What Is The Goal?

- We are given two vectors $x, y$
- We sample a random vector $r$, where each entry of $r$ is drawn from a normal distribution
- What is the probability that the sign of $x \cdot r$ is the same as the sign of $y \cdot r$?
- We will show that the following holds, where $\Theta$ is the angle between the vectors:

$$Pr[\text{Sign}(x \cdot r) = \text{Sign}(y \cdot r)] = 1 - \frac{\theta_{xy}}{\pi}$$
Why do we care?

- Useful for locality sensitive hashing
- Also relevant in political science (Majority is Stablest Theorem)
  - Take 2 voting populations p, q, where votes are random but have correlation ρ
  - The probability that p and q elect the same candidate under majority rule is
  - It turns out this is the highest probability possible for the results to align under reasonable assumptions about the voting rule

Quick Review of Dot Products

\[
x \cdot y = \sum_{i=1}^{d} x_i y_i \\
\cos \theta_{xy} = \frac{x \cdot y}{\|x\|_2 \|y\|_2}
\]

- The cosine above is positive when the angle between x and y is in the range -90 to 90 degrees (-π/2 to π/2 radians)
- Equivalently, the dot product is positive when the normal vector to one of the two vectors does not fall between the two.
When do two Random Vector Dot Products Share a sign?

- The random vector must fall within $\pi/2$ radians of both vectors, or more than $\pi/2$ radians away from both vectors.
- Equivalently, the normal vector to the random vector cannot fall in the acute angle formed by the two vectors.
- Because entries are gaussian, the angle of $r$ and its normal vector will be uniformly distributed over $[0, 2\pi]$.
- The probability of having the same sign is just the probability the normal vector does not fall in the acute part $1 - \frac{\theta}{2\pi} = 1 - \frac{\theta}{\pi}$.

Where will $r$'s normal vector fall in this diagram?

$r$'s angle will be distributed uniformly on $[0, 2\pi]$. 

Where will $r$'s normal vector fall in this diagram?