

*Prepared by*

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# Probability Theory Topics

a) Random Variables: Binomial and Geometric
b) Useful Probabilistic Bounds and Inequalities

# Probability Measures

- A probability measure (Prob) is a mapping from a set of events to the reals such that:

1) For any event A

$$0 \leq \text{Prob}(A) \leq 1$$

2) Prob (all possible events) = 1

3) If A, B are mutually exclusive events, then

$$\text{Prob}(A \cup B) = \text{Prob}(A) + \text{Prob}(B)$$

# Conditional Probability

- Define

$$\text{Prob}(A \mid B) = \frac{\text{Prob}(A \wedge B)}{\text{Prob}(B)}$$

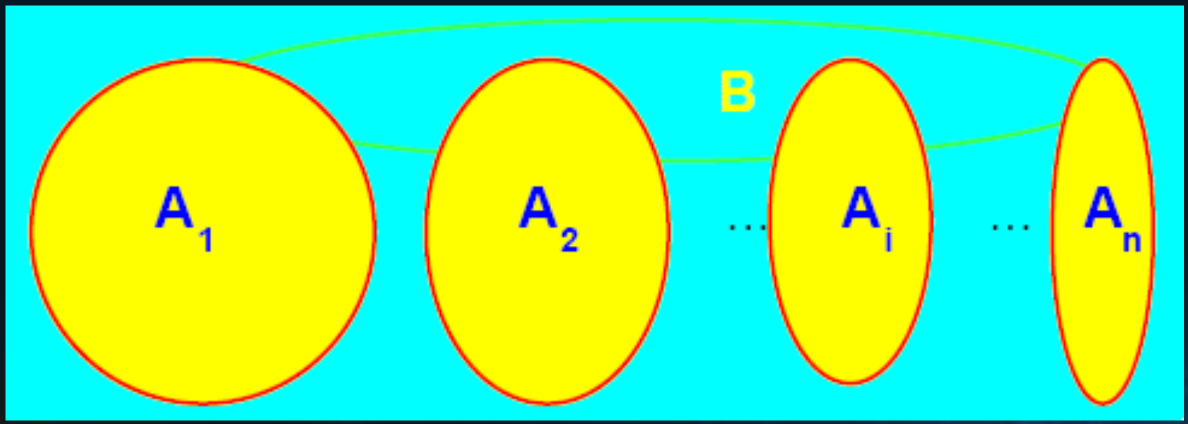
$$\text{for Prob}(B) > 0$$

# Bayes' Theorem

- If $A_1, \ldots, A_n$ are mutually exclusive and contain all events then

$$\text{Prob}(A_i \mid B) = \frac{P_i}{\sum_{j=1}^{n} P_j}$$

$$\text{where } P_j = \text{Prob}(B \mid A_j) \cdot \text{Prob}(A_j)$$



# An Quick Proof That This Works

- P(B and A) = P(B|A)P(A)
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- P(A|B)P(B) = P(B|A)P(A)
- P(A|B) = $\frac{P(B|A)P(A)}{P(B)}$
- P(B) = $\sum_{j=1}^{n} P(B|Aj)P(Aj)$

# Bayes' Theorem Example

- (This example from https://dlsun.github.io/probability/bayes.html)
- ELIZA HIV testing
- 1% of adult males have HIV
- If a patient has HIV, ELIZA is positive 98% of the time
- If a patient does not have HIV, ELIZA is negative 94% of the time
- Say a patient gets a positive result from ELIZA, what is the probability he has HIV?

# Working out the Example

- A is event that patient has HIV
- B is event that patient gets positive test
- P(A) = 1%
- P(B|A) = 98%
- P(B) = (.98)(.01)+(.06)(.99)=6.92%
- P(A|B) = $\frac{P(B|A)P(A)}{P(B)} = \frac{(.98)(.01)}{.0692}$

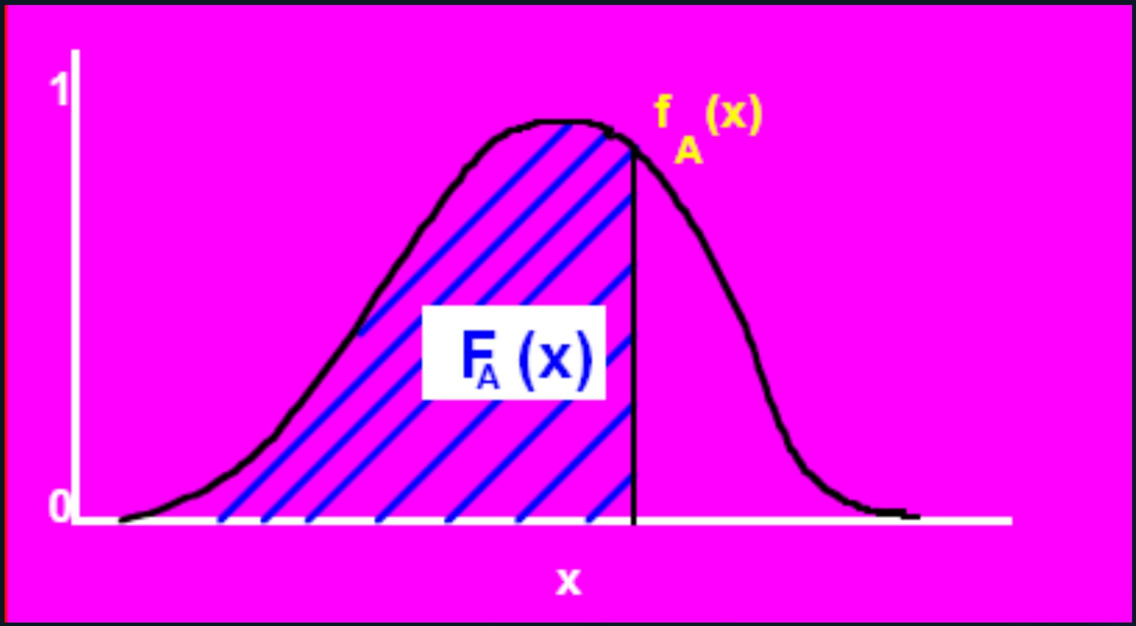
# The Result:

- Patient only has a 14.6% chance of having HIV despite receiving a positive test.
- Test is pretty good, but with a very small initial probability, it is not good enough
- Applications to medical practice?
- Applications to Duke COVID testing policy?

# Random Variable A (Over Real Numbers)

- Density Function

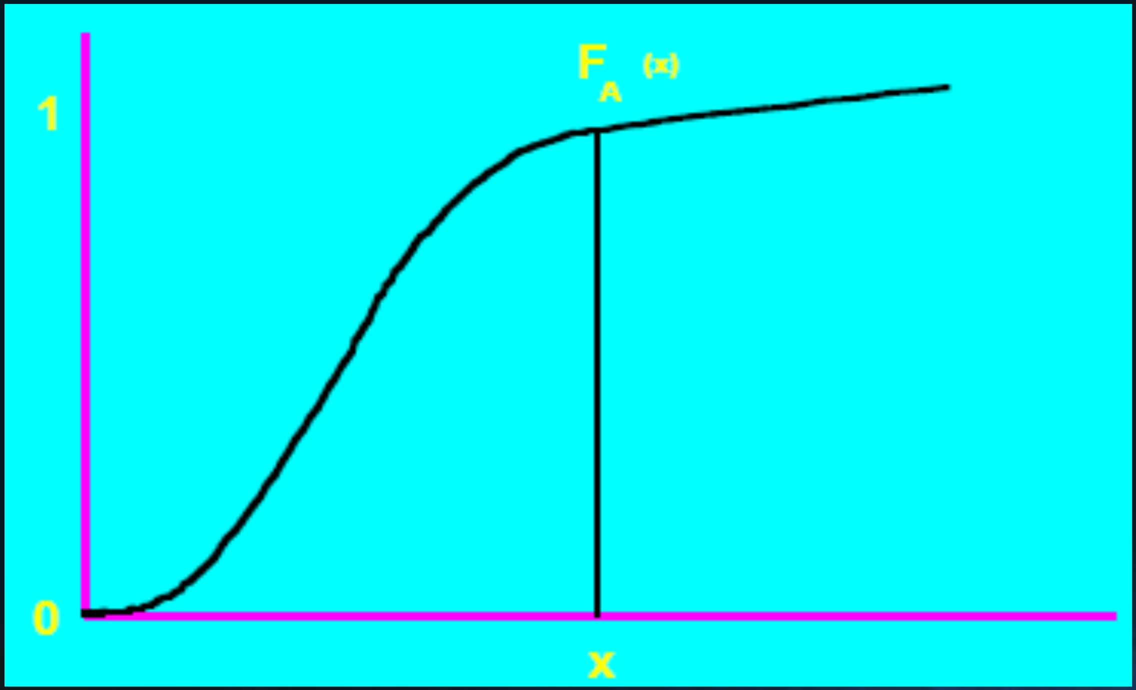
$$f_A(x) = \text{Prob}(A=x)$$



# Random Variable A (cont’ d)

- Prob Distribution Function

$$F_A(x) = \mathrm{Prob}(A \le x) = \int_{-\infty}^{x} f_A(x)\,dx$$

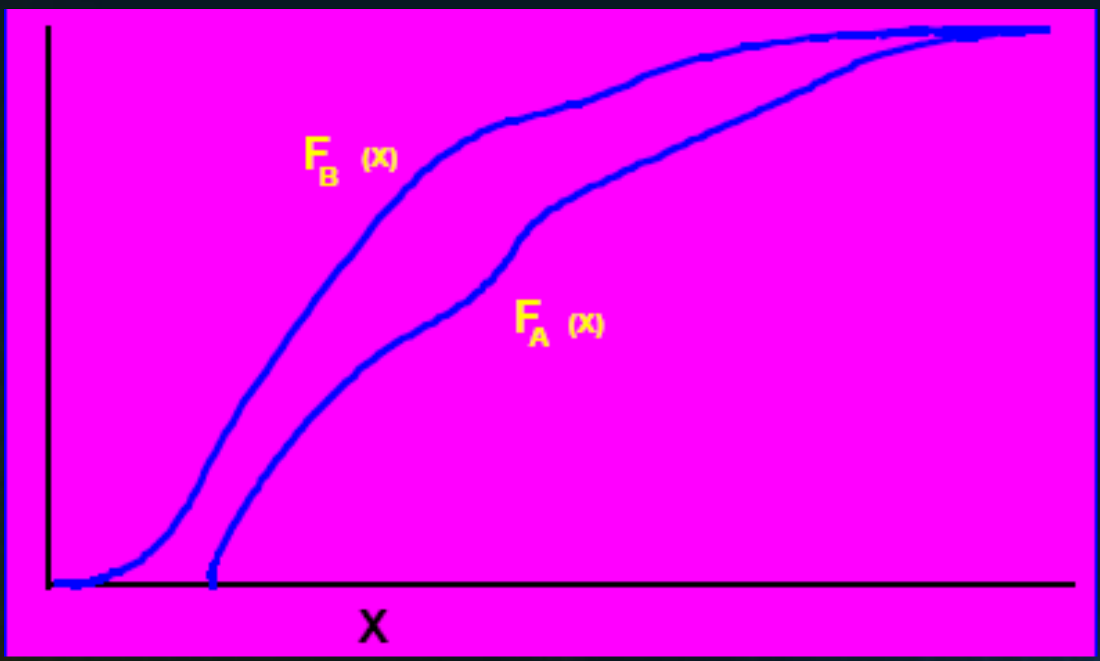


# Random Variable A (cont' d)

- If for Random Variables A,B

$$\forall x \quad F_A(x) \leq F_B(x)$$

- Then "A upper bounds B" and "B lower bounds A"

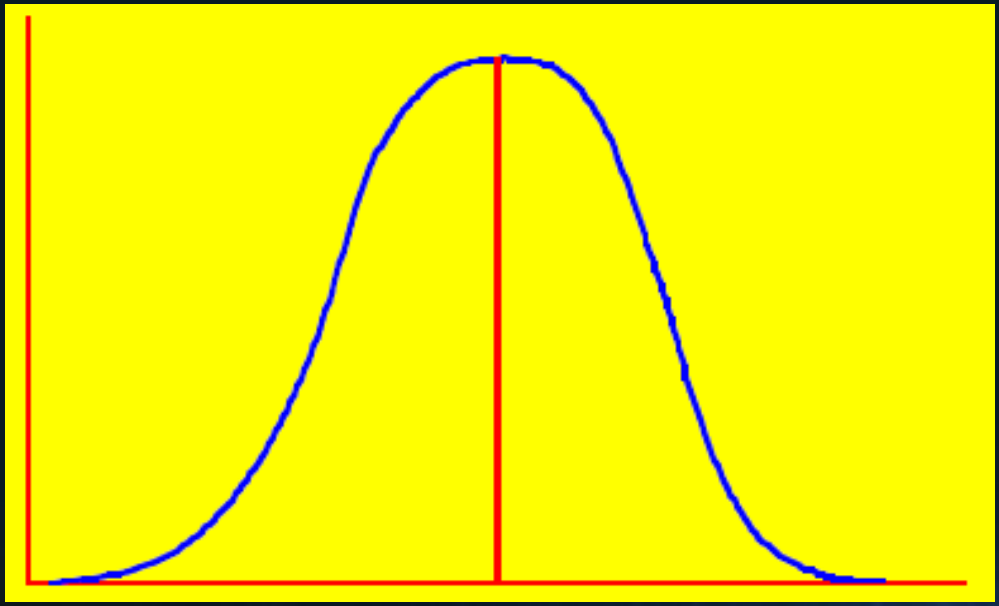



$$F_A(x) = \text{Prob}\ (A \leq x)$$
$$F_B(x) = \text{Prob}\ (B \leq x)$$

# Expectation of Random Variable A

$$E(A) = \bar{A} = \int_{-\infty}^{\infty} x \, f_A(x) \, dx$$

- $\bar{A}$ is also called "average of A" and "mean of A" = $\mu_A$

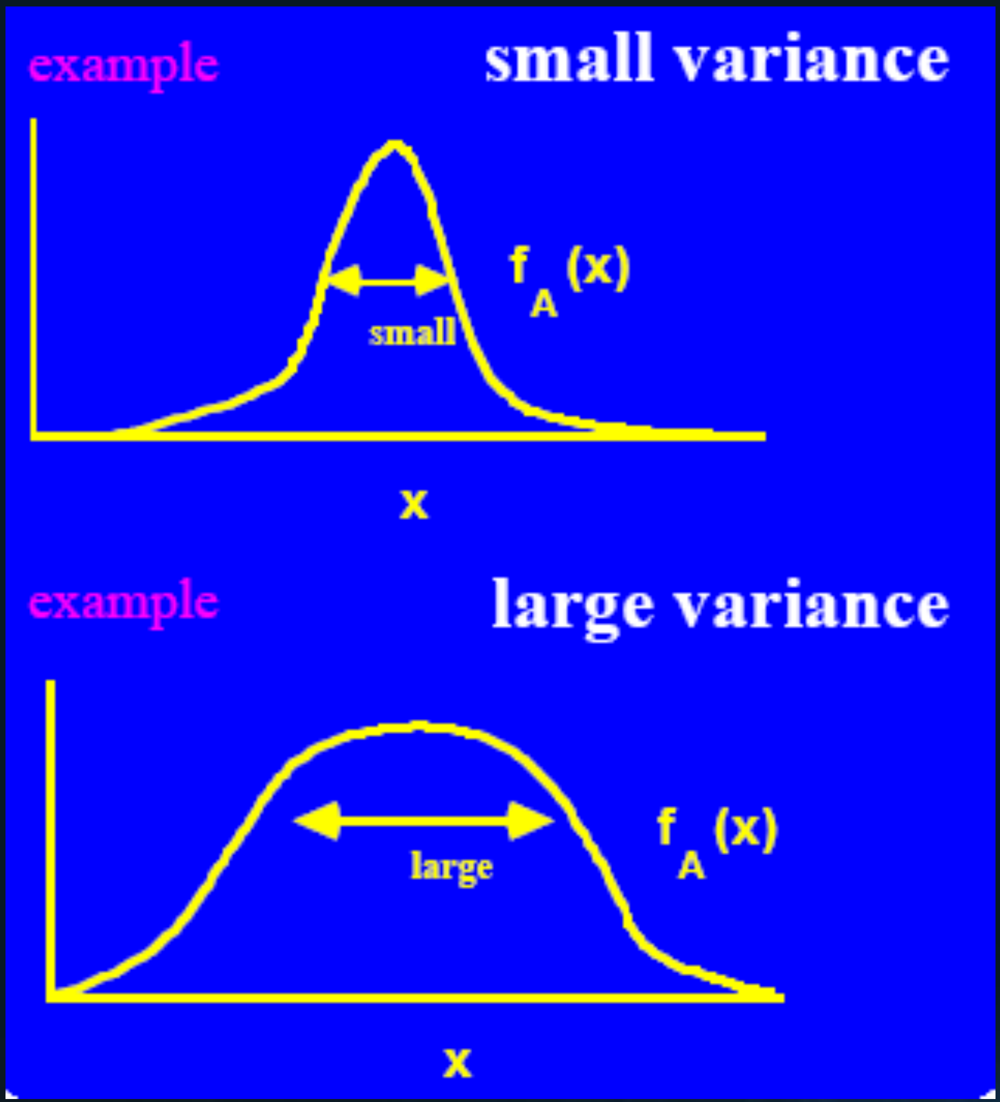


# Variance of Random Variable A

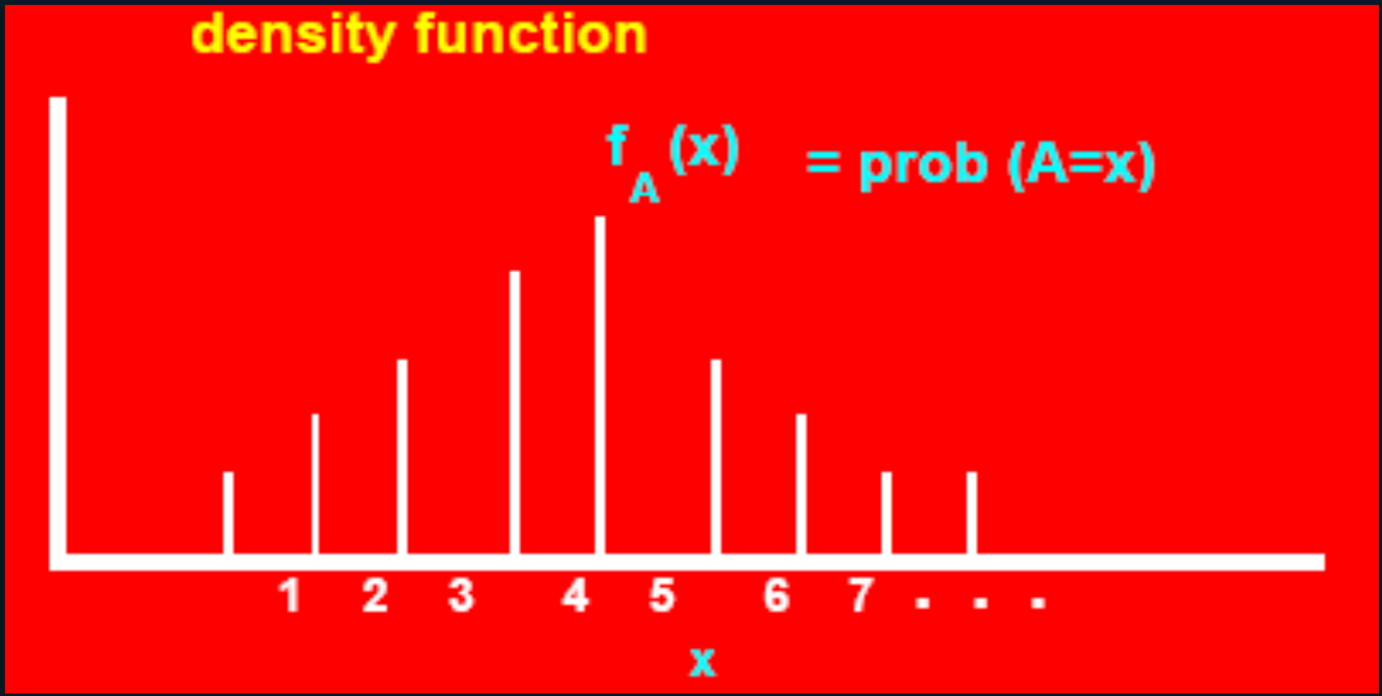
$$\sigma_A^2 = \overline{(A - \overline{A})^2} = \overline{A^2} - (\overline{A})^2$$

where 2nd momoment $\overline{A^2} = \int_{-\infty}^{\infty} x^2 \, f_A(x)\, dx$

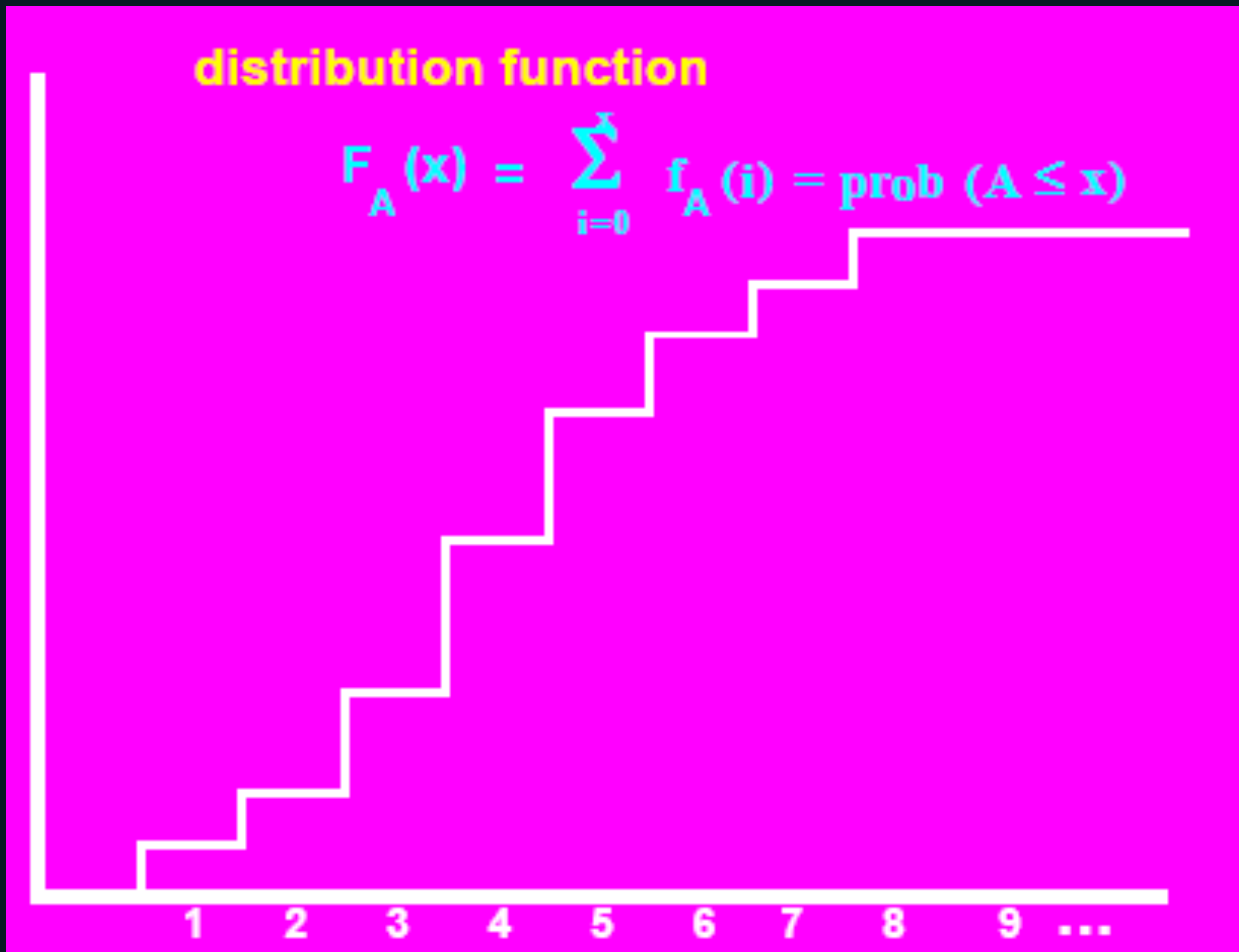
# Variance of Random Variable A (cont' d)



# Discrete Random Variable A



# Discrete Random Variable A (cont' d)



# Discrete Random Variable A Over Nonnegative Numbers

- Expectation

$$E(A) = \bar{A} = \sum_{x=0}^{\infty} x\, f_A(x)$$

# Pair-Wise Independent Random Variables

- A,B independent if

$$\text{Prob}(A \wedge B) = \text{Prob}(A) * \text{Prob}(B)$$

# Bounding Numbers of Permutations

- n! = n * (n-1) * 2 * 1
  = number of permutations of n objects

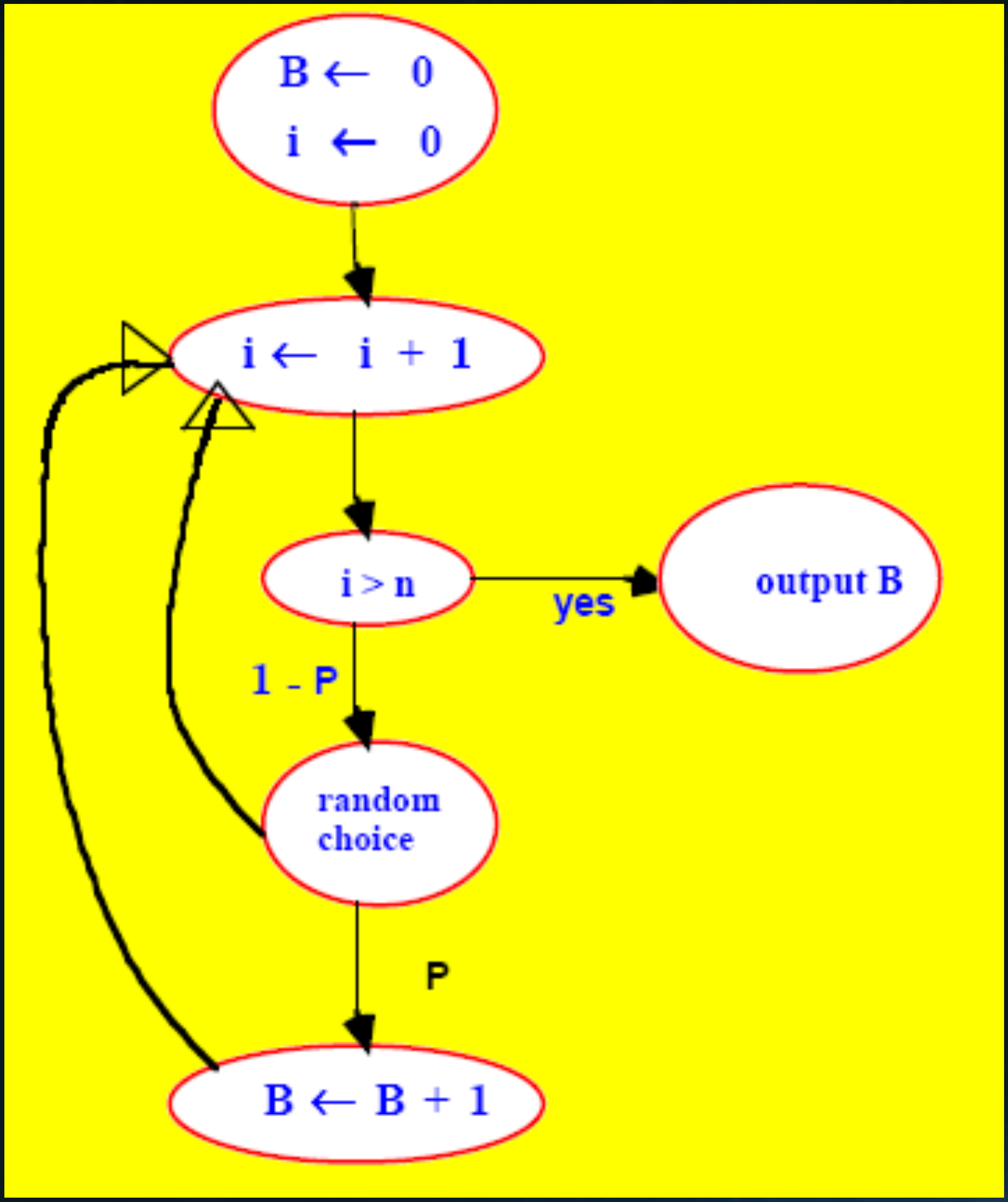
- Stirling's formula
  n! = f(n) (1+o(1)), where

$$f(n) = n^n e^{-n} \sqrt{2\pi n}$$

# Bernoulli Variable

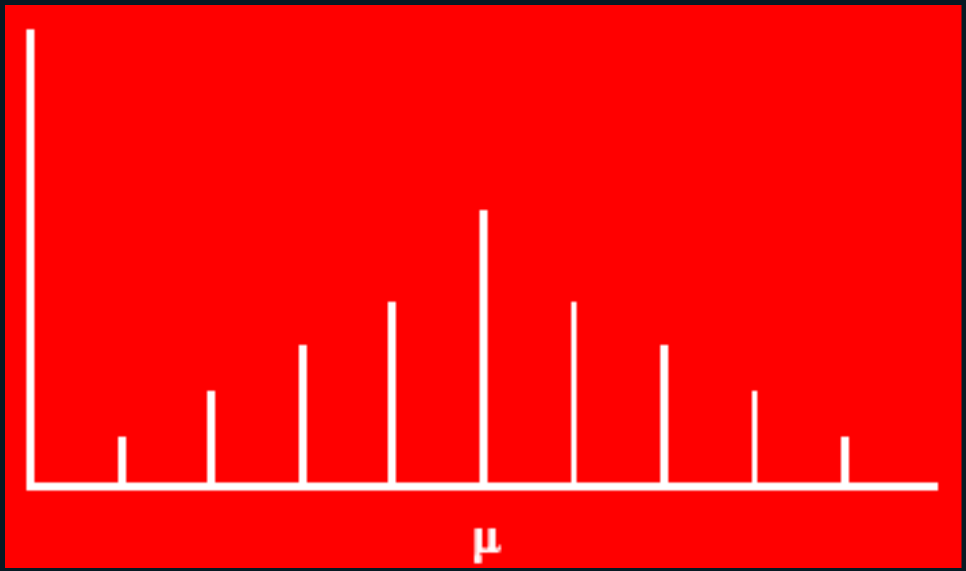
- $A_i$ is 1 with prob P and 0 with prob 1-P
- Binomial Variable
  - B is sum of n independent Bernoulli variables $A_i$ each with some probability p

```
procedure   BINOMIAL with parameters n,p
  begin     B ← 0
            for i=1 to n do
                 with probability P do B ← B+1
            output
  end
```



B ← 0
i ← 0
i ← i + 1
i > n
yes
output B
1 - P
random
choice
P
B ← B + 1

# B is Binomial Variable with Parameters n,p




$$mean \quad \mu = \mathrm{n} \cdot \mathrm{p}$$

$$variance \quad \sigma^2 = \mathrm{np}\ (1\text{-}\mathrm{p})$$

# B is Binomial Variable with Parameters n,p (cont'd)

$$\textit{density fn} = \text{Prob}(B=x) = \binom{n}{x} p^{x} \, (1\text{-}p)^{n\text{-}x}$$

$$\textit{distribution fn} = \text{Prob}(B \leq x) = \sum_{k=0}^{x} \binom{n}{k} p^{k} \, (1\text{-}p)^{n\text{-}k}$$

# Geometric Variable V

- parameter p

$$\forall x \geq 0 \quad \text{Prob}(V=x) = p(1-p)^x$$

*procedure*

GEOMETRIC parameter p

*begin* $V \leftarrow 0$

loop with probability 1-p

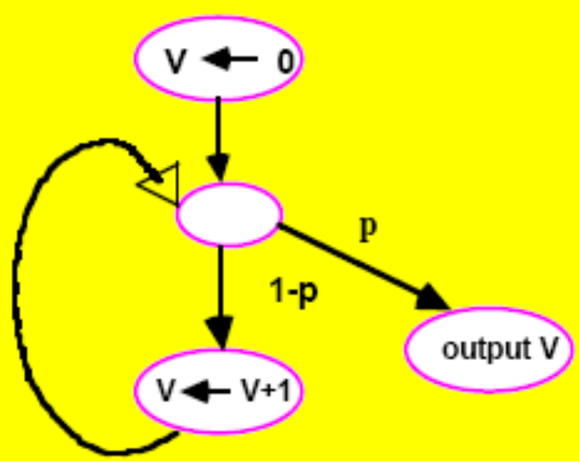
*goto* exit

$V \leftarrow V+1$

*goto* loop

exit: *output* V

*mean* $\mu = \frac{1-p}{p}$

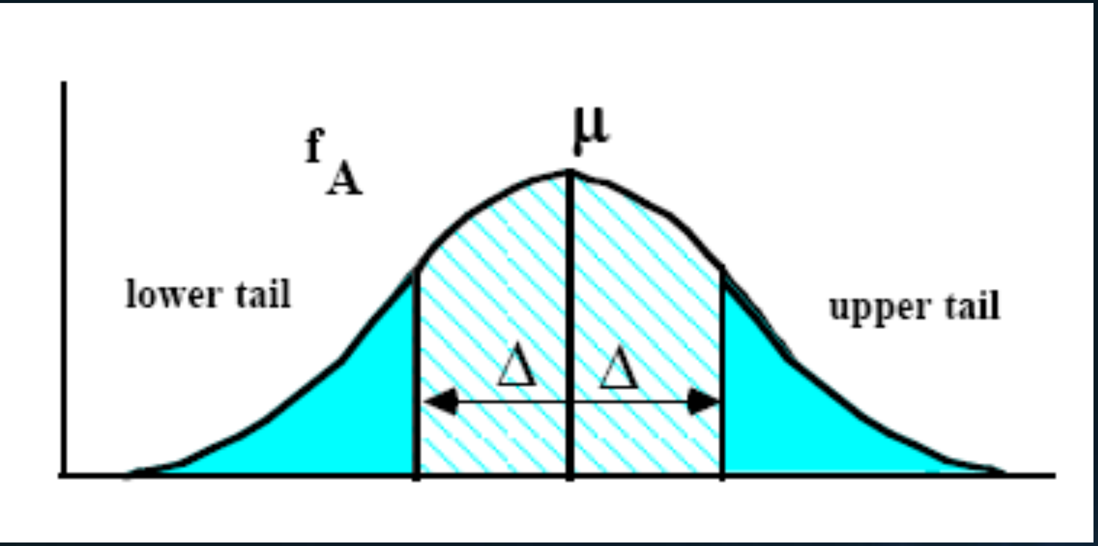


# Probabilistic Inequalities

- For Random Variable A

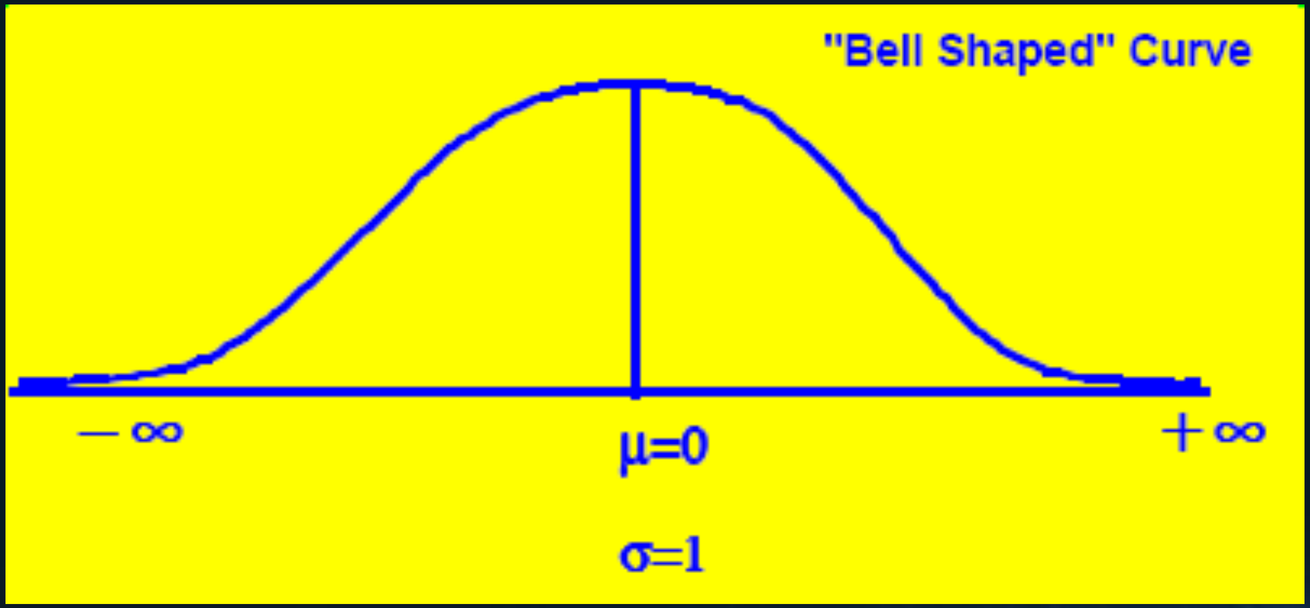
$$mean \quad \mu = \overline{A}$$

$$variance \quad \sigma^2 = \overline{A^2} - (\overline{A})^2$$



# Gaussian Density Function

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



# Normal Distribution

$$\Phi(X) = \int_{-\infty}^{x} \Psi(Y)\, dY$$

- Bounds x > 0 (Feller, p. 175)

$$\Psi(x)\left(\frac{1}{x} - \frac{1}{x^3}\right) \le 1 - \Phi(x) \le \frac{\Psi(x)}{x}$$

$$\forall x\ \varepsilon\, [0,1]$$

$$\frac{x}{\sqrt{2\pi e}} = x\,\Psi(1) \le \Phi(x) - \frac{1}{2} \le \Psi(0) = \frac{x}{\sqrt{2\pi}}$$

# Sums of Independently Distributed Variables

- Let $S_n$ be the sum of n independently distributed variables $A_1, \ldots, A_n$
- Each with mean $\frac{\mu}{n}$ and variance $\frac{\sigma^2}{n}$
- So $S_n$ has mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\sigma}^2$

# Strong Law of Large Numbers: Limiting to Normal Distribution

- The probability density function of

$$T_n = \frac{(S_n - \mu)}{\sigma} \ \textit{limits} \text{ as } n \to \infty$$

to normal distribution Φ(x)

- Hence

$$\text{Prob}(|S_n - \mu| \le \sigma x) \to \Phi(x) \text{ as } n \to \infty$$